Part A Quantum Theory: Additional questions

1. Consider the one-dimensional harmonic oscillator with stationary state Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \frac{1}{2}m\omega^2 x^2\psi \quad = \quad E\psi \ .$$

Show that $\psi_0(x) = \exp\left(-\xi^2/2\right)$ solves this equation for appropriate energy E_0, where $\xi = \sqrt{\frac{m\omega}{\hbar}}x$.

Given that $\psi_2(x) = \left(\xi^2 + b\right)\exp[-\xi^2/2]$ is orthogonal to $\psi_0(x)$, in that

$$\int_{-\infty}^{\infty} \psi_0(x)\psi_2(x)\,\mathrm{d}x \quad = \quad 0 \ ,$$

hence determine the constant b.

2. Let $\psi(x)$ be a normalized stationary state wave function for a particle moving on the x-axis. Define

   (a) the probability density function $\rho(x)$;
   (b) the expectation value $\langle f(x)\rangle_\psi$ of a function $f(x)$ in the state ψ.

Given that the normalized stationary state wave functions for a particle confined to the interval $[0, a] \subset \mathbb{R}$ are

$$\psi_n(x) \quad = \quad \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a} \ ,$$

for $x \in [0, a]$ and zero otherwise, where n is a positive integer, show that

$$\langle x^2\rangle_{\psi_n} - \left(\langle x\rangle_{\psi_n}\right)^2 \quad = \quad \frac{a^2}{12}\left(1 - \frac{6}{n^2\pi^2}\right) \ .$$

Comment on the limit $n \to \infty$.

3. The potential function for a *quantum bouncing ball* of mass m is

$$V(z) \quad = \quad \begin{cases} mgz \ , & z > 0 \ , \\ \infty \ , & z \leq 0 \ , \end{cases}$$

where g is (the usual positive gravitational) constant. What is the wave function $\psi(z)$ for $z \leq 0$? Explain the physical significance of this.

By an appropriate change of variables, show that the corresponding one-dimensional stationary state Schrödinger equation (in the z direction) may be written as

$$\frac{\mathrm{d}^2 f}{\mathrm{d}y^2} - yf \quad = \quad 0 \ .$$

[*Hint: write $z = ay + b$.*]

This equation has a normalizable solution $f(y) = \mathrm{Ai}(y)$, where Ai is called the *Airy* function. Show that the energy levels of the bouncing ball are

$$E_n \quad = \quad -\left(\frac{mg^2\hbar^2}{2}\right)^{1/3} y_n \ ,$$

where y_n are the zeros of the Airy function. [*In fact there are a countably infinite number of such zeros* $\ldots < y_3 < y_2 < y_1 < 0$.]

4. Verify that the *Gaussian wave packet*

$$\Psi(x,t) \quad = \quad \frac{1}{\pi^{1/4}\left[1+(\mathrm{i}\hbar t/m)\right]^{1/2}} \exp\left[\frac{-x^2}{2[1+(\mathrm{i}\hbar t/m)]}\right]$$

satisfies the free Schrödinger equation and is normalized for all times t.

5. Write down the stationary state Schrödinger equation for a one-dimensional harmonic oscillator with coordinate x, mass m and angular frequency ω. Show that for an appropriate constant C the wave function

$$\psi(x) = x\,\mathrm{e}^{-Cx^2}$$

is a stationary state of energy E, which you should identify.

6. (i) Write down the stationary state Schrödinger equation for a particle of mass m and energy E moving in $\mathbb{R}^3$ under the influence of a potential $V(\mathbf{r})$. What does it mean to say that the wave function is *normalized*? How is this condition related to the probabilistic interpretation of the wave function?

   (ii) A particle of mass m moves inside a ball of radius a, centred on the origin of $\mathbb{R}^3$, such that the potential $V = 0$ inside the ball and the wave function vanishes on and outside the boundary. Show that there are continuous stationary state wave functions of the form $\psi(r)$, depending only on distance $r = |\mathbf{r}|$ from the origin, that satisfy the stationary state Schrödinger equation with energies

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}\ ,$$

   for $n = 1, 2, \ldots$ a positive integer. Determine the probability that a particle in such a state is found within a distance $\frac{1}{4}a$ of the origin. Comment on the limit of this formula as $n \to \infty$.

Please send comments and corrections to `sparks@maths.ox.ac.uk`.