Sasaki-Einstein Manifolds

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A Sasaki-Einstein manifold is a Riemannian manifold (S, g) that is both Sasakian and Einstein.

Sasakian geometry is the odd-dimensional cousin of Kähler geometry. Indeed, just as Kähler geometry is the natural intersection of complex, symplectic, and Riemannian geometry, so Sasakian geometry is the natural intersection of CR, contact, and Riemannian geometry. Perhaps the most straightforward definition is the following: a Riemannian manifold (S, g) is Sasakian if and only if its metric cone $\big(C(S) = \mathbb{R}_{>0} \times S, \bar{g} = \mathrm{d}r^2 + r^2 g\big)$ is Kähler. In particular, (S, g) has odd dimension $2n - 1$, where n is the complex dimension of the Kähler cone.

A metric g is Einstein if $\mathrm{Ric}_g = \lambda g$ for some constant λ. It turns out that a Sasakian manifold can be Einstein only for $\lambda = 2(n-1)$, so that g has positive Ricci curvature. Assuming, as we shall do throughout, that (S, g) is complete, it follows from Myers' Theorem that S is compact with finite fundamental group. Moreover, a simple calculation shows that a Sasakian metric g is Einstein with $\mathrm{Ric}_g = 2(n-1)g$ if and only if the cone metric $\bar{g}$ is Ricci-flat, $\mathrm{Ric}_{\bar{g}} = 0$. It immediately follows that for a Sasaki-Einstein manifold the restricted holonomy group of the cone $\mathrm{Hol}^0(\bar{g}) \subset SU(n)$.

The canonical example of a Sasaki-Einstein manifold is the odd dimensional sphere S^{2n-1}, equipped with its standard Einstein metric. In this case the Kähler cone is $\mathbb{C}^n \setminus \{0\}$, equipped with its flat metric.

A Sasakian manifold (S, g) inherits a number of geometric structures from the Kähler structure of its cone. In particular, an important role is played by the Reeb vector field. This may be defined as $\xi = J(r\partial_r)$, where J denotes the integrable complex structure of the Kähler cone. The restriction of ξ to $S = \{r = 1\} = \{1\} \times S \subset C(S)$ is a unit length Killing vector field, and its orbits thus define a one-dimensional foliation of S called the Reeb foliation. There is then a classification of Sasakian manifolds, and hence also Sasaki-Einstein manifolds, according to the global properties of this foliation. If all the orbits of ξ are compact, and hence circles, then ξ

1991 *Mathematics Subject Classification.* Primary 53C25, 53C15.

Key words and phrases. Sasakian geometry, Sasaki-Einstein manifolds.

The author is supported by a Royal Society University Research Fellowship.

integrates to a locally free isometric action of $U(1)$ on (S, g). If this action is free, then the Sasakian manifold is said to be regular; otherwise it is said to be quasi-regular. On the other hand, if ξ has a non-compact orbit the Sasakian manifold is said to be irregular.

For the purposes of this introduction it is convenient to focus on the case of dimension 5 ($n = 3$), since this is the lowest non-trivial dimension. Prior to the 21st century, the only known examples of Sasaki-Einstein 5-manifolds were regular. As we shall explain, a regular Sasaki-Einstein manifold is the total space of a principal $U(1)$ bundle over a Kähler-Einstein manifold of positive Ricci curvature. The classification of such Fano Kähler-Einstein surfaces due to Tian-Yau then leads to a classification of all regular Sasaki-Einstein 5-manifolds. Passing to their simply-connected covers, these are connected sums $S^5\#k(S^2 \times S^3)$ where $k = 0, 1, 3, 4, 5, 6, 7, 8$. For each of $k = 0, 1, 3, 4$ there is a unique such regular Sasaki-Einstein structure, while for $5 \leq k \leq 8$ there are continuous families of complex dimension $2(k - 4)$. However, before 2001 it was not known whether quasi-regular Sasaki-Einstein 5-manifolds existed, and indeed there was even a conjecture that irregular Sasaki-Einstein manifolds do not exist.

The progress over the last decade has been dramatic. Again, focusing on dimension 5, it is now known that there exist Sasaki-Einstein structures on $\#k(S^2 \times S^3)$ for all values of k. These include infinitely many toric Sasaki-Einstein metrics, meaning that the torus $\mathbb{T}^3$ acts isometrically on the Sasakian structure, for every value of k. Indeed, for $k = 1$ these metrics are known completely explicitly, giving countably infinite families of quasi-regular and irregular Sasaki-Einstein structures on $S^2 \times S^3$. The list of Sasaki-Einstein structures on $\#k(S^2 \times S^3)$ also includes examples with the smallest possible isometry group, namely the $\mathbb{T} \cong U(1)$ generated by a quasi-regular Reeb vector field. In particular, there are known to exist infinitely many such structures for every k, except $k = 1, 2$, and these often come in continuous families. Moreover, S^5 itself admits at least 80 inequivalent quasi-regular Sasaki-Einstein structures. Again, some of these come in continuous families, the largest known having complex dimension 5. There are also quasi-regular Sasaki-Einstein metrics on 5-manifolds which are not connected sums of $S^2 \times S^3$, including infinitely many rational homology 5-spheres, as well as infinitely many connected sums of these. Similar abundant results hold also in higher dimensions. In particular, all 28 oriented diffeomorphism classes on S^7 admit Sasaki-Einstein metrics.

In this article I will review these developments, as well as other results not mentioned above. It is important to mention those topics that will not be covered. As in Kähler geometry, one can define extremal Sasakian metrics and Sasaki-Ricci solitons. Both of these generalize the notion of a Sasaki-Einstein manifold in different directions. Perhaps the main reason for not discussing these topics, other than reasons of time and space, is that so far there are not any obvious applications of these geometries to supergravity theories and string theory, which is the author's main interest.

The article is arranged as follows. Section 1 is a review of all the necessary background in Sasakian geometry; section 2 covers regular Sasaki-Einstein manifolds; section 3 describes the construction of quasi-regular Sasaki-Einstein structures, focusing in particular on links of weighted homogeneous hypersurface singularities; in section 4 we describe what is known about explicit constructions of Sasaki-Einstein manifolds; section 5 covers toric Sasakian geometry and the classification of toric Sasaki-Einstein manifolds; section 6 describes obstructions; and section 7 concludes with some open problems.

Acknowledgements. I would like to thank Charles Boyer for comments on the first version of this article.

1. Sasakian geometry

1.1. Sasakian basics. We begin with an introduction to Sasakian geometry. Many of the results here are elementary and follow almost immediately from the definitions. The reader is referred to [**14, 48, 75**] or the recent monograph [**21**] for detailed proofs.

DEFINITION 1.1. A compact Riemannian manifold (S, g) is *Sasakian* if and only if its metric cone $\big(C(S) = \mathbb{R}_{>0} \times S, \bar{g} = \mathrm{d}r^2 + r^2 g\big)$ is Kähler.

It follows that S has odd dimension $2n-1$, where n denotes the complex dimension of the Kähler cone. Notice that the Sasakian manifold (S, g) is naturally isometrically embedded into the cone via the inclusion $S = \{r = 1\} = \{1\} \times S \subset C(S)$. We shall often regard S as embedded into $C(S)$ this way. There is also a canonical projection $p : C(S) \to S$ which forgets the r coordinate. Being Kähler, the cone $(C(S), \bar{g})$ is equipped with an integrable complex structure J and a Kähler 2-form ω, both of which are parallel with respect to the Levi-Civita connection $\bar{\nabla}$ of $\bar{g}$. The Kähler structure of $(C(S), \bar{g})$, combined with its cone structure, induces the Sasakian structure on $S = \{1\} \times S \subset C(S)$.

The following equations are useful in proving many of the formulae that follow:

$$\begin{aligned} &\bar{\nabla}_{r\partial_r}(r\partial_r) = r\partial_r~, \qquad \bar{\nabla}_{r\partial_r} X = \bar{\nabla}_X (r\partial_r) = X~, \\ &\bar{\nabla}_X Y = \nabla_X Y - g(X,Y) r\partial_r~. \end{aligned} \tag{1.1}$$

Here X and Y denote vector fields on S, appropriately interpreted also as vector fields on $C(S)$, and ∇ is the Levi-Civita connection of g, .

The canonical vector field $r\partial_r$ is known as the *homothetic* or *Euler vector field*. Using the relations (1.1), together with the fact that J is parallel, $\bar{\nabla} J = 0$, one easily shows that $r\partial_r$ is real holomorphic, $\mathcal{L}_{r\partial_r} J = 0$. It is then natural to define the *characteristic vector field*

$$\xi = J(r\partial_r)~. \tag{1.2}$$

Again, elementary calculations show that ξ is real holomorphic and also Killing, $\mathcal{L}_\xi \bar{g} = 0$. Moreover, ξ is clearly tangent to surfaces of constant r and has square length $\bar{g}(\xi, \xi) = r^2$.

We may similarly define the 1-form

$$\eta = \mathrm{d}^c \log r = \mathrm{i}(\bar{\partial} - \partial) \log r \ , \tag{1.3}$$

where as usual $\mathrm{d}^c = J \circ \mathrm{d}$ denotes the composition of exterior derivative with the action of J on 1-forms, and ∂, $\bar{\partial}$ are the usual Dolbeault operators, with $\mathrm{d} = \partial + \bar{\partial}$. It follows straightforwardly from the definition that

$$\eta(\xi) = 1, \qquad i_\xi \mathrm{d}\eta = 0 \ . \tag{1.4}$$

Here we have introduced the interior contraction: if α is a $(p+1)$-form and X a vector field then $i_X \alpha$ is the p-form defined via $i_X\alpha(X_1, \ldots, X_p) = \alpha(X, X_1, \ldots, X_p)$. Moreover, it is also clear that

$$\eta(X) = \frac{1}{r^2} \bar{g}\left(J(r\partial_r), X\right) = \frac{1}{r^2}\bar{g}(\xi, X) \ . \tag{1.5}$$

Using this last formula one can show that the Kähler 2-form on $C(S)$ is

$$\omega = \frac{1}{2}\mathrm{d}(r^2 \eta) = \frac{1}{2}\mathrm{i}\partial\bar{\partial} r^2 \ . \tag{1.6}$$

The function $\frac{1}{2}r^2$ is hence a global Kähler potential for the cone metric.

The 1-form η restricts to a 1-form $\eta \mid_S$ on $S \subset C(S)$. One checks from $\mathcal{L}_{r\partial_r}\eta = 0$ that in fact $\eta = p^*(\eta \mid_S)$. In a standard abuse of notation, we shall then not distinguish between the 1-form η on the cone and its restriction to the Sasakian manifold $\eta \mid_S$. Similar remarks apply to the Reeb vector field ξ: by the above comments this is tangent to S, where it defines a unit length Killing vector field, so $g(\xi, \xi) = 1$ and $\mathcal{L}_\xi g = 0$. Notice from (1.5) that $\eta(X) = g(\xi, X)$ holds for all vector fields X on S.

Since the Kähler 2-form ω is in particular symplectic, it follows from (1.6) that the top degree form $\eta \wedge (\mathrm{d}\eta)^{n-1}$ on S is nowhere zero; that is, it is a volume form on S. By definition, this makes η a *contact 1-form* on S. Indeed, the open symplectic manifold $\left(C(S) = \mathbb{R}_{>0} \times S, \omega = \frac{1}{2}\mathrm{d}(r^2\eta)\right)$ is called the *symplectization* of the contact manifold (S, η). The relations $\eta(\xi) = 1$, $i_\xi \mathrm{d}\eta = 0$ from (1.4) imply that ξ is the unique *Reeb vector field* for this contact structure. We shall hence also refer to ξ as the Reeb vector field of the Sasakian structure.

The contact subbundle $D \subset TS$ is defined as $D = \ker \eta$. The tangent bundle of S then splits as

$$TS = D \oplus L_\xi \ , \tag{1.7}$$

where L_ξ denotes the line tangent to ξ. This splitting is easily seen to be orthogonal with respect to the Sasakian metric g.

Next define a section Φ of $\mathrm{End}(TS)$ via $\Phi\mid_D = J\mid_D$, $\Phi\mid_{L_\xi}=0$. Using $J^2=-1$ and that the cone metric $\bar{g}$ is Hermitian one shows that

$$\Phi^2 = -1+\eta\otimes\xi \ , \tag{1.8}$$

$$g\left(\Phi(X),\Phi(Y)\right) = g(X,Y)-\eta(X)\eta(Y) \ , \tag{1.9}$$

where X, Y are any vector fields on S. In fact a triple (η,ξ,Φ), with η a contact 1-form with Reeb vector field ξ and Φ a section of $\mathrm{End}(TS)$ satisfying (1.8), is known as an *almost contact structure*. This implies that $(D, J_D\equiv \Phi\mid_D)$ defines an almost CR structure. Of course, this has been induced by embedding S as a real hypersurface $\{r=1\}$ in a complex manifold, so this almost CR structure is of hypersurface type (the bundle D has rank one less than that of TS) and is also integrable. The Levi form may be taken to be $\frac{1}{2}\mathrm{i}\bar{\partial}\partial r^2$, which is the Kähler form of the cone (1.6). Since this is positive, by definition we have a strictly pseudo-convex CR structure. The second equation (1.9) then says that $g\mid_D$ is a Hermitian metric on D. Indeed, an almost contact structure (η,ξ,Φ) together with a metric g satisfying (1.9) is known as a *metric contact structure*. Sasakian manifolds are thus special types of metric contact structures, which is how Sasaki originally introduced them [**91**]. Since

$$g(X,Y)=\frac{1}{2}\mathrm{d}\eta(X,\Phi(Y))+\eta(X)\eta(Y) \ ,$$

we see that $\frac{1}{2}\mathrm{d}\eta\mid_D$ is the fundamental 2-form associated to $g\mid_D$. The contact subbundle D is symplectic with respect to this 2-form.

The tensor Φ may also be defined via

$$\Phi(X)=\nabla_X\xi \ .$$

This follows from the last equation in (1.1), together with the calculation $\bar{\nabla}_X\xi=\bar{\nabla}_X\left(J\left(r\partial_r\right)\right)=J(X)$. Then a further calculation gives

$$\left(\nabla_X\Phi\right)Y = g(\xi,Y)X-g(X,Y)\xi \ , \tag{1.10}$$

where X and Y are any vector fields on S. This leads to the following equivalent definitions of a Sasakian manifold [**14**], the first of which is perhaps the closest to the original definition of Sasaki [**91**].

PROPOSITION 1.2. *Let (S,g) be a Riemannian manifold, with ∇ the Levi-Civita connection of g and $R(X,Y)$ the Riemann curvature tensor. Then the following are equivalent:*

(1) *There exists a Killing vector field ξ of unit length so that the tensor field $\Phi(X)=\nabla_X\xi$ satisfies (1.10) for any pair of vector fields X, Y on S.*
(2) *There exists a Killing vector field ξ of unit length so that the Riemann curvature satisfies*

$$R(X,\xi)Y=g(\xi,Y)X-g(X,Y)\xi \ ,$$

for any pair of vector fields X, Y on S.

(3) *The metric cone* $(C(S), \bar{g}) = (\mathbb{R}_{>0} \times S, \mathrm{d}r^2 + r^2 g)$ *over* S *is Kähler.*

The equivalence of (1) and (2) follows from an elementary calculation relating $(\nabla_X \Phi)Y$ to $R(X, \xi)Y$. We have already sketched the proof that (3) implies (1). To show that (1) implies (3) one defines an almost complex structure J on $C(S)$ via

$$J\left(r\partial_r\right) = \xi \ , \quad J(X) = \Phi(X) - \eta(X) r\partial_r \ ,$$

where X is a vector field on S, appropriately interpreted as a vector field on $C(S)$, and $\eta(X) = g(\xi, X)$. It is then straightforward to check from the definitions that the cone is indeed Kähler.

We may think of a Sasakian manifold as the collection $\mathcal{S} = (S, g, \eta, \xi, \Phi)$. As mentioned, this is a special type of metric contact structure.

1.2. The Reeb foliation. The Reeb vector field ξ has unit length on (S, g) and in particular is nowhere zero. Its integral curves are geodesics, and the corresponding foliation $\mathcal{F}_\xi$ is called the *Reeb foliation*. Notice that, due to the orthogonal splitting (1.7), the contact subbundle D is the normal bundle to $\mathcal{F}_\xi$. The leaf space is clearly identical to that of the complex vector field $\xi - \mathrm{i}J(\xi) = \xi + \mathrm{i}r\partial_r$ on the cone $C(S)$. Since this complex vector field is holomorphic, the Reeb foliation thus naturally inherits a transverse holomorphic structure. In fact the leaf space also inherits a Kähler metric, giving rise to a transversely Kähler foliation, as we now describe.

Introduce a foliation chart $\{U_\alpha\}$ on S, where each U_α is of the form $U_\alpha = I \times V_\alpha$ with $I \subset \mathbb{R}$ an open interval and $V_\alpha \subset \mathbb{C}^{n-1}$ open. We may introduce coordinates $(x, z_1, \ldots, z_{n-1})$ on U_α, where $\xi = \partial_x$ and $z_1, \ldots, z_{n-1}$ are complex coordinates on V_α. The fact that the cone is complex implies that the transition functions between the V_α are holomorphic. More precisely, if U_β has corresponding coordinates $(y, w_1, \ldots, w_{n-1})$, with $U_\alpha \cap U_\beta \neq \emptyset$, then

$$\frac{\partial z_i}{\partial \bar{w}_j} = 0 \ , \qquad \frac{\partial z_i}{\partial y} = 0 \ .$$

Recall that the contact subbundle D is equipped with the almost complex structure J_D, so that on $D \otimes \mathbb{C}$ we may define the $\pm\mathrm{i}$ eigenspaces of J_D as the $(1,0)$ and $(0,1)$ vectors, respectively. Then in the above foliation chart U_α, $(D \otimes \mathbb{C})^{(1,0)}$ is spanned by

$$\partial_{z_i} - \eta\left(\partial_{z_i}\right)\xi \ .$$

Since ξ is a Killing vector field, and so preserves the metric g, it follows that $g \mid_D$ gives a well-defined Hermitian metric g^T_α on the patch V_α by restricting to a slice $\{x = \text{constant}\}$. Moreover, (1.4) implies that

$$\mathrm{d}\eta\left(\partial_{z_i} - \eta\left(\partial_{z_i}\right)\xi, \partial_{\bar{z}_j} - \eta\left(\partial_{\bar{z}_j}\right)\xi\right) = \mathrm{d}\eta\left(\partial_{z_i}, \partial_{\bar{z}_j}\right) \ .$$

The fundamental 2-form ω^T_α for the Hermtian metric g^T_α in the patch V_α is hence equal to the restriction of $\frac{1}{2}\mathrm{d}\eta$ to a slice $\{x = \text{constant}\}$. Thus ω^T_α

is closed, and the transverse metric g^T_α is Kähler. Indeed, in the chart U_α notice that we may write

$$\eta = \mathrm{d}x + \mathrm{i}\sum_{i=1}^{n-1} \partial_{z_i} K_\alpha \mathrm{d}z_i - \mathrm{i}\sum_{i=1}^{n-1} \partial_{\bar{z}_i} K_\alpha \mathrm{d}\bar{z}_i \ ,$$

where K_α is a function on U_α with $\partial_x K_\alpha = 0$. The local function K_α is a Kähler potential for the transverse Kähler structure in the chart U_α, as observed in [**56**]. Such a structure is called a *transversely Kähler foliation*. We denote the collection of transverse metrics by $g^T = \{g^T_\alpha\}$. Although g^T so defined is really a collection of metrics in each coordinate chart, notice that we may identify it with the global tensor field on S defined via

$$g^T(X,Y) = \frac{1}{2}\mathrm{d}\eta(X, \Phi(Y)) \ .$$

That is, the restriction of g^T to a slice $\{x = \text{constant}\}$ in the patch U_α is equal to g^T_α. Similarly, the transverse Kähler form ω^T may be defined globally as $\frac{1}{2}\mathrm{d}\eta$.

The *basic forms* and *basic cohomology* of the Reeb foliation $\mathcal{F}_\xi$ play an important role (for further background, the reader might consult [**104**]):

Definition 1.3. A p-form α on S is called *basic* if

$$i_\xi \alpha = 0 \ , \qquad \mathcal{L}_\xi \alpha = 0 \ .$$

We denote by Λ^p_B the sheaf of germs of basic p-forms and Ω^p_B the set of global sections of Λ^p_B.

If α is a basic form then it is easy to see that $\mathrm{d}\alpha$ is also basic. We may thus define $\mathrm{d}_B = \mathrm{d}\mid_{\Omega^*_B}$, so that $\mathrm{d}_B : \Omega^p_B \to \Omega^{p+1}_B$. The corresponding complex $(\Omega^*_B, \mathrm{d}_B)$ is called the *basic de Rham complex*, and its cohomology $H^*_B(\mathcal{F}_\xi)$ the *basic cohomology*.

Let U_α and U_β be coordinate patches as above, with coordinates adapted to the Reeb foliation $(x, z_1, \ldots, z_{n-1})$ and $(y, w_1, \ldots, w_{n-1})$, respectively. Then a form of Hodge type (p,q) on U_α

$$\alpha = \alpha_{i_1\cdots i_p \bar{j}_1 \cdots \bar{j}_q} \mathrm{d}z_{i_1} \wedge \cdots \wedge \mathrm{d}z_{i_p} \wedge \mathrm{d}\bar{z}_{\bar{j}_1} \wedge \cdots \wedge \mathrm{d}\bar{z}_{\bar{j}_q} \ ,$$

is also of type (p,q) with respect to the w_i coordinates on U_β. Moreover, if α is basic then $\alpha_{i_1\cdots i_p \bar{j}_1 \cdots \bar{j}_q}$ is independent of x. There are hence globally well-defined Dolbeault operators

$$\begin{aligned} \partial_B &: \Omega^{p,q}_B \to \Omega^{p+1,q}_B \ , \\ \bar{\partial}_B &: \Omega^{p,q}_B \to \Omega^{p,q+1}_B \ . \end{aligned}$$

Both are nilpotent of degree 2, so that one can define the *basic Dolbeault complex* $(\Omega^{p,*}, \bar{\partial}_B)$ and corresponding cohomology groups $H^{p,*}_B(\mathcal{F}_\xi)$. These invariants of the Reeb foliation are important invariants of the Sasakian manifold. Clearly $\mathrm{d}_B = \partial_B + \bar{\partial}_B$, and we may similarly define the operator $\mathrm{d}^c_B = \mathrm{i}(\bar{\partial}_B - \partial_B)$.

In each local leaf space V_α we may define the Ricci forms as $\rho^T_\alpha = -\mathrm{i}\partial\bar{\partial}\log\det g^T_\alpha$. These are the $(1,1)$-forms associated to the Ricci tensors Ric^T_α of g^T_α via the complex structure, in the usual way. Via pull-back to U_α under the projection $U_\alpha \to V_\alpha$ that identifies points on the same Reeb orbit, one sees that these patch together to give global tensors ρ^T and Ric^T on S, just as was the case for the transverse Kähler form and metric. In particular, the *transverse Ricci form* ρ^T is a global basic 2-form of Hodge type $(1,1)$ which is closed under d_B. The corresponding basic cohomology class $[\rho^T/2\pi]$ is denoted by $c_1^B(\mathcal{F}_\xi) \in H^{1,1}_B(\mathcal{F}_\xi)$, or simply c_1^B, and is called the *basic first Chern class of* $\mathcal{F}_\xi$. Again, this is an important invariant of the Sasakian structure. We say that $c_1^B > 0$ or $c_1^B < 0$ if c_1^B or $-c_1^B$ is represented by a transverse Kähler form, respectively. In particular, the Sasakian structure will be called *transverse Fano* if $c_1^B > 0$.

Thus far we have considered a fixed Sasakian manifold $\mathcal{S} = (S, g, \eta, \xi, \Phi)$. It will be important later to understand how one can deform such a structure to another Sasakian structure on the same manifold S. Since a Sasakian manifold and its corresponding Kähler cone have several geometric structures, one can fix some of these whilst deforming others. An important class of such deformations, which are analogous to deformations in Kähler geometry, is summarized by the following result:

PROPOSITION 1.4. *Fix a Sasakian manifold $\mathcal{S} = (S, g, \eta, \xi, \Phi)$. Then any other Sasakian structure on S with the same Reeb vector field ξ, the same holomorphic structure on the cone $C(S) = \mathbb{R}_{>0} \times S$, and the same transversely holomorphic structure of the Reeb foliation $\mathcal{F}_\xi$ is related to the original structure via the deformed contact form $\eta' = \eta + \mathrm{d}^c_B\phi$, where ϕ is a smooth basic function that is sufficiently small.*

PROOF. The proof is straightforward. We fix the holomorphic structure on the cone, but replace r by $r' = r\exp\phi$, where $\frac{1}{2}(r')^2$ will be the new global Kähler potential for the cone metric. Since J and ξ are held fixed, from (1.2) we have $r'\partial_{r'} = r\partial_r$, which implies that ϕ is a function on S. The new metric on $C(S)$ will be $\bar{g}'(X,Y) = \omega'(X, JY)$ where $\omega' = \frac{1}{2}\mathrm{i}\partial\bar{\partial}r'^2$. Since $r\partial_r$ is holomorphic, it is clear that $\bar{g}'$ is homogeneous degree 2. Moreover, one easily checks that the necessary condition $\mathcal{L}_\xi r' = 0$, which means that ϕ is basic, implies that $\bar{g}'$ is also a cone. Then

$$\eta' = \mathrm{d}^c\log r' = \eta + \mathrm{d}^c_B\phi\ .$$

For small enough φ, $\eta'\wedge(\mathrm{d}\eta')^{n-1}$ is still a volume form on $S' = \{r' = 1\} \cong S$, or in other words ω' is still non-degenerate on the cone, and similarly $\bar{g}'$ will be a non-degenerate metric. Thus we have defined a new Kähler cone, and hence new Sasakian structure.

This deformation is precisely a deformation of the transverse Kähler metric, holding fixed its basic cohomology class in $H^{1,1}_B(\mathcal{F}_\xi)$. Indeed, clearly

$$\left(\omega'\right)^T = \omega^T + \frac{1}{2}\mathrm{d}_B\mathrm{d}^c_B\phi\ .$$

In fact the converse is also true by the transverse $\partial\bar{\partial}$ lemma proven in [**43**], which is thus an equivalent way to characterize these *transverse Kähler deformations*. Notice that the basic first Chern class $c_1^B(\mathcal{F}_\xi)$ is invariant under such deformations, although the contact subbundle D will change. □

Later we shall also consider deforming the Reeb vector field ξ whilst holding the holomorphic structure of the cone fixed.

1.3. Regularity. There is a classification of Sasakian manifolds according to the global properties of the Reeb foliation $\mathcal{F}_\xi$. If the orbits of the Reeb vector field ξ are all closed, and hence circles, then ξ integrates to an isometric $U(1)$ action on (S, g). Since ξ is nowhere zero this action is locally free; that is, the isotropy group of every point in S is finite. If the $U(1)$ action is in fact free then the Sasakian structure is said to be *regular*. Otherwise, it is said to be *quasi-regular*. If the orbits of ξ are not all closed the Sasakian structure is said to be *irregular*. In this case the closure of the 1-parameter subgroup of the isometry group of (S, g) is isomorphic to a torus $\mathbb{T}^k$, for some positive integer k called the *rank* of the Sasakian structure. In particular, irregular Sasakian manifolds have at least a $\mathbb{T}^2$ isometry.

In the regular or quasi-regular case, the leaf space $Z = S/\mathcal{F}_\xi = S/U(1)$ has the structure of a compact manifold or orbifold, respectively. In the latter case the orbifold singularities of Z descend from the points in S with non-trivial isotropy subgroups. Notice that, being finite subgroups of $U(1)$, these will all be isomorphic to cyclic groups. The transverse Kähler structure described above then pushes down to a Kähler structure on Z, so that Z is a compact complex manifold or orbifold equipped with a Kähler metric h.

Digression on orbifolds. The reader will not need to know much about orbifolds in order to follow this article. However, we briefly digress here to sketch some basics, referring to [**14**] or [**21**] for a much more detailed account in the current context.

Just as a manifold M is a topological space that is locally modelled on $\mathbb{R}^k$, so an orbifold is a topological space locally modelled on $\mathbb{R}^k/\Gamma$, where Γ is a finite group of diffeomorphisms. The local Euclidean charts $\{U_i, \varphi_i\}$ of a manifold are replaced with *local uniformizing systems* $\{\tilde{U}_i, \Gamma_i, \varphi_i\}$. Here $\tilde{U}_i$ is an open subset of $\mathbb{R}^k$ containing the origin; Γ_i is, without loss of generality, a finite subgroup of $O(k)$ acting effectively on $\mathbb{R}^k$; and $\varphi_i : \tilde{U}_i \to U_i$ is a continuous map onto the open set $U_i \subset M$ such that $\varphi_i \circ \gamma = \varphi_i$ for all $\gamma \in \Gamma_i$ and the induced map $\tilde{U}_i/\Gamma_i \to U_i$ is a homeomorphism. These charts are then glued together in an appropriate way. The groups Γ_i are called the *local uniformizing groups*. The least common multiple of the orders of the local uniformizing groups Γ_i, when it is defined, is called the *order* of M and denoted $\mathrm{ord}(M)$. In particular, M is a manifold if and only if $\mathrm{ord}(M) = 1$. One can similarly define complex orbifolds, where one may take the $\Gamma_i \subset U(k)$ acting on $\mathbb{C}^k$.

If $x \in M$ is point and $p = \varphi_i^{-1}(x)$ then the conjugacy class of the isotropy subgroup $\Gamma_p \subset \Gamma_i$ depends only on x, not on the chart $\tilde{U}_i$. One denotes this Γ_x, so that the non-singular points of M are those for which Γ_x is trivial. The set of such points is dense in M. In the case at hand, where M is realized as the leaf space Z of a quasi-regular Reeb foliation, Γ_x is the same as the leaf holonomy group of the leaf x.

For orbifolds the notion of fibre bundle is modified to that of a fibre *orbibundle*. These consist of bundles over the local uniformizing neighbourhoods $\tilde{U}_i$ that patch together in an appropriate way. In particular, part of the data specifying an orbibundle with structure group G are group homomorphisms $h_i \in \mathrm{Hom}(\Gamma_i, G)$. The local uniformizing systems of an orbifold are glued together with the property that if $\phi_{ji} : \tilde{U}_i \to \tilde{U}_j$ is a diffeomorphism into its image then for each $\gamma_i \in \Gamma_i$ there is a unique $\gamma_j \in \Gamma_j$ such that $\phi_{ji} \circ \gamma_i = \gamma_j \circ \phi_{ji}$. The patching condition is then that if B_i is a fibre bundle over $\tilde{U}_i$ there should exist a corresponding bundle map $\phi^*_{ij} : B_j \mid_{\phi_{ji}(\tilde{U}_i)} \to B_i$ such that $h_i(\gamma_i) \circ \phi^*_{ij} = \phi^*_{ij} \circ h_j(\gamma_j)$. Of course, by choosing an appropriate refinement of the cover we may assume that $B_i = \tilde{U}_i \times F$ where F is the fibre on which G acts. The total space is then itself an orbifold in which the B_i may form the local uniformizing neighbourhoods. The group Γ_i acts on B_i by sending $(p_i, f) \in \tilde{U}_i \times F$ to $(\gamma^{-1}p_i, fh_i(\gamma))$, where $\gamma \in \Gamma_i$. Thus the local uniformizing groups of the total space may be taken to be subgroups of the Γ_i. In particular, when $F = G$ is a Lie group so that we have a principal G orbibundle, then the image $h_i(\Gamma_i)$ acts freely on the fibre. Thus provided the group homomorphisms h_i *inject* into the structure group G, the total space will in fact be a smooth manifold. This will be important in what follows.

The final orbinotion we need is that of *orbifold cohomology*, introduced by Haefliger [**58**]. One may define the orbibundle P of orthonormal frames over a Riemannian orbifold (M, g) in the usual way. This is a principal $O(n)$ orbibundle, and the discussion in the previous paragraph implies that the total space P is in fact a smooth manifold. One can then introduce the *classifying space* BM of the orbifold in an obvious way by defining $BM = (EO(n) \times P)/O(n)$, where $EO(n)$ denotes the universal $O(n)$ bundle and the action of $O(n)$ is diagonal. One then defines the orbifold homology, cohomology and homotopy groups as those of BM, respectively. In particular, the orbifold cohomology groups are denoted $H^*_{\mathrm{orb}}(M, \mathbb{Z}) = H^*(BM, \mathbb{Z})$, and these reduce to the usual cohomology groups of M when M is a manifold. The projection $BM \to M$ has generic fibre the contractible space $EO(n)$, and this then induces an isomorphism $H^*_{\mathrm{orb}}(M, \mathbb{R}) \to H^*(M, \mathbb{R})$. Typically integral classes map to rational classes under the natural map $H^*_{\mathrm{orb}}(M, \mathbb{Z}) \to H^*_{\mathrm{orb}}(M, \mathbb{R}) \xrightarrow{\sim} H^*(M, \mathbb{R})$.

Returning to Sasakian geometry, in the regular or quasi-regular case the leaf space $Z = S/\mathcal{F}_\xi = S/U(1)$ is a manifold or orbifold, respectively.

The Gysin sequence for the corresponding $U(1)$ (orbi)bundle then implies that the projection map $\pi : S \to Z$ gives rise to a ring isomorphism $\pi^* : H^*(Z,\mathbb{R}) \cong H^*_B(\mathcal{F}_\xi)$, thus relating the cohomology of the leaf space Z to the basic cohomology of the foliation.

We may now state the following result [**15**]:

THEOREM 1.5. *Let $\mathcal{S}$ be a compact regular or quasi-regular Sasakian manifold. Then the space of leaves of the Reeb foliation $\mathcal{F}_\xi$ is a compact Kähler manifold or orbifold (Z, h, ω_Z, J_Z), respectively. The corresponding projection*

$$\pi : (S, g) \to (Z, h) \ ,$$

is a (*orbifold*) *Riemannian submersion, with fibres being totally geodesic circles. Moreover, the cohomology class $[\omega_Z]$ is proportional to an integral class in the* (*orbifold*) *cohomology group $H^2_{\mathrm{orb}}(Z,\mathbb{Z})$.*

In either the regular or quasi-regular case, ω_Z is a closed 2-form on Z which thus defines a cohomology class $[\omega_Z] \in H^2(Z,\mathbb{R})$. In the regular case, the projection π defines a principal $U(1)$ bundle, and ω_Z is proportional to the curvature 2-form of a unitary connection on this bundle. Thus $[\omega_Z]$ is proportional to a class in the image of the natural map $H^2(Z,\mathbb{Z}) \to H^2(Z,\mathbb{R})$, since the curvature represents $2\pi c_1$ where c_1 denotes the first Chern class of the principal $U(1)$ bundle. In the quasi-regular case, the projection π is instead a principal $U(1)$ orbibundle, with ω_Z again proportional to a curvature 2-form. The orbifold cohomology group $H^2_{\mathrm{orb}}(Z,\mathbb{Z})$ classifies isomorphism classes of principal $U(1)$ orbibundles over an orbifold Z, just as in the regular manifold case the first Chern class in $H^2(Z,\mathbb{Z})$ classifies principal $U(1)$ bundles. The Kähler form ω_Z then defines a cohomology class $[\omega_Z] \in H^2(Z,\mathbb{R})$ which is proportional to a class in the image of the natural map $H^2_{\mathrm{orb}}(Z,\mathbb{Z}) \to H^2_{\mathrm{orb}}(Z,\mathbb{R}) \to H^2(Z,\mathbb{R})$.

A Kähler manifold or orbifold whose Kähler class is proportional to an integral cohomology class in this way is called a *Hodge* orbifold. There is no restriction on this constant of proportionality in Sasakian geometry: it may be changed via the D-homothetic transformation defined in the next section.

The converse is also true [**15**]:

THEOREM 1.6. *Let (Z, h) be a compact Hodge orbifold. Let $\pi : S \to Z$ be a principal $U(1)$ orbibundle over Z whose first Chern class is an integral class defined by $[\omega_Z]$, and let η be a 1-form on S with $\mathrm{d}\eta = 2\pi^*\omega_Z$ (η is then proportional to a connection 1-form). Then $(S, \pi^*h + \eta \otimes \eta)$ is a Sasakian orbifold. Furthermore, if all the local uniformizing groups inject into the structure group $U(1)$* (*the $h_i \in \mathrm{Hom}(\Gamma_i, U(1))$ are all injective*)*, then the total space S is a smooth manifold.*

We close this subsection by noting that

$$i_\xi \omega = -\frac{1}{2}\mathrm{d}r^2 \ , \tag{1.11}$$

where recall that ω is the Kähler form on the cone $(C(S), \bar{g})$. Thus $\frac{1}{2}r^2$ is precisely the *Hamiltonian function* for the Reeb vector field ξ. In the regular/quasi-regular case the Kähler manifold/orbifold (Z, h, ω_Z, J_Z) may then be viewed as the *Kähler reduction* of the Kähler cone with respect to the corresponding Reeb $U(1)$ action.

1.4. The Einstein condition. We begin with the following more general definition:

Definition 1.7. A Sasakian manifold $\mathcal{S} = (S, g, \eta, \xi, \Phi)$ is said to be *η-Sasaki-Einstein* if there are constants λ and ν such that

$$\mathrm{Ric}_g = \lambda g + \nu \eta \otimes \eta \ .$$

An important fact is that $\lambda + \nu = 2(n-1)$. This follows from the second condition in Proposition 1.2, which implies that for a Sasakian manifold $\mathrm{Ric}_g(\xi, \xi) = 2(n-1)$. In particular, Sasaki-Einstein manifolds, with $\nu = 0$, necessarily have $\lambda = 2(n-1)$.

Definition 1.8. A Sasaki-Einstein manifold is a Sasakian manifold (S, g) with $\mathrm{Ric}_g = 2(n-1)g$.

It is easy to see that the η-Sasaki-Einstein condition is equivalent to the transverse Kähler metric being Einstein, so that $\mathrm{Ric}^T = \kappa g^T$ for some constant κ. To see the equivalence one notes that

$$\mathrm{Ric}_g(\tilde{X}, \tilde{Y}) = \mathrm{Ric}^T(X, Y) - 2g^T(X, Y) \ , \tag{1.12}$$

where X, Y are vector fields on the local leaf spaces $\{V_\alpha\}$ and $\tilde{X}$, $\tilde{Y}$ are lifts to D. Then $\mathrm{Ric}^T = \kappa g^T$ together with (1.12) implies that

$$\mathrm{Ric}_g = (\kappa - 2)g + (2n - \kappa)\eta \otimes \eta \ .$$

Given a Sasakian manifold $\mathcal{S}$ one can check that for a constant $a > 0$ the rescaling

$$g' = ag + (a^2 - a)\eta \otimes \eta \ , \quad \eta' = a\eta \ , \quad \xi' = \frac{1}{a}\xi \ , \quad \Phi' = \Phi \ , \tag{1.13}$$

gives a Sasakian manifold $(S, g', \eta', \xi', \Phi')$ with the same holomorphic structure on $C(S)$, but with $r' = r^a$. This is known as a *D-homothetic transformation* [**97**]. Using the above formulae, together with the fact that the Ricci tensor is invariant under scaling the metric by a positive constant, it is then straightforward to show that if $\mathcal{S} = (S, g, \eta, \xi, \Phi)$ is η-Sasaki-Einstein with constant $\lambda > -2$, then a D-homothetic transformation with $a = (\lambda + 2)/2n$ gives a Sasaki-Einstein manifold. Thus any Sasakian structure which is transversely Kähler-Einstein with $\kappa > 0$ may be transformed via this scaling to a Sasaki-Einstein structure.

The Sasaki-Einstein case may be summarized by the following:

Proposition 1.9. *Let (S, g) be a Sasakian manifold of dimension $2n-1$. Then the following are equivalent*

(1) (S, g) *is Sasaki-Einstein with* $\mathrm{Ric}_g = 2(n-1)g$.
(2) *The Kähler cone* $(C(S), \bar{g})$ *is Ricci-flat,* $\mathrm{Ric}_{\bar{g}} = 0$.
(3) *The transverse Kähler structure to the Reeb foliation* $\mathcal{F}_\xi$ *is Kähler-Einstein with* $\mathrm{Ric}^T = 2ng^T$.

It immediately follows that the restricted holonomy group $\mathrm{Hol}^0(\bar{g}) \subset SU(n)$. Notice that a Sasaki-Einstein 3-manifold has a universal covering space which is isometric to the standard round sphere, so the first interesting dimension is $n = 3$, or equivalently real dimension $\dim S = 5$.

Since ρ^T represents $2\pi c_1^B(\mathcal{F}_\xi) \in H_B^{1,1}(\mathcal{F}_\xi)$, clearly a necessary condition for a Sasakian manifold to admit a transverse Kähler deformation to a Sasaki-Einstein structure, in the sense of Proposition 1.4, is that $c_1^B = c_1^B(\mathcal{F}_\xi) > 0$. Indeed, we have the following result, formalized in [**48**]:

PROPOSITION 1.10. *The following necessary conditions for a Sasakian manifold* $\mathcal{S}$ *to admit a deformation of the transverse Kähler structure to a Sasaki-Einstein metric are equivalent:*
(1) $c_1^B = a[\mathrm{d}\eta] \in H_B^{1,1}(\mathcal{F}_\xi)$ *for some positive constant* a.
(2) $c_1^B > 0$ *and* $c_1(D) = 0 \in H^2(S, \mathbb{R})$.
(3) *For some positive integer* $\ell > 0$*, the* ℓ*th power of the canonical line bundle* $K^\ell_{C(S)}$ *admits a nowhere vanishing holomorphic section* Ω *with* $\mathcal{L}_\xi \Omega = \mathrm{i}n\ell\Omega$.

As described in [**106**], the space $X = C(S) \cup \{r = 0\}$, obtained by adding the cone point at $\{r = 0\}$ to $C(S) \cong \mathbb{R}_{>0} \times S$, can be made into a complex analytic space in a unique way. In fact it is simple to see that X is Stein, and the point $o = \{r = 0\} \in X$ is an isolated singularity. Then (3) above implies that, by definition, X is ℓ-Gorenstein:

DEFINITION 1.11. An analytic space X with an isolated singularity $o \in X$ is said to be *ℓ-Gorenstein* if $K^\ell_{X \setminus \{o\}}$ is trivial. In particular, if $\ell = 1$ one says that X is *Gorenstein*.

PROOF. (Proposition 1.10) The equivalence of (1) and (2) follows immediately from the long exact sequence [**104**] relating the basic cohomology of the foliation $\mathcal{F}_\xi$ to the cohomology of S (see [**48**]). The Ricci form ρ of the cone $(C(S), \bar{g})$ is related to the transverse Ricci form, by an elementary calculation, via $\rho = \rho^T - n\mathrm{d}\eta$. Here we are regarding ρ^T as a global basic 2-form on S, pulled back to the cone $C(S)$. If condition (1) holds then by the above comments there is a D-homothetic transformation so that $[\rho^T] = n[\mathrm{d}\eta] \in H_B^{1,1}(\mathcal{F}_\xi)$. It now follows from the transverse $\partial\bar{\partial}$ lemma [**43**] that there is a smooth function f on $C(S)$ with $r\partial_r f = \xi f = 0$ and $\rho = \mathrm{i}\partial\bar{\partial} f$ (f is the pull-back of a basic function on S). But now $\mathrm{e}^f \omega^n / n!$ defines a flat metric on $K_{C(S)}$, where recall that ω is the Kähler form for $\bar{g}$. There is thus a multi-valued section $\tilde{\Omega}$ of $K_{C(S)}$ such that $\Omega = \tilde{\Omega}^{\otimes \ell}$ is a global holomorphic section of $K^\ell_{C(S)}$, for some positive integer $\ell > 0$, with $\|\Omega\| = 1$. Using the fact that f is invariant under $r\partial_r$ and that ω is homogeneous degree 2, the equality

$$\frac{\mathrm{i}^n}{2^n}(-1)^{n(n-1)/2}\tilde{\Omega}\wedge\bar{\tilde{\Omega}} = \mathrm{e}^f\frac{\omega^n}{n!} \ ,$$

implies that $\mathcal{L}_{r\partial_r}\Omega = n\ell\Omega$. □

1.5. 3-Sasakian manifolds. In dimensions of the form $n = 2p$, so $\dim S = 4p - 1$, there exists a special class of Sasaki-Einstein manifolds called *3-Sasakian* manifolds:

Definition 1.12. A Riemannian manifold (S, g) is *3-Sasakian* if and only if its metric cone $\big(C(S) = \mathbb{R}_{>0}\times S, \bar{g} = \mathrm{d}r^2 + r^2 g\big)$ is hyperKähler.

This implies that the cone has complex dimension $n = 2p$, or real dimension $4p$, and that the holonomy group $\mathrm{Hol}(\bar{g}) \subset Sp(p) \subset SU(2p)$. Thus 3-Sasakian manifolds are automatically Sasaki-Einstein. The hyperKähler structure on the cone descends to a 3-Sasakian structure on the base of the cone (S, g). In particular, the triplet of complex structures gives rise to a triplet of Reeb vector fields (ξ_1, ξ_2, ξ_3) whose Lie brackets give a copy of the Lie algebra $\mathtt{su}(2)$. There is then a corresponding 3-dimensional foliation, whose leaf space is a *quaternionic Kähler* manifold or orbifold. This extra structure means that 3-Sasakian geometry is rather more constrained, and it is somewhat more straightforward to construct examples. In particular, rich infinite classes of examples were produced in the 1990s via a quotient construction (essentially the hyperKähler quotient). A review of those developments was given in a previous article in this journal series [**14**], with a more recent account appearing in [**19**]. We note that the first non-trivial dimension for a 3-Sasakian manifold is $\dim S = 7$, and also that 3-Sasakian manifolds are automatically regular or quasi-regular as Sasaki-Einstein manifolds (indeed, the first quasi-regular Sasaki-Einstein manifolds constructed were 3-Sasakian 7-manifolds). We will therefore not discuss 3-Sasakian geometry any further in this article, but focus instead on the construction of Sasaki-Einstein manifolds that are not 3-Sasakian.

1.6. Killing spinors. For applications to supergravity theories one wants a slightly stronger definition of Sasaki-Einstein manifold than we have given above. This is related to the following:

Definition 1.13. Let (S, g) be a complete Riemannian spin manifold. Denote the spin bundle by $\mathscr{S}S$ and let ψ be a smooth section of $\mathscr{S}S$. Then ψ is said to be a *Killing spinor* if for some constant α

$$\nabla_Y\psi = \alpha Y\cdot\psi \ , \tag{1.14}$$

for every vector field Y, where ∇ denotes the spin connection of g and $Y\cdot\psi$ is Clifford multiplication of Y on ψ. One says that ψ is *imaginary* if $\alpha \in \mathrm{Im}(\mathbb{C}^*)$, *parallel* if $\alpha = 0$, or *real* if $\alpha \in \mathrm{Re}(\mathbb{C}^*)$.

It is a simple exercise to show that the existence of such a Killing spinor implies that g is Einstein with constant $\lambda = 4(m-1)\alpha^2$, where $m = \dim S$. In particular, the existence of a real Killing spinor implies that (S, g) is a compact Einstein manifold with positive Ricci curvature. The relation to Sasaki-Einstein geometry is given by the following result of [**6**]:

THEOREM 1.14. *A complete simply-connected Sasaki-Einstein manifold admits at least 2 linearly independent real Killing spinors with* $\alpha = +\frac{1}{2}, -\frac{1}{2}$ *for* $n = 2p - 1$ *and* $\alpha = +\frac{1}{2}, +\frac{1}{2}$ *for* $n = 2p$*, respectively. Conversely, a complete Riemannian spin manifold admitting such Killing spinors in these dimensions is Sasaki-Einstein with* $\mathrm{Hol}(\bar{g}) \subset SU(n)$*.*

Notice that in both cases $\mathrm{Hol}(\bar{g}) \subset SU(n)$, so that in particular a simply-connected Sasaki-Einstein manifold is indeed spin. Moreover, in this case $\ell = 1$ in Proposition 1.10 so that the singularity $X = C(S) \cup \{r = 0\}$ is Gorenstein. Indeed, a real Killing spinor on (S, g) lifts to a *parallel* spinor on $(C(S), \bar{g})$ [**6**], and from this parallel spinor one can construct a nowhere zero holomorphic $(n, 0)$-form by "squaring" it. We refer the reader to [**75**] for details.

When a Sasaki-Einstein manifold is not simply-connected the existence of Killing spinors is more subtle. An instructive example is S^5, equipped with its standard metric. Here $X = \mathbb{C}^3$ is equipped with its flat Kähler metric. Denoting standard complex coordinates on $\mathbb{C}^3$ by (z_1, z_2, z_3) we may consider the quotient $S^5/\mathbb{Z}_q$, where $\mathbb{Z}_q$ acts by sending $(z_1, z_2, z_3) \mapsto (\zeta z_1, \zeta z_2, \zeta z_3)$ with ζ a primitive qth root of unity. For $q = 2$ this is the antipodal map, giving $\mathbb{RP}^5$ which is not even a spin manifold. In fact of all these quotients only S^5 and $S^5/\mathbb{Z}_3$ admit Killing spinors.

For applications to supergravity theories, a Sasaki-Einstein manifold is in fact defined to satisfy this stronger requirement that it admits Killing spinors. Of course, since $\pi_1(S)$ is finite by Myers' Theorem [**80**], one may always lift to a simply-connected cover, where Theorem 1.14 implies that the two definitions coincide. We shall thus not generally emphasize this distinction.

The reader might wonder what happens to Theorem 1.14 when the number of linearly independent Killing spinors is not 2. For simplicity we focus on the simply-connected case. When $n = 2p - 1$, the existence of 1 Killing spinor in fact implies the existence of 2 with opposite sign of α, so that (S, g) is Sasaki-Einstein. If there are more than 2, or at least 2 with the same sign of α, then (S, g) is the round sphere. When $n = 4$, so that $\dim S = 7$, it is possible for a Riemannian spin 7-manifold (S, g) to admit a single real Killing spinor, in which case (S, g) is said to be a *weak* G_2 *holonomy manifold*; the metric cone then has holonomy contained in the group $Spin(7) \subset SO(8)$. In all other dimensions of the form $n = 2p$, the existence of 1 Killing spinor again implies the existence of 2, implying (S, g) is Sasaki-Einstein. A simply-connected 3-Sasakian manifold has 3 linearly independent Killing spinors, all with $\alpha = +\frac{1}{2}$. If there are more than 3, or at least 2 with opposite sign of α, then again (S, g) is necessarily the round sphere. For further details, and a list of references, the reader is referred to [**14**].

2. Regular Sasaki-Einstein manifolds

2.1. Fano Kähler-Einstein manifolds. Theorem 1.5, together with Proposition 1.9, implies that any regular Sasaki-Einstein manifold is the total space of a principal $U(1)$ bundle over a Kähler-Einstein manifold (Z, h). On the other hand, Theorem 1.6 implies that the converse is also true. In fact this construction of Einstein metrics on the total spaces of principal $U(1)$ bundles over Kähler-Einstein manifolds is in the very early paper of Kobayashi [**62**].

THEOREM 2.1. *A complete regular Sasaki-Einstein manifold (S, g) of dimension $(2n-1)$ is the total space of a principal $U(1)$ bundle over a compact Kähler-Einstein manifold (Z, h, ω_Z) with positive Ricci curvature $\mathrm{Ric}_h = 2nh$, which is the leaf space of the Reeb foliation $\mathcal{F}_\xi$. If S is simply-connected then this $U(1)$ bundle has first Chern class $-[n\omega_Z/\pi I(Z)] = -c_1(Z)/I(Z)$, where $I(Z) \in \mathbb{Z}_{>0}$ is the* Fano index *of Z.*

Conversely, if (Z, h, ω_Z) is a complete simply-connected Kähler-Einstein manifold with positive Ricci curvature $\mathrm{Ric}_h = 2nh$, then let $\pi : S \to Z$ be the principal $U(1)$ bundle with first Chern class $-c_1(Z)/I(Z)$. Then $g = \pi^*h + \eta \otimes \eta$ is a regular Sasaki-Einstein metric on the simply-connected manifold S, where η is the connection 1-form on S with curvature $\mathrm{d}\eta = 2\pi^*\omega_Z$.

Recall here:

DEFINITION 2.2. A *Fano manifold* is a compact complex manifold Z with positive first Chern class $c_1(Z) > 0$. The *Fano index $I(Z)$* is the largest positive integer such that $c_1(Z)/I(Z)$ is an integral class in the group of holomorphic line bundles $\mathrm{Pic}(Z) = H^2(Z, \mathbb{Z}) \cap H^{1,1}(Z, \mathbb{R})$.

In particular, Kähler-Einstein manifolds with positive Ricci curvature are Fano. Notice that the principal $U(1)$ bundle in Theorem 2.1 is that associated to the line bundle $K_Z^{1/I(Z)}$, where K_Z is the canonical line bundle of Z. Also notice that by taking a $\mathbb{Z}_m \subset U(1)$ quotient of a simply-connected S in Theorem 2.1, where $U(1)$ acts via the free Reeb action, we also obtain a regular Sasaki-Einstein manifold with $\pi_1(S/\mathbb{Z}_m) \cong \mathbb{Z}_m$; this is equivalent to taking the mth power of the principal $U(1)$ bundle, which has associated line bundle $K_Z^{m/I(Z)}$. However, the Killing spinors on (S, g) guaranteed by Theorem 1.14 are invariant under $\mathbb{Z}_m$ only when m divides the Fano index $I(Z)$. Only in these cases is the quotient Sasaki-Einstein in the stronger sense of admitting a real Killing spinor.

Via Theorem 2.1 the classification of regular Sasaki-Einstein manifolds effectively reduces to classifying Fano Kähler-Einstein manifolds. This is a rich and deep subject, which is still very much an active area of research. Below we give a brief overview of some key results.

2.2. Homogeneous Sasaki-Einstein manifolds.

Definition 2.3. A Sasakian manifold $\mathcal{S}$ is said to be *homogeneous* if there is a transitively acting group G of isometries preserving the Sasakian structure.

If S is compact, then G is necessarily a compact Lie group. We then have the following theorem of [**15**]:

Theorem 2.4. *Let (S, g') be a complete homogeneous Sasakian manifold with $\mathrm{Ric}_{g'} \geq \epsilon > -2$. Then (S, g') is a compact regular homogeneous Sasakian manifold, and there is a homogeneous Sasaki-Einstein metric g on S that is compatible with the underlying contact structure. Moreover, S is the total space of a principal $U(1)$ bundle over a generalized flag manifold K/P, equipped with its Kähler-Einstein metric. Via Theorem 2.1, the converse is also true.*

Recall here that a *generalized flag manifold* K/P is a homogeneous space where K is a complex semi-simple Lie group, and P is any complex subgroup of K that contains a Borel subgroup (so that P is a *parabolic* subgroup of K). It is well-known that K/P is Fano and admits a homogeneous Kähler-Einstein metric [**12**]. Conversely, any compact homogeneous simply-connected Kähler-Einstein manifold is a generalized flag manifold. The metric on K/P is G-invariant, where G is a maximal compact subgroup of K, and one can write $K/P = G/H$ for appropriate subgroup H.

In low dimensions Theorem 2.4 leads [**14**] to the following list, well-known to supergravity theorists:

Corollary 2.5. *Let (S, g) be a complete homogeneous Sasaki-Einstein manifold of dimension $2n-1$. Then S is a principal $U(1)$ bundle over*

(1) $\mathbb{CP}^1$ *when* $n = 2$,
(2) $\mathbb{CP}^2$ *or* $\mathbb{CP}^1 \times \mathbb{CP}^1$ *when* $n = 3$,
(3) $\mathbb{CP}^3$, $\mathbb{CP}^2 \times \mathbb{CP}^1$, $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$, $SU(3)/\mathbb{T}^2$, *or the real Grassmannian* $\mathrm{Gr}_2(\mathbb{R}^5)$ *of 2-planes in* $\mathbb{R}^5$ *when* $n = 4$.

2.3. Regular Sasaki-Einstein 5-manifolds. As mentioned in the introduction, regular Sasaki-Einstein 5-manifolds are classified [**45, 9**]. This is thanks to the classification of Fano Kähler-Einstein surfaces due to Tian-Yau [**99, 100, 103**].

Theorem 2.6. *Let (S, g) be a regular Sasaki-Einstein 5-manifold. Then $S = \tilde{S}/\mathbb{Z}_m$, where the universal cover $(\tilde{S}, g)$ is one of the following:*

(1) S^5 *equipped with its standard round metric. Here* $Z = \mathbb{CP}^2$ *equipped with its standard Fubini-Study metric.*

(2) *The Stiefel manifold* $V_2(\mathbb{R}^4) \cong S^2 \times S^3$ *of 2-frames in* $\mathbb{R}^4$. *Here* $Z = \mathbb{CP}^1 \times \mathbb{CP}^1$ *equipped with the symmetric product of round metrics on each* $\mathbb{CP}^1 \cong S^2$.

(3) *The total space* S_k *of the principal* $U(1)$ *bundles* $S_k \to P_k$, *for* $3 \leq k \leq 8$, *where* $P_k = \mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$ *is the k-point blow-up of* $\mathbb{CP}^2$. *For each complex structure on these del Pezzo surfaces there is a unique Kähler-Einstein metric, up to automorphism* [**93, 99, 100, 103**], *and a corresponding unique Sasaki-Einstein metric g on* $S_k \cong \# k \left(S^2 \times S^3\right)$. *In particular, for* $5 \leq k \leq 8$ *by varying the complex structure this gives a complex* $2(k-4)$*-dimensional family of regular Sasaki-Einstein structures.*

Cases (1) and (2) are of course the 2 homogeneous spaces listed in (2) of Corollary 2.5, and so the metrics are easily written down explicitly. The Sasaki-Einstein metric in case (2) was first noted by Tanno in [**98**], although in the physics literature the result is often attributed to Romans [**89**]. In the latter case the manifold is referred to as T^{11}, the T^{pq} being homogeneous Einstein metrics on principal $U(1)$ bundles over $\mathbb{CP}^1 \times \mathbb{CP}^1$ with Chern numbers $(p, q) \in \mathbb{Z} \oplus \mathbb{Z} \cong H^2(\mathbb{CP}^1 \times \mathbb{CP}^1, \mathbb{Z})$. This is a generalization of the Kobayashi construction [**62**], and was further generalized to torus bundles by Wang-Ziller in [**109**]. The corresponding Ricci-flat Kähler cone over T^{11} has the complex structure of the quadric singularity $\{z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\} \subset \mathbb{C}^4$ minus the isolated singular point at the origin. This hypersurface singularity is called the "conifold" in the string theory literature, and there are literally hundreds of papers that study different aspects of string theory on this space.

The Kähler-Einstein metrics on del Pezzo surfaces in (3) are known to exist, but are not known in explicit form. The complex structure moduli simply correspond to moving the blown-up points. The fact that S_k is diffeomorphic to $\# k \left(S^2 \times S^3\right)$ follows from Smale's Theorem [**95**]:

Theorem 2.7. *A compact simply-connected spin 5-manifold S with no torsion in* $H_2(S, \mathbb{Z})$ *is diffeomorphic to* $\# k \left(S^2 \times S^3\right)$.

The homotopy and homology groups of S_k are of course straightforward to compute using their description as principal $U(1)$ bundles over P_k.

By Theorem 1.14, in each case the simply-connected cover $(\tilde{S}, g)$ admits 2 real Killing spinors. Only for $m = 3$ in case (1) and $m = 2$ in case (2) do the $\mathbb{Z}_m$ quotients also admit such Killing spinors [**45**].

2.4. Existence of Kähler-Einstein metrics. For a Kähler manifold (Z, h, ω, J) the Einstein equation $\mathrm{Ric}_h = \kappa h$ is of course equivalent to the 2-form equation $\rho_h = \kappa\omega_h$, where $\rho_h = -\mathrm{i}\partial\bar{\partial} \log \det h$ denotes the Ricci form of the metric h. Since the cohomology class $[\rho] \in H^{1,1}(Z, \mathbb{R})$ of the Ricci form equals $2\pi c_1(Z)$, it follows that on such a Kähler-Einstein manifold $2\pi c_1(Z) = \kappa[\omega]$. Notice that for $\kappa = 0$ Z is Calabi-Yau ($c_1(Z) = 0$) and there is no restriction on the Kähler class, while for $\kappa > 0$ or $\kappa < 0$ instead

Z must be either Fano or anti-Fano ($c_1(Z) < 0$), respectively, and in either case here the Kähler class is fixed uniquely.

Suppose that $\omega = \omega_h$ is a Kähler 2-form on Z with $\kappa[\omega] = 2\pi c_1(Z) \in H^{1,1}(Z, \mathbb{R})$. By the $\partial\bar{\partial}$ lemma, there exists a global real function $f \in C^\infty(Z)$ such that

$$\rho_h - \kappa\omega_h = \mathrm{i}\partial\bar{\partial} f \ . \tag{2.1}$$

The function f is often called the *discrepancy potential.* It is unique up to an additive constant, and the latter may be conveniently fixed by requiring, for example, $\int_Z \left(\mathrm{e}^f - 1\right)\omega_h^{n-1} = 0$, where $\dim_{\mathbb{C}} Z = n-1$. Notice that f is essentially the same function f appearing in the proof of Proposition 1.10. (More precisely, the function there is the pull-back of f here under the $\mathbb{C}^* = \mathbb{R}_{>0} \times U(1)$ quotient $C(S) \to Z$ for a regular Sasakian structure with leaf space Z.)

On the other hand, if g is a Kähler-Einstein metric, with $[\omega_g] = [\omega_h]$ and $\rho_g = \kappa\omega_g$, then the $\partial\bar{\partial}$ lemma again gives a real function $\phi \in C^\infty(Z)$ such that

$$\omega_g - \omega_h = \mathrm{i}\partial\bar{\partial}\phi \ . \tag{2.2}$$

Thus $\rho_h - \rho_g = \mathrm{i}\partial\bar{\partial}(f - \kappa\phi)$, or relating the volume forms as $\omega_g^{n-1} = \mathrm{e}^F \omega_h^{n-1}$ with $F \in C^\infty(Z)$ equivalently

$$\mathrm{i}\partial\bar{\partial} F = \mathrm{i}\partial\bar{\partial}(f - \kappa\phi) \ .$$

This implies $F = f - \kappa\phi + c$ with c a constant. Again, this may be fixed by requiring, for example,

$$\int_Z \left(\mathrm{e}^{f-\kappa\phi} - 1\right)\omega_h^{n-1} = 0 \ . \tag{2.3}$$

We have then shown the following:

Proposition 2.8. *Let (Z, J) be a compact Kähler manifold, of dimension $\dim_{\mathbb{C}} Z = n-1$, with Kähler metrics h, g in the same Kähler class, $[\omega_h] = [\omega_g] \in H^{1,1}(Z, \mathbb{R})$, and with $\kappa[\omega_h] = 2\pi c_1(Z)$. Let $f, \phi \in C^\infty(Z)$ be the functions defined via (2.1) and (2.2), and with the relative constant of $f - \kappa\phi$ fixed by (2.3). Then the metric g is Kähler-Einstein with constant κ if and only if ϕ satisfies the Monge-Ampère equation*

$$\omega_g^{n-1} = \mathrm{e}^{f-\kappa\phi}\omega_h^{n-1} \ ,$$

or equivalently

$$\frac{\det\left(h_{i\bar{j}} + \frac{\partial^2\phi}{\partial z_i \partial \bar{z}_j}\right)}{\det h_{i\bar{j}}} = \mathrm{e}^{f-\kappa\phi} \ , \tag{2.4}$$

where $z_1, \ldots, z_{n-1}$ are local complex coordinates on Z.

For $\kappa < 0$ this problem was solved independently by Aubin [**4**] and Yau [**110**]. Without loss of generality, we may rescale the metric so that $\kappa = -1$ and then state:

THEOREM 2.9. *Let (Z, J) be a compact Kähler manifold with $c_1(Z) < 0$. Then there exists a unique Kähler-Einstein metric with $\rho_g = -\omega_g$.*

The proof relies on the Maximum Principle. The Calabi-Yau case $\kappa = 0$ is substantially harder, and was proven in Yau's celebrated paper:

THEOREM 2.10. *Let (Z, J) be a compact Kähler manifold with $c_1(Z) = 0$. Then there exists a unique Ricci-flat Kähler metric in each Kähler class.*

On the other hand, the problem for $\kappa > 0$ is still open. In particular, there are known obstructions to solving the Monge-Ampère equation (2.4) in this case. On the other hand, it is known that if there is a solution, it is unique up to automorphism [**5**]. In the remainder of this section we give a very brief overview of the Fano $\kappa > 0$ case, referring the reader to the literature for further details.

In [**76**] Matsushima proved that for a Fano Kähler-Einstein manifold the complex Lie algebra $\mathfrak{a}(Z)$ of holomorphic vector fields is the complexification of the Lie algebra of *Killing* vector fields. Since the isometry group of a compact Riemannian manifold is a compact Lie group, in particular this implies that $\mathfrak{a}(Z)$ is necessarily *reductive*; that is, $\mathfrak{a}(Z) = \mathcal{Z}(\mathfrak{a}(Z)) \oplus [\mathfrak{a}(Z), \mathfrak{a}(Z)]$, where $\mathcal{Z}(\mathfrak{a}(Z))$ denotes the centre. The simplest such obstructed examples are in fact the 1-point and 2-point blow-ups of $\mathbb{CP}^2$ that are not listed in Theorem 2.6, despite these being Fano surfaces. Matsushima's result implies that the isometry group of a Fano Kähler-Einstein manifold is a maximal compact subgroup of the automorphism group.

Another obstruction, also related to holomorphic vector fields on Z, is the Futaki invariant of [**47**]. If $\zeta \in \mathfrak{a}(Z)$ is a holomorphic vector field then define

$$\mathcal{F}(\zeta) = \int_Z \zeta(f)\, \omega_h^{n-1}, \tag{2.5}$$

where f is the discrepancy potential defined via (2.1). The function $\mathcal{F}$ is independent of the choice of Kähler metric h in the Kähler class $[\omega_h]$, and defines a Lie algebra homomorphism $\mathcal{F} : \mathfrak{a}(Z) \to \mathbb{C}$. For this reason it is also sometimes called the *Futaki character*. Moreover, Mabuchi [**69**] proved that the nilpotent radical of $\mathfrak{a}(Z)$ lies in $\ker \mathcal{F}$, so that $\mathcal{F}$ is completely determined by its restriction to the maximal reductive subalgebra. Since $\mathcal{F}$ is a Lie algebra character, it also vanishes on the derived algebra $[\mathfrak{a}(Z), \mathfrak{a}(Z)]$, and therefore the Futaki invariant is determined entirely by its restriction to the centre of $\mathfrak{a}(Z)$. In practice, $\mathcal{F}$ may be computed via localization; see, for example, the formula in [**102**]. Clearly f is constant for a Kähler-Einstein metric, and thus the Futaki invariant must vanish in this case. Indeed, both the 1-point and 2-point blow-ups of $\mathbb{CP}^2$ also have non-zero Futaki invariants, and are thus obstructed this way also. The Futaki invariant will turn out to be closely related to a natural construction in Sasakian geometry, that generalizes to the quasi-regular and irregular cases.

Since both obstructions above are related to the Lie algebra of holomorphic vector fields on the Fano Z, there was a conjecture that in the absence of holomorphic vector fields there would be no obstruction to the existence of a Kähler-Einstein metric. However, a counterexample was later given by Tian in [**101**].

In fact it is currently believed that a Fano manifold Z admits a Kähler-Einstein metric if and only if it is *stable*, in an appropriate geometric invariant theory sense. This idea goes back to Yau [**111**], and has been developed by Donaldson, Tian, and others. It is clearly beyond the scope of this article to describe this still very active area of research. However, the basic idea is to use K_Z^{-k} for $k \gg 0$ to embed Z into a large complex projective space $\mathbb{CP}^{N_k}$ via its space of sections (the Kodaira embedding). Then stability of Z is taken in the geometric invariant theory sense, for the automorphisms of these projective spaces, as $k \to \infty$. A stable orbit should contain a zero of a corresponding moment map, and in the present case this amounts to saying that by acting with an appropriate automorphism of $\mathbb{CP}^{N_k}$ a *stable* Z can be moved to a *balanced embedding*, in the sense of Donaldson [**40**]. For a sequence of balanced embeddings, the pull-back of the Fubini-Study metric on $\mathbb{CP}^{N_k}$ as $k \to \infty$ should then approach the Kähler-Einstein metric on Z. The precise notion of stability here is called *K-stability*.

In practice, even if the above stability conjecture was settled it is difficult to check in practice for a given Fano manifold. More practically, one can sometimes prove existence of solutions to (2.4) in appropriate examples using the continuity method. Thus, for appropriate classes of examples, one can often write down sufficient conditions for a solution, although these conditions are in general not expected to be necessary. This will be the pragmatic approach followed in the next section when we come to discuss the extension to quasi-regular Sasakian manifolds, or rather their associated Fano Kähler-Einstein orbifolds.

However, there are two classes of examples in which necessary and sufficient conditions are known. The first is the classification of Fano Kähler-Einstein surfaces already mentioned. One may describe this result by saying that a Fano surface admits a Kähler-Einstein metric if and only if its Futaki invariant is zero. The second class of examples are the *toric* Fano manifolds. Here a complex $(n-1)$-manifold is said to be *toric* if there is a biholomorphic $(\mathbb{C}^*)^{n-1}$ action with a dense open orbit. Then a toric Fano manifold admits a Kähler-Einstein metric if and only if its Futaki invariant is zero [**108**]. The Kähler-Einstein metric is invariant under the real torus subgroup $\mathbb{T}^{n-1} \subset (\mathbb{C}^*)^{n-1}$. We shall see that this result generalizes to the quasi-regular and irregular Sasakian cases, and so postpone further discussion to later in the article.

Otherwise, examples are somewhat sporadic (as will be the case also in the next section). As an example, the *Fermat hypersurfaces* $F_{d,n} = \{z_0^d + \cdots + z_n^d = 0\} \subset \mathbb{CP}^n$ are Fano provided $d \leq n$, and Nadel [**81**] has shown that these admit Kähler-Einstein metrics if $n/2 \leq d \leq n$. It is straightforward to compute the homology groups of the corresponding regular Sasaki-Einstein manifolds $S_{d,n}$ [**15**]. In particular, in dimension 7 ($n = 4$) one finds examples of Sasaki-Einstein 7-manifolds with third Betti numbers $b_3(S_{4,4}) = 60$, $b_3(S_{3,4}) = 10$ (notice $S_{2,n}$ corresponds to a quadric, and is homogeneous).

Finally, we stress that in any given dimension there are only finitely many (deformation classes of) Fano manifolds [**65**]. Thus there are only finitely many Kähler-Einstein structures, and hence finitely many *regular* Sasaki-Einstein structures, up to continuous deformations of the complex structure on the Kähler-Einstein manifold. This result is no longer true when one passes to the orbifold category, or quasi-regular case.

3. Quasi-regular Sasaki-Einstein manifolds and hypersurface singularities

3.1. Fano Kähler-Einstein orbifolds. Recall that a quasi-regular Sasakian manifold is a Sasakian manifold whose Reeb foliation has compact leaves, but such that the corresponding $U(1)$ action is only locally free, rather than free. As in the previous section, Theorem 1.5 and Proposition 1.9 imply that the leaf space of a quasi-regular Sasaki-Einstein manifold is a compact Kähler-Einstein orbifold (Z, h). The main tool in this section will be the converse result obtained using the inversion Theorem 1.6:

THEOREM 3.1. *Let (Z, h, ω_Z) be a compact simply-connected ($\pi_1^{\mathrm{orb}}(Z)$ trivial) Kähler-Einstein orbifold with positive Ricci curvature* $\mathrm{Ric}_h = 2nh$. *Let $\pi : S \to Z$ be the principal $U(1)$ orbibundle with first Chern class $-c_1(Z)/I(Z) \in H^2_{\mathrm{orb}}(Z, \mathbb{Z})$. Then $(S, g = \pi^* h + \eta \otimes \eta)$ is a compact simply-connected quasi-regular Sasaki-Einstein orbifold, where η is the connection 1-form on S with curvature* $\mathrm{d}\eta = 2\pi^*\omega_Z$. *Furthermore, if all the local uniformizing groups inject into $U(1)$ then the total space S is a smooth manifold.*

Here the *orbifold Fano index* $I(Z)$ is defined in a precisely analogous way to the manifold case: it is the largest positive integer such that $c_1(Z)/I(Z)$ defines an integral class in the orbifold Picard group $H^2_{\mathrm{orb}}(Z, \mathbb{Z}) \cap H^{1,1}(Z, \mathbb{R})$. As in the regular case, the principal $U(1)$ orbibundle appearing here is that associated to the complex line orbibundle $\left(K_Z^{\mathrm{orb}}\right)^{1/I(Z)}$.

Here we make the important remark that canonical line orbibundle of Z, K_Z^{orb}, is not necessarily the same as the canonical line bundle defined in the algebro-geometric sense. The difference between the two lies in the fact that complex codimension one orbifold singularities are not seen by the canonical line bundle, owing to the simple fact that $\mathbb{C}/\mathbb{Z}_m \cong \mathbb{C}$ as an algebraic variety. More specifically, let the complex codimension one singularities of Z be along divisors D_i, and suppose that D_i has multiplicity m_i in the above sense. In particular, a Kähler-Einstein orbifold metric on Z will have a $2\pi/m_i$ conical singularity along D_i. Then

$$K_Z^{\mathrm{orb}} = K_Z + \sum_i \left(1 - \frac{1}{m_i}\right) D_i \ .$$

The D_i are known as the *ramification divisors*. A Fano Kähler-Einstein orbifold is then Fano in the sense that $\left(K_Z^{\mathrm{orb}}\right)^{-1}$ is positive, which is not the same condition as K_Z^{-1} being positive. Also, the Fano indices in the two senses will not in general agree.

3.2. The join operation. As already mentioned, the first examples of quasi-regular Sasaki-Einstein manifolds were the quasi-regular 3-Sasakian manifolds constructed in [**24**]. The first examples of quasi-regular Sasaki-Einstein manifolds that are *not* 3-Sasakian were in fact constructed using these examples, together with the following Theorem of [**15**]:

THEOREM 3.2. *Let $\mathcal{S}_1$, $\mathcal{S}_2$ be two simply-connected quasi-regular Sasaki-Einstein manifolds of dimensions $2n_1 - 1$, $2n_2 - 1$, respectively. Then there is a natural operation called the* join *which produces in general a simply-connected quasi-regular Sasaki-Einstein orbifold $\mathcal{S}_1 \star \mathcal{S}_2$ of dimension $2(n_1 + n_2) - 3$. The join is a smooth manifold if and only if*

$$\gcd\left(\mathrm{ord}(Z_1) l_2, \mathrm{ord}(Z_2) l_1\right) = 1 \ , \tag{3.1}$$

where $l_i = I(Z_i)/\gcd(I(Z_1), I(Z_2))$ are the relative orbifold Fano indices of the Kähler-Einstein leaf spaces Z_1, Z_2.

Recall here that $\mathrm{ord}(M)$ denotes the order of M as an orbifold (see section 1.3).

The proof of this result follows from the simple observation that given two Kähler-Einstein orbifolds (Z_1, h_1), (Z_2, h_2), the product $Z_1 \times Z_2$ carries a direct product Kähler-Einstein metric which is the sum of h_1 and h_2, after an appropriate constant rescaling of each. The join is then the unique simply-connected Sasaki-Einstein orbifold obtained by applying the inversion Theorem 3.1. The smoothness condition (3.1) is simply a rewriting of the condition that the local uniformizing groups inject into $U(1)$, given that this is true for each of $\mathcal{S}_1$, $\mathcal{S}_2$. In particular, note that if $\mathcal{S}_1$ is a regular Sasaki-Einstein manifold ($\mathrm{ord}(Z_1) = 1$) and $I(Z_2)$ divides $I(Z_1)$ then the join is smooth whatever the value of the order of Z_2.

The join construction can produce interesting non-trivial examples. For example, the homogeneous Sasaki-Einstein manifold in (2) of Theorem 2.6 is simply $S^3 \star S^3$, with the round metric on each S^3. More importantly, the join of a quasi-regular 3-Sasakian manifold with a regular Sasaki-Einstein manifold (such as S^3) gives rise to a quasi-regular Sasaki-Einstein manifold by the observation at the end of the previous paragraph. However, this particular construction produces new examples of Sasaki-Einstein manifolds only in dimension 9 and higher.

3.3. The continuity method for Kähler-Einstein orbifolds. The Proposition 2.8 holds also for compact Kähler orbifolds, with an identical proof. Thus also in the orbifold category, to find a Kähler-Einstein metric on a Fano orbifold one must similarly solve the Monge-Ampère equation (2.4). Of course, since necessary and sufficient algebraic conditions on Z are not even known in the smooth manifold case, for orbifolds the pragmatic approach of Boyer, Galicki, Kollár and their collaborators is to find a sufficient condition in appropriate classes of examples. Also as in the smooth manifold case, one can use the continuity method to great effect. Our discussion in the remainder of this section will closely follow [**19**].

Suppose, without loss of generality, that we are seeking a solution to (2.4) with $\kappa = 1$. Then the continuity method works here by introducing the more general equation

$$\frac{\det\left(h_{i\bar{j}} + \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j}\right)}{\det h_{i\bar{j}}} = \mathrm{e}^{f - t\phi} \ , \tag{3.2}$$

where now $t \in [0, 1]$ is a constant parameter. We wish to solve the equation with $t = 1$. We know from Yau's proof of the Calabi conjecture [**110**] that there is a solution with $t = 0$. The classic continuity argument works by trying to show that the subset of $[0, 1]$ where solutions exist is both open and closed. In fact openness is a straightforward application of the implicit function theorem. On the other hand, closedness is equivalent to the integrals

$$\int_Z \mathrm{e}^{-\gamma t \phi_t} \omega_0^{n-1}$$

being uniformly bounded, for any constant $\gamma \in ((n-1)/n, 1)$. Here ω_0 denotes the Kähler form for h_0, the metric given by Yau's result. Nadel interprets this condition in terms of *multiplier ideal sheaves* [**81**].

This was first studied in the case of Fano orbifolds by Demailly-Kollár [**38**], and indeed it is their results that led to the first examples of quasi-regular Sasaki-Einstein 5-manifolds in [**17**]. The key result is the following:

THEOREM 3.3. *Let Z be a compact Fano orbifold of dimension* $\dim_{\mathbb{C}} Z = n-1$*. Then the continuity method produces a Kähler-Einstein orbifold metric on Z if there is a $\gamma > (n-1)/n$ such that for every $s \geq 1$ and for every holomorphic section $\tau_s \in H^0\left(Z, \left(K_Z^{\mathrm{orb}}\right)^{-s}\right)$*

$$\int_Z |\tau_s|^{-2\gamma/s} \, \omega_0^{n-1} < \infty \ . \tag{3.3}$$

For appropriate classes of examples, the condition (3.3) is not too difficult to check. We next introduce such a class.

3.4. Links of weighted homogeneous hypersurface singularities. Let $w_i \in \mathbb{Z}_{>0}$, $i = 0, \ldots, n$, be a set of positive integers. We regard these as a vector $\mathbf{w} \in (\mathbb{Z}_{>0})^{n+1}$. There is an associated weighted $\mathbb{C}^*$ action on $\mathbb{C}^{n+1}$ given by

$$\mathbb{C}^{n+1} \ni (z_0, \ldots, z_n) \mapsto (\lambda^{w_0} z_0, \ldots, \lambda^{w_n} z_n) \ , \tag{3.4}$$

where $\lambda \in \mathbb{C}^*$ and the w_i are referred to as the *weights*. Without loss of generality one can assume that $\gcd(w_0, \ldots, w_n) = 1$, so that the $\mathbb{C}^*$ action is effective, although this is not necessary.

Definition 3.4. A polynomial $F \in \mathbb{C}[z_0, \ldots, z_n]$ is said to be a *weighted homogeneous polynomial with weights* $\mathbf{w}$ *and degree* $d \in \mathbb{Z}_{>0}$ if

$$F\left(\lambda^{w_0} z_0, \ldots, \lambda^{w_n} z_n\right) = \lambda^d F(z_0, \ldots, z_n) \ .$$

We shall always assume that F is chosen so that the affine algebraic variety

$$X_F = \{F = 0\} \subset \mathbb{C}^{n+1} \tag{3.5}$$

is smooth everywhere except at the origin in $\mathbb{C}^{n+1}$.

Definition 3.5. The hypersurface given by (3.5) is called a *quasi-homogeneous hypersurface singularity.* The *link* L_F of the singularity is defined to be

$$L_F = \{F = 0\} \cap S^{2n+1} \ , \tag{3.6}$$

where $S^{2n+1} = \left\{\sum_{i=0}^n |z_i|^2 = 1\right\} \subset \mathbb{C}^{n+1}$ is the unit sphere.

L_F is a smooth $(2n-1)$-dimensional manifold, and it is a classic result of Milnor [**78**] that L_F is $(n-2)$-connected. Indeed, the homology groups of L_F were computed in [**78, 79**] in terms of the so-called *monodromy map.*

A particularly nice such set of singularities are the so-called Brieskorn-Pham singularities. These take the particular form

$$F = \sum_{i=0}^{n} z_i^{a_i} \ , \tag{3.7}$$

where $\mathbf{a} \in (\mathbb{Z}_{>0})^{n+1}$. Thus the weights of the $\mathbb{C}^*$ action are $w_i = d/a_i$ where $d = \mathrm{lcm}\{a_i\}$. The corresponding hypersurface singularities are always isolated, as is easy to check. In this case it is convenient to denote the link by $L_F = L(\mathbf{a})$. Moreover, to the vector $\mathbf{a}$ one associates a graph $G(\mathbf{a})$ with $n+1$ vertices labelled by the a_i. Two vertices a_i, a_j are connected if and only if $\gcd(a_i, a_j) > 1$. We denote the connected component of $G(\mathbf{a})$ determined by the even integers by C_{even}; all even integer vertices are contained in C_{even}, although C_{even} may of course contain odd integer vertices also. The following result is due to Brieskorn [**30**] (although see also [**39**]):

Theorem 3.6. *The following are true:*

(1) *The link $L(\mathbf{a})$ is a rational homology sphere if and only if either $G(\mathbf{a})$ contains at least one isolated point, or C_{even} has an odd number of vertices and for any distinct $a_i, a_j \in C_{\text{even}}$, $\gcd(a_i, a_j) = 2$.*

(2) *The link* $L(\mathbf{a})$ *is an integral homology sphere if and only if either* $G(\mathbf{a})$ *contains at least two isolated points, or* $G(\mathbf{a})$ *contains one isolated point and* C_{even} *has an odd number of vertices and* $a_i, a_j \in C_{\mathrm{even}}$ *implies* $\gcd(a_i, a_j) = 2$ *for any distinct* i, j.

A simply-connected integral homology sphere is also a *homotopy sphere*, by the Hurewicz isomorphism theorem and the Whitehead theorem. Hence by the higher-dimensional versions of the Poincaré conjecture, a simply-connected integral homology sphere is in fact homeomorphic to the sphere. In particular, Theorem 3.6 says which $\mathbf{a}$ lead to homotopy spheres $L(\mathbf{a})$. Again, classical results going back to Milnor [**77**] and Smale [**94**] show that in every dimension greater than 4 the differentiable homotopy spheres form an Abelian group, where the group operation is given by the connected sum. There is a subgroup consisting of those which bound parallelizable manifolds, and these groups are known in every dimension [**61**]. They are distinguished by the signature τ of a parallelizable manifold whose boundary is the homotopy sphere. There is a natural choice in fact (the Milnor fibre in Milnor's fibration theorem [**78**]), and Brieskorn computed the signature in terms of a combinatorial formula involving the $\{a_i\}$. Many of the results quoted in the introduction involving rational homology spheres and exotic spheres are proven this way. Let us give a simple example:

Example 3.7. By Theorem 3.6 the link $L(6k-1, 3, 2, 2, 2)$ is a homotopy 7-sphere. Using Brieskorn's formula one can compute the signature of the associated Milnor fibre, with the upshot being that all 28 oriented diffeomorphism classes on S^7 are realized by taking $k = 1, 2, \ldots, 28$.

Returning to the general case in Definition 3.5, the fact that L_F has a natural Sasakian structure was observed as long ago as reference [**96**]. We begin by noting that $\mathbb{C}^{n+1}$ (minus the origin) has a Kähler cone metric that is a cone with respect to the *weighted* Euler vector field

$$\tilde{r}\partial_{\tilde{r}} = \sum_{i=0}^{n} w_i \rho_i \partial_{\rho_i} \ , \tag{3.8}$$

where $z_i = \rho_i \exp(\mathrm{i}\theta_i)$, $i = 0, \ldots, n$. The Kähler form is $\frac{1}{2}\mathrm{i}\partial\bar{\partial}\tilde{r}^2$ where $\tilde{r}^2$ is a homogeneous degree 2 function under (3.8), and a natural choice is $\tilde{r}^2 = \sum_{i=0}^{n} \rho_i^{2/w_i}$. The holomorphic vector field $(\mathrm{i}\,1 + J)\,\tilde{r}\partial_{\tilde{r}}$ of course generates the weighted $\mathbb{C}^*$ action (3.4), and by construction the hypersurface X_F is invariant under this $\mathbb{C}^*$ action. Thus the Kähler metric inherited by X_F via its embedding (3.5) is also a Kähler cone with respect to this $\mathbb{C}^*$ action, which in turn gives rise to a Sasakian structure on L_F.

On the other hand, the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by the weighted $\mathbb{C}^*$ action is by definition the weighted projective space $\mathbb{P}(\mathbf{w}) = \mathbb{CP}^n_{[w_0,\ldots,w_n]}$. There is a corresponding commutative square

$$
\begin{array}{ccc}
L_F & \longrightarrow & S^{2n+1} \\
\Big\downarrow \pi & & \Big\downarrow \\
Z_F & \longrightarrow & \mathbb{P}(\mathbf{w}),
\end{array}
\tag{3.9}
$$

where the horizontal arrows are Sasakian and Kählerian embeddings, respectively, and the vertical arrows are orbifold Riemannian submersions. Here Z_F is simply the hypersurface $\{F = 0\}$, now regarded as defined in the weighted projective space, so $Z_F = \{F = 0\} \subset \mathbb{P}(\mathbf{w})$. Thus Z_F is a weighted projective variety.

We are of course interested in the case in which Z_F is Fano:

PROPOSITION 3.8. *The orbifold Z_F is Fano if and only if $|\mathbf{w}| - d > 0$, where $|\mathbf{w}| = \sum_{i=0}^{n} w_i$.*

This was proven in [**22**], but a simpler method of proof [**53**] that bypasses the orbifold subtleties is to use Proposition 1.10. Indeed, X_F is Gorenstein since the smooth locus $X_F \setminus \{0\}$ is equipped with a nowhere zero holomorphic $(n, 0)$-form, given explicitly in a coordinate chart in which the denominator is nowhere zero by

$$
\Omega = \frac{\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n}{\partial F/\partial z_0} \ . \tag{3.10}
$$

One has similar expressions in charts in which $\partial F/\partial z_i \neq 0$, and it is straightforward to check that these glue together into a global holomorphic volume form on $X_F \setminus \{0\}$. Since $\tilde{r}\partial_{\tilde{r}} z_i = w_i z_i$ and F has degree d, it follows from (3.10) that

$$
\mathcal{L}_{\tilde{r}\partial_{\tilde{r}}} \Omega = (|\mathbf{w}| - d)\, \Omega \ . \tag{3.11}
$$

As in the proof of Proposition 1.10, positivity of $|\mathbf{w}| - d$ is then equivalent to the Ricci form on Z_F being positive. Indeed, via a D-homothetic transformation we may define $r = \tilde{r}^a$ to be a new Kähler potential, where $a = (|\mathbf{w}| - d)/n$, so that the new Reeb vector field is

$$
\xi = \frac{n}{|\mathbf{w}| - d}\, \tilde{\xi} \ ,
$$

and $\mathcal{L}_\xi \Omega = i n \Omega$. The resulting Kähler metric on Z_F now satisfies $[\rho_Z] = 2n[\omega_Z] \in H^{1,1}(Z, \mathbb{R})$. One may then ask when this metric can be deformed to a Kähler-Einstein metric, thus giving a quasi-regular Sasaki-Einstein metric on L_F. Although in general necessary and sufficient conditions are not known, Theorem 3.3 gives a sufficient condition that is practical to check.

3.5. Quasi-regular Sasaki-Einstein metrics on links. Theorem 3.3 was used by Demailly-Kollár in their paper [**38**] to prove the existence of Kähler-Einstein orbifold metrics on certain orbifold del Pezzo surfaces, realized as weighted hypersurfaces in $\mathbb{CP}^3_{[w_0,w_1,w_2,w_3]}$. More precisely, they produced precisely 3 such examples. The very first examples of quasi-regular

Sasaki-Einstein 5-manifolds were constructed in [**17**] using this result, together with the inversion Theorem 3.1. The differential topology of the corresponding links can be analyzed using the results described in the previous section, together with Smale's Theorem 2.7, resulting in 2 non-regular Sasaki-Einstein metrics on $S^2 \times S^3$, and 1 on $\#2\left(S^2 \times S^3\right)$. An avalanche of similar results followed [**3, 18, 25, 26, 27, 59, 60**], classifying all such log del Pezzo surfaces (Fano orbifold surfaces with only isolated orbifold singularities) for which Theorem 3.3 produces a Kähler-Einstein orbifold metric. This led to quasi-regular Sasaki-Einstein structures on $\#k\left(S^2 \times S^3\right)$ for all $1 \leq k \leq 9$. Compare to Theorem 2.6.

In [**22**] Theorem 3.3 was applied to the Brieskorn-Pham links $L(\mathbf{a})$, giving the following remarkable result:

THEOREM 3.9. *Let $L(\mathbf{a})$ be a Brieskorn-Pham link, with weighted homogeneous polynomial given by (3.7). Denote $c_i = \mathrm{lcm}(a_0, \ldots, \hat{a}_i, \ldots, a_n)$, $b_i = \gcd(a_i, c_i)$, where as usual a hat denotes omission of the entry. Then $L(\mathbf{a})$ admits a quasi-regular Sasaki-Einstein metric if the following conditions hold:*

(1) $\sum_{i=0}^n \frac{1}{a_i} > 1$,
(2) $\sum_{i=0}^n \frac{1}{a_i} < 1 + \frac{n}{n-1} \min_i \{\frac{1}{a_i}\}$,
(3) $\sum_{i=0}^n \frac{1}{a_i} < 1 + \frac{n}{n-1} \min_{i,j} \{\frac{1}{b_i b_j}\}$.

The isometry group of the Sasaki-Einstein manifold is then 1-dimensional, generated by the Reeb vector field.

The condition (1) is simply the Fano condition in Proposition 3.8, rewritten in terms of the $a_i = d/w_i$. Conditions (2) and (3) result from the condition in Theorem 3.3. The statement about the isometry group follows from the fact that the Kähler orbifolds $Z(\mathbf{a})$ have no continuous automorphisms. It follows that the isometry group of the Kähler-Einstein metric on $Z(\mathbf{a})$ is finite.

An important modification of this result follows from perturbing the defining polynomial F to

$$F(\mathbf{a}, p) = \sum_{i=0}^n z_i^{a_i} + p(z_0, \ldots, z_n) \ , \tag{3.12}$$

where p is a weighted homogeneous polynomial of degree d. Then the above Theorem holds also for the links of $X_F = \{F(\mathbf{a}, p) = 0\} \subset \mathbb{C}^{n+1}$, provided the intersection of X_F with any number of hyperplanes $\{z_i = 0\}$ are all smooth outside the origin. The polynomials p may depend on complex parameters, which will then lead to continuous families of quasi-regular Sasaki-Einstein manifolds in which there is a corresponding family of complex structures on X_F or Z_F. In fact similar remarks apply also to the log del Pezzo examples that produce non-regular Sasaki-Einstein structures on $\#k\left(S^2 \times S^3\right)$ for $1 \leq k \leq 9$.

Theorem 3.9 gives only necessary conditions for existence; it is not expected that this result is sharp. On the other hand, subsequent work of Ghigi-Kollár [**54**], combined with the Lichnerowicz obstruction of [**53**] that we will describe in section 6, leads to the following:

THEOREM 3.10. *Let $L(\mathbf{a})$ be a Brieskorn-Pham link such that the $\{a_i\}$ are pairwise relatively prime. Then $L(\mathbf{a})$ is homeomorphic to S^{2n-1} and admits a Sasaki-Einstein metric if and only if*

$$1 < \sum_{i=0}^{n} \frac{1}{a_i} < 1 + n \min_i \left\{ \frac{1}{a_i} \right\} .$$

Of course, the topological statement here follows from the Brieskorn Theorem 3.6.

Using Theorems 3.9 and 3.10 it is now straightforward to construct vast numbers of new quasi-regular Sasaki-Einstein manifolds by simply finding those $\mathbf{a}$ which satisfy the given inequalities. For example, for fixed dimension n one can show that there are only finitely many such $\mathbf{a}$ that, via the Brieskorn result, give rise to homotopy spheres. With the aid of a computer one can easily list all such examples in dimensions 5 and 7.

EXAMPLE 3.11. It is simple to check that the links $L(2,3,7,k)$ are homotopy spheres for any k that is relatively prime to at least 2 of $\{2,3,7\}$. Moreover, for $5 \leq k \leq 41$ these satisfy the conditions in Theorem 3.9, giving 27 examples of quasi-regular Sasaki-Einstein metrics on S^5.

There are other examples also, given in [**22**], some of which have deformations in the sense of (3.12). There are 12 further examples satisfying the conditions in Theorem 3.10 of the form $L(2,3,5,k)$ for appropriate k. Combining all these results [**21**] gives:

COROLLARY 3.12. *There are at least 80 inequivalent families of Sasaki-Einstein structures on S^5. Some of these admit continuous Sasaki-Einstein deformations, the largest known of which depends on 5 complex parameters.*

In dimension 7 the authors of [**23**] showed that there are 8610 inequivalent families of Sasaki-Einstein structures on homotopy 7-spheres given by Theorem 3.9, and moreover there are examples for each of the 28 oriented diffeomorphism classes. More examples are produced using Theorem 3.10. In particular:

EXAMPLE 3.13. The link $L(2,3,7,43,1333)$ is diffeomorphic to the standard 7-sphere. It admits a complex 41-dimensional family of Sasaki-Einstein deformations.

Summarizing these results [**21**] gives:

COROLLARY 3.14. *Each of the 28 oriented diffeomorphism classes on S^7 admit several hundred inequivalent families of Sasaki-Einstein structures. In each class, some of these admit continuous deformations. The largest such family is that in Example 3.13.*

Similar results hold also in higher dimensions, although the numbers of solutions grows rapidly, prohibiting a complete list even in dimension 9. However, one can show in particular that Sasaki-Einstein structures exist, in vast numbers, on both the standard and exotic Kervaire spheres in every dimension of the form $4m+1$. We refer the reader to [**22**] for details.

In [**64**] Kollár considered Kähler-Einstein orbifold metrics on orbifolds whose singularities are purely of complex codimension 1. Thus as algebraic varieties these are smooth surfaces. This allows for considerably more complicated topology than the log del Pezzo surfaces described at the beginning of this section, and leads to:

THEOREM 3.15. *For every $k \geq 6$ there are infinitely many complex $(k-1)$-dimensional families of Sasaki-Einstein structures on $\#k\left(S^2 \times S^3\right)$.*

Thus all of the manifolds in Theorem 2.7 admit Sasaki-Einstein structures. We shall give a very different proof of this in section 5, for which the Sasaki-Einstein manifolds have an isometry group containing the torus $\mathbb{T}^3$.

We turn next to rational homology spheres. We begin with an example:

EXAMPLE 3.16. In every dimension the links $L(m, m, \ldots, m, k)$ are rational homology spheres provided $\gcd(m, k) = 1$. The conditions in Theorem 3.9 are satisfied if $k > m(m-1)$. The homology groups $H_{m-1}(L, \mathbb{Z})$ are torsion groups of order $k^{b_{m-1}}$ where b_{m-1} is the $(m-1)$th Betti number of the link $L(m, m, \ldots, m)$ [**20**].

The torsion groups here may be computed using [**88**]. This leads to the following result of [**20**]:

COROLLARY 3.17. *In every odd dimension greater than 3 there are infinitely many smooth compact simply-connected rational homology spheres admiting Sasaki-Einstein structures.*

In dimension 5 the classification Theorem 2.7 was extended to general oriented simply-connected 5-manifolds by Barden [**7**]. With the additional assumption of being spin, such 5-manifolds fall into 3 classes: rational homology spheres; connected sums of $S^2 \times S^3$, as in Theorem 2.7; and connected sums of these first two classes. Which of these diffeomorphism types of general simply-connected spin 5-manifolds admit Sasaki-Einstein structures has been investigated, with the most recent results appearing in [**29**]. In particular, there is the following interesting result of Kollár [**63**]:

THEOREM 3.18. *Let S be a simply-connected 5-manifold admitting a transverse Fano Sasakian structure. Then $H_2(S, \mathbb{Z})_{\rm tor}$ is isomorphic to one of the following groups:*

$$\mathbb{Z}_m^2 \ , \ \mathbb{Z}_2^{2n} \ , \ \mathbb{Z}_3^4 \ , \ \mathbb{Z}_3^6 \ , \ \mathbb{Z}_3^8 \ , \ \mathbb{Z}_4^4 \ , \ \mathbb{Z}_5^4 \ , \quad m \geq 1 \ , \ n > 1 \ .$$

In particular, this determines also the possible torsion groups for simply-connected Sasaki-Einstein 5-manifolds. Precisely which of the manifolds in the Smale-Barden classification could admit Sasaki-Einstein structures is listed in Corolloary 11.4.14 of the monograph [**21**], together with those for which existence has been shown. Further results, again using weighted homogeneous hypersurface singularities, have been presented recently in [**29**].

4. Explicit constructions

4.1. Cohomogeneity one Sasaki-Einstein 5-manifolds. In the last section we saw that quasi-regular Sasaki-Einstein structures exist in abundance, in every odd dimension. It is important to stress that these are existence results, based on sufficient algebro-geometric conditions for solving the Monge-Ampère equation (2.4) on the orbifold leaf space of a quasi-regular Sasakian manifold that is transverse Fano. Indeed, the isometry groups of the Sasaki-Einstein manifolds produced via this method in Theorem 3.9 are as small as possible, and this lack of symmetry suggests that it will be difficult to write down solutions in explicit form.

On the other hand, given enough symmetry one might hope to find examples of Sasaki-Einstein manifolds for which the metric and Sasakian structure can be written down explicitly in local coordinates. Of course, such examples will be rather special. We have already mentioned in Theorem 2.4 that homogeneous Sasaki-Einstein manifolds are classified. The next simplest case, in terms of symmetries, is that of *cohomogeneity one*. By definition this means there is a compact Lie group G of isometries preserving the Sasakian structure which acts such that the generic orbit has real codimension 1. In fact the first *explicit* quasi-regular Sasaki-Einstein 5-manifolds constructed were of this form. The construction also gave the very first examples of *irregular* Sasaki-Einstein manifolds, which had been conjectured by Cheeger-Tian [**32**] not to exist. The following result was presented in [**50**]:

THEOREM 4.1. *There exist countably infinitely many Sasaki-Einstein metrics on $S^2 \times S^3$, labelled by two positive integers $p, q \in \mathbb{Z}_{>0}$, $\gcd(p,q) = 1$, $q < p$, given explicitly in local coordinates by*

$$\begin{aligned} g \;&=\; \frac{1-y}{6}\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right) + \frac{1}{w(y)q(y)}\mathrm{d}y^2 + \frac{q(y)}{9}\left(\mathrm{d}\psi - \cos\theta\mathrm{d}\phi\right)^2 \\ &\quad + w(y)\left[\mathrm{d}\alpha + f(y)\left(\mathrm{d}\psi - \cos\theta\mathrm{d}\phi\right)\right]^2 \;, \end{aligned} \tag{4.1}$$

where

$$w(y) \;=\; \frac{2(a-y^2)}{1-y} \;, \quad q(y) = \frac{a - 3y^2 + 2y^3}{a - y^2} \;, \quad f(y) = \frac{a - 2y + y^2}{6(a-y^2)} \;,$$

and the constant $a = a_{p,q}$ is

$$a = a_{p,q} = \frac{1}{2} - \frac{(p^2 - 3q^2)}{4p^3}\sqrt{4p^2 - 3q^2} \;. \tag{4.2}$$

The manifolds are cohomogeneity one under the isometric action of a Lie group with Lie algebra $\mathtt{su}(2) \oplus \mathtt{u}(1) \oplus \mathtt{u}(1)$. The Sasakian structures are quasi-regular if and only if $4p^2 - 3q^2 = m^2$, $m \in \mathbb{Z}$; otherwise they are *irregular of rank 2. In particular, there are countably infinite numbers of quasi-regular and irregular Sasaki-Einstein structures on $S^2 \times S^3$.*

We discovered these manifolds quite by accident, whilst trying to classify a certain class of supergravity solutions [**49**]. It is not too difficult to check that the metric in (4.1) is indeed Sasaki-Einstein, although a key point is that the local coordinate system here is not in fact well-adapted to the Sasakian structure. For example, the Reeb vector field is $\xi = 3\partial_\psi - \frac{1}{2}\partial_\alpha$. Instead these local coordinates are convenient for analysing when and how this metric extends to a smooth complete metric on a compact manifold.

The metric in the first line of (4.1) can in fact be shown to be a smooth complete metric on $S^2 \times S^2$, for any value of the constant $a \in (0,1)$, by taking $\theta \in [0,\pi]$, $y \in [y_1, y_2]$, and the coordinates ϕ and ψ to be periodic with period 2π. Here $y_1 < y_2$ are the two smallest roots of the cubic appearing in the numerator of the function $q(y)$, the condition that $a \in (0,1)$ guaranteeing in particular that these roots are real. Geometrically, these coordinates naturally describe a 4-manifold which is given by the 1-point compactifications of the fibres of the tangent bundle of S^2, TS^2. This results in an S^2 bundle over S^2 that is topologically trivial. There is a natural action of $SO(3) \times U(1)$, under which the metric is invariant, in which $SO(3)$ acts in the obvious way on TS^2 and the $U(1)$ acts on the fibre, the latter $U(1)$ being generated by the Killing vector field ∂_ψ. The 1-form in square brackets appearing in the second line of (4.1) can then be shown to be proportional to a connection 1-form on the total space of a principal $U(1)$ bundle over $S^2 \times S^2$, provided $a = a_{p,q}$ is given by (4.2). The integers p and q are simply the Chern numbers of this $U(1)$ bundle, so naturally $(p,q) \in \mathbb{Z} \oplus \mathbb{Z} \cong H^2(S^2 \times S^2, \mathbb{Z})$. It is important here that $w(y) > 0$ for all $y \in [y_1, y_2]$. It is also important to stress that this principal $U(1)$ bundle is *not* generated by the Reeb vector field, and indeed the metric on $S^2 \times S^2$ in the first line of (4.1) is neither Kähler nor Einstein. Via the Gysin sequence and Smale's Theorem 2.7, the total space is diffeomorphic to $S^2 \times S^3$ provided $\gcd(p,q) = 1$.

In a sense, the Sasaki-Einstein manifolds of Theorem 4.1 interpolate between the two 5-dimensional homogeneous Sasaki-Einstein manifolds given in (1) and (2) of Theorem 2.6. More precisely, setting $p = q$ leads to a Sasaki-Einstein orbifold $S^5/\mathbb{Z}_{2p}$, with the round metric on S^5, while $q = 0$ instead leads to a Sasaki-Einstein orbifold which is a non-freely acting $\mathbb{Z}_p$ quotient of the homogeneous Sasaki-Einstein metric on $V_2(\mathbb{R}^4)$.

As stated in Theorem 4.1, the resulting Sasaki-Einstein manifolds are cohomogeneity one under the effective isometric action of a compact Lie group G with Lie algebra $\mathtt{su}(2) \oplus \mathtt{u}(1) \oplus \mathtt{u}(1)$. In fact we have the following classification result [**36**]:

THEOREM 4.2. *Let (S, g) be a compact simply-connected Sasaki-Einstein 5-manifold for which the isometry group acts with cohomogeneity one. Then (S, g) is isometric to one of the manifolds in Theorem 4.1.*

Much is known about the structure of cohomogeneity one manifolds, and also the Einstein equations in this case; a review was presented in a previous article in this journal series [**107**]. The cohomogeneity one assumption reduces the conditions for having a complete G-invariant Sasaki-Einstein metric to solving a system of ordinary differential equations on an interval, with certain boundary conditions at the endpoints of this interval. Here the interval is parametrized by distance t along a geodesic transverse to a generic orbit of G. Denoting the stabilizer group of a point on a generic orbit by $H \subset G$, then the manifold S has a dense open subset that is equivariantly diffeomorphic to $(t_0, t_1) \times G/H$. At the boundaries $t = t_0$, $t = t_1$ of the interval the generic orbit collapses to 2 special orbits G/H_0, G/H_1. For this to happen smoothly, H_1/H and H_2/H must both be diffeomorphic to spheres of positive dimension. Choosing an Ad_H-invariant decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, a G-invariant metric on S is determined by a map from $[t_0, t_1] \to S^2_+(\mathfrak{m})^H$, where the latter is the space of Ad_H-invariant symmetric positive bilinear maps on $\mathfrak{m}$. This has appropriate boundary conditions at $t = t_0$ and $t = t_1$ that guarantee the metric compactifies to a smooth metric on S. We refer the reader to [**36**] for a complete discussion in the case of Sasaki-Einstein 5-manifolds. For the cohomogeneity one manifolds in Theorem 4.1 the special orbits are located at $y = y_1$, $y = y_2$.

4.2. A higher dimensional generalization. Being Sasaki-Einstein, the Reeb foliation $\mathcal{F}_\xi$ for the manifolds in Theorem 4.1 is transversely Kähler-Einstein. The Kähler-Einstein metric on a local leaf space is, after an appropriate local change of coordinates, given by

$$
\begin{aligned}
g^T \;&=\; \frac{1-y}{6}\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right) + \frac{1}{w(y)q(y)}\mathrm{d}y^2 \\
&\quad + \frac{w(y)q(y)}{4}\left(\mathrm{d}\gamma + \tfrac{1}{3}\cos\theta \mathrm{d}\phi\right)^2 \; .
\end{aligned} \tag{4.3}
$$

This local Kähler metric is of *Calabi* form. By definition, the Calabi ansatz [**31**] takes a Kähler manifold (V, g_V, ω_V) of complex dimension m and produces a local Kähler metric in complex dimension $m+1$ given by

$$
\begin{aligned}
h \;&=\; (\beta - y) g_V + \frac{1}{4Y(y)}\mathrm{d}y^2 + Y(y)\left(\mathrm{d}\gamma + A\right)^2 \; , \\
\omega_h \;&=\; (\beta - y)\omega_V - \frac{1}{2}\mathrm{d}y \wedge (\mathrm{d}\gamma + A) \; .
\end{aligned} \tag{4.4}
$$

Here A is a local 1-form on V with $\mathrm{d}A = 2\omega_V$, Y is an arbitrary function, and β is a constant. If V is compact of course the 1-form A cannot be globally defined on V. However, if (V, ω_V) is a Hodge manifold then by definition ω_V is proportional to the curvature 2-form of a Hermitian line bundle over V. In this case the 1-form $\mathrm{d}\gamma + A$ may be interpreted globally as being proportional to the connection 1-form on the total space P of the associated principal $U(1)$ bundle. The Kähler metric (4.4) is then defined on $(y_1, y_2) \times P$, where the function $Y(y)$ is strictly positive on this interval and $y_2 < \beta$ so that $(\beta - y)$ is also strictly positive.

The Kähler-Einstein metric (4.3) is locally of this form, where one takes (V, g_V) to be the standard Fubini-Study Kähler-Einstein metric on $\mathbb{CP}^1$, normalized to have volume $2\pi/3$, $4Y(y) = w(y)q(y)$, and $\beta = 1$. The metric is also cohomogeneity one, where the isometry group has Lie algebra $\mathbf{su}(2) \oplus \mathbf{u}(1)$. In fact this metric was constructed as early as [**55**], with some global properties being discussed in [**85**]. One can replace $(V, g_V) = (\mathbb{CP}^1, g_{\text{Fubini-Study}})$ by a general Fano Kähler-Einstein manifold of complex dimension m. The metric h in (4.4) is then itself Kähler-Einstein provided Y satisfies an appropriate ordinary differential equation. Remarkably, this equation can be solved explicitly in every dimension [**10, 84**], leading to a local 1-parameter family of Kähler-Einstein metrics. However, in the latter reference it was shown that this local metric extends to a complete metric on a compact Fano manifold only for a particular member of this family, which is then simply the homogeneous Fubini-Study metric on $\mathbb{CP}^{m+1}$. This family of local metrics was subsequently forgotten.

Given the above discussion, it is perhaps then unsurprising that the construction of Sasaki-Einstein manifolds in Theorem 4.1 extends to a construction of infinitely many Sasaki-Einstein manifolds, in every odd dimension $2n - 1 \geq 5$, for every complete Fano Kähler-Einstein manifold (V, g_V) with $\dim_{\mathbb{C}} V = n - 2$. The following result was shown in [**51**], although the more precise statement given here appeared later in [**73**]:

THEOREM 4.3. *Let (V, g_V) be a complete Fano Kähler-Einstein manifold of complex dimension $\dim_{\mathbb{C}} V = n-2$ with Fano index $I = I(V)$. Then for every choice of positive integers $p, k \in \mathbb{Z}_{>0}$ satisfying $Ip/2 < k < pI$, $\gcd(p,k) = 1$, there is an associated explicit complete simply-connected Sasaki-Einstein manifold in dimension $2n-1$.*

Theorem 4.1 is the special case in which $(V, g_V) = (\mathbb{CP}^1, g_{\text{Fubini-Study}})$ and where $k = p + q$. The proof is almost identical to the proof of Theorem 4.1. One uses the local 1-parameter family of Kähler-Einstein metrics of [**10, 84**] to write down a local 1-parameter family of Sasaki-Einstein metrics in dimension $2n - 1$. This is a local version of the inversion theorem: given a (local) Kähler-Einstein metric h with positive Ricci curvature $\text{Ric}_h = 2nh$ the local metric

$$g = h + (\mathrm{d}\psi + A)^2 , \tag{4.5}$$

is Sasaki-Einstein, where A is a local 1-form with $\mathrm{d}A = 2\omega_h$. After an appropriate change of local coordinates, one sees that this local metric can be made into a complete metric on a compact manifold for a countably infinite number of members of the family. This manifold is the total space of a principal $U(1)$ bundle over a manifold which is itself an S^2 bundle over V. The latter is obtained from K_V^{-1} by compactifying each fibre, and the integers p, k specify the first Chern class of the principal $U(1)$ bundle.

Unlike the case $n = 3$ in Theorem 4.1, the homology groups of these Sasaki-Einstein manifolds in general depend on p and k. Determining which of the Sasakian structures are quasi-regular and which are irregular is equivalent to determining whether a certain polynomial of degree $n-1$, with integer coefficients depending on p and k, has a certain root which is rational or irrational, respectively [**72**].

4.3. Transverse Hamiltonian 2-forms. In [**52, 34**] it was noted that the last construction may be extended further by replacing the Kähler-Einstein manifold (V, g_V) by a finite set of Fano Kähler-Einstein manifolds (V_a, g_a), $a = 1, \ldots, N$, and correspondingly extending Calabi's Kähler metric ansatz. Moreover, in [**37**] (see also [**70**]) an infinite set of explicit *cohomogeneity two* Sasaki-Einstein metrics were presented on $S^2 \times S^3$. These have isometry group $\mathbb{T}^3$. In [**70**] it was realized that there is a single geometric structure that underlies all of these explicit constructions of Sasaki-Einstein manifolds, namely a transverse Hamiltonian 2-form on the Kähler leaf space of the Reeb foliation.

Hamiltonian 2-forms were introduced in [**2**]:

Definition 4.4. Let (Z, h, ω, J) be a Kähler manifold. A *Hamiltonian 2-form* ϕ is a real $(1,1)$-form that solves non-trivially the equation [**2**]

$$\nabla_Y \phi = \frac{1}{2}\left(\mathrm{d}\,\mathrm{Tr}_\omega \phi \wedge JY^\flat - \mathrm{d}^c\,\mathrm{Tr}_\omega \phi \wedge Y^\flat\right) \ .$$

Here Y is any vector field, ∇ denotes the Levi-Civita connection of h, and $Y^\flat = h(Y, \cdot)$ is the 1-form dual to Y.

In fact Hamiltonian 2-forms on Kähler manifolds are related to another structure that is perhaps rather more well-known, especially to relativists:

Proposition 4.5. *A* $(1,1)$*-form* ϕ *is Hamiltonian if and only if* $\phi + (\mathrm{Tr}_\omega \phi)\,\omega$ *is closed and the symmetric 2-tensor* $S = J(\phi - (\mathrm{Tr}_\omega\phi)\,\omega)$ *is a* Killing tensor*; that is,* $\mathrm{Sym}\,\nabla S = 0$ *(in components,* $\nabla_{(i} S_{jk)} = 0$*).*

The proof is an elementary calculation and may be found in [**2**]. In fact if ϕ is Hamiltonian then the $(1,1)$-form $\phi - \frac{1}{2}(\mathrm{Tr}_\omega\phi)\,\omega$ a is conformal Killing 2-form in the sense of [**92**]. In the relativity literature such a form is also called a *conformal Killing-Yano* form. Again, this leads to an equivalence between conformal Killing 2-forms of type $(1,1)$ and Hamiltonian 2-forms. Conformal Killing tensors and forms generalize the notion of conformal Killing vectors. The latter generate symmetries of the metric, and the same is true also of Killing tensors, albeit in a more subtle way. For example, a classic early result was that a Killing form gives rise to a quadratic first integral of the geodesic equation [**86**].

The key result about Hamiltonian 2-forms is that their existence leads to a very specific form for the Kähler metric h. Moreover, particularly relevant for us is that the Kähler-Einstein condition is then equivalent to solving a simple set of decoupled ordinary differential equations. Below we just sketch how this works, referring the reader to [**2**] for details. We note that many of the resulting ansätze for Kähler metrics had been arrived at prior to the work of [**2**], both in the mathematics literature (as pointed out in [**2**]), and also in the physics literature. The theory of Hamiltonian 2-forms unifies these various approaches.

One first notes that if ϕ is a Hamiltonian 2-form, then so is $\phi_t = \phi - t\omega$ for any $t \in \mathbb{R}$. One then defines the *momentum polynomial* of ϕ to be

$$p(t) = \frac{(-1)^m}{m!} * \phi_t^m \ .$$

Here m is the complex dimension of the Kähler manifold and $*$ is the Hodge operator with respect to the metric h. It is then straightforward to show that $\{p(t)\}$ are a set of Poisson-commuting Hamiltonian functions for the 1-parameter family of Killing vector fields $K(t) = J\,\mathrm{grad}_h p(t)$. For a given point in the Kähler manifold, these Killing vectors will span a vector subspace of the tangent space of the point; the maximum dimension of this subspace, taken over all points, is called the *order* s of ϕ. This leads to a local Hamiltonian $\mathbb{T}^s$ action on the Kähler manifold, and one may take a (local) Kähler quotient by this torus action. The reduced Kähler space is a direct product of N Kähler manifolds that depends on the moment map level at which one reduces, but only very weakly. The $2s$-dimensional fibres turn out to be *orthotoric*, which is a rather special type of toric Kähler structure. For further details, we refer the reader to reference [**2**]. However, the above should give some idea of how one arrives at the following structure theorem of [**2**]:

THEOREM 4.6. *Let (Z, h, ω, J) be a Kähler manifold of complex dimension m with a Hamiltonian 2-form ϕ of order s. This means that the momentum polynomial $p(t)$ has s non-constant roots $y_1, \ldots, y_s$. Denote the remaining* distinct *constant roots by $\zeta_1, \ldots, \zeta_N$, where ζ_a has multiplicity m_a, so that $p(t) = p_{\mathrm{nc}}(t)p_{\mathrm{c}}(t)$ where $p_{\mathrm{nc}}(t) = \prod_{i=1}^s (t - y_i)$ and $p_{\mathrm{c}}(t) = \prod_{a=1}^N (t - \zeta_a)^{m_a}$. Then there are functions $F_1, \ldots, F_s$ of one variable such that on a dense open subset the Kähler structure may be written*

$$\begin{aligned} h &= \sum_{a=1}^N p_{\mathrm{nc}}(\zeta_a) g_a + \sum_{i=1}^s \left[\frac{p'(y_i)}{F_i(y_i)} \mathrm{d}y_i^2 + \frac{F_i(y_i)}{p'(y_i)} \left(\sum_{j=1}^s \sigma_{j-1}(\hat{y}_i)\theta_j \right)^2 \right] \ . \\ \omega &= \sum_{a=1}^N p_{\mathrm{nc}}(\zeta_a)\omega_a + \sum_{i=1}^s \mathrm{d}\sigma_i \wedge \theta_i \ , \qquad \mathrm{d}\theta_i = \sum_{a=1}^N (-1)^i \zeta_a^{s-i} \omega_a \ . \end{aligned}$$

Here σ_i denotes the ith elementary symmetric function of the non-constant roots $y_1, \ldots, y_s$, and $\sigma_{j-1}(\hat{y}_i)$ denotes the $(j-1)$th elementary symmetric function of the $s-1$ roots $\{y_k \mid k \neq i\}$. Moreover, (g_a, ω_a) is a positive (or negative) definite Kähler metric on a manifold V_a with $\dim_{\mathbb{C}} V_a = m_a$.

In fact the Hamiltonian 2-form is simply

$$\phi = \sum_{a=1}^{N} \zeta_a p_{\mathrm{nc}}(\zeta_a)\omega_a + \sum_{i=1}^{s} (\sigma_i \mathrm{d}\sigma_1 - \mathrm{d}\sigma_{i+1}) \wedge \theta_i \ ,$$

where $\sigma_{s+1} \equiv 0$. What is remarkable about this ansatz for a Kähler structure is the following [**2**]:

Proposition 4.7. *The Kähler metric in Theorem 4.6 is Kähler-Einstein if for all $i = 1, \ldots, s$ the functions F_i satisfy*

$$F_i'(t) = p_{\mathrm{c}}(t) \sum_{j=0}^{s} b_j t^{s-j} \ , \tag{4.6}$$

where b_j are arbitrary constants (independent of i), and for all $a = 1, \ldots, N$ $\pm(g_a, \omega_a)$ is Kähler-Einstein with scalar curvature

$$\mathrm{Scal}_{\pm g_a} = \mp m_a \sum_{i=0}^{s} b_i \zeta_a^{s-i} \ .$$

In this case the Ricci form is $\rho_h = -\frac{1}{2} b_0 \omega$.

Of course, this result follows from direct local calculations. Notice that (4.6) may immediately be integrated to obtain a local Kähler-Einstein metric that is completely explicit, up to the Kähler metrics g_a.

By taking $m = n - 1$ and $b_0 = -4n$ one can lift such local Kähler-Einstein metrics of positive Ricci curvature to local Sasaki-Einstein metrics in dimension $2n - 1$ using (4.5). One may then ask when this local metric extends to a complete metric on a compact manifold.

In fact all known (or at least known to the author) explicit constructions of Sasaki-Einstein manifolds are of this form. The Sasaki-Einstein manifolds in Theorem 4.3 are constructed this way, with $s = 1$, $N = 1$. Indeed, this case is precisely the Calabi ansatz (4.4) for the local Kähler-Einstein metric, as already mentioned. The generalization in [**52, 34**] mentioned at the beginning of this section is $s = 1$ but $N \geq 1$. Finally, most interesting is to take $s > 1$. In particular, for a Sasaki-Einstein 5-manifold this means that necessarily $N = 1$ and moreover $m_1 = 0$. In other words, the transverse Kähler-Einstein metric is *orthotoric* in the sense of reference [**2**]. The results described in this section give the explicit local form of such a metric, although it must be stressed that this is not how they were first derived. In fact in [**37**] the local family of orthotoric Kähler-Einstein metrics was obtained by taking a certain limit of a family of black hole metrics. These black hole solutions themselves possess Killing tensors. On the other hand, in [**70**] the same local metrics were obtained by taking a limit of the Plebanski-Demianski metrics [**87**], again a result in general relativity. It is then simply a matter of analyzing when these local metrics extend to complete metrics on a compact manifold. The result is the following [**37, 70**]:

THEOREM 4.8. *There exist a countably infinite number of explicit Sasaki-Einstein metric on $S^2 \times S^3$, labelled naturally by 3 positive integers $a, b, c \in \mathbb{Z}_{>0}$ with $a \leq b$, $c \leq b$, $d = a + b - c$, $\gcd(a, b, c, d) = 1$, and also such that each of the pair $\{a, b\}$ is coprime to each of $\{c, d\}$. For $a = p - q$, $b = p + q$, $c = p$ these reduce to the Sasaki-Einstein structures in Theorem 4.1. Otherwise they are cohomogeneity two with isometry group $\mathbb{T}^3$.*

The proof here is rather different to that of the proof of Theorem 4.1. In fact it is easiest to understand the global structure using the toric methods developed in the next section. For integers $\{a, b, c\}$ not satisfying some of the coprime conditions one obtains Sasaki-Einstein orbifolds. The conditions under which the Sasakian structures are quasi-regular is not simple to determine explicitly in general, and involves a quartic Diophantine equation. Generically one expects them to be irregular. The next section allows one to characterize the Sasaki-Einstein manifolds in Theorem 4.8: they are all of the simply-connected toric Sasaki-Einstein manifolds with second Betti number $b_2(S) = 1$.

5. Toric Sasaki-Einstein manifolds

5.1. Toric Sasakian geometry. We begin with the following:

DEFINITION 5.1. A Sasakian manifold (S, g) is said to be *toric* if there is an effective, holomorphic and Hamiltonian action of the torus $\mathbb{T}^n$ on the corresponding Kähler cone $(C(S), \bar{g}, \omega, J)$ with Reeb vector field $\xi \in \mathfrak{t}_n =$ Lie algebra of $\mathbb{T}^n$.

Here we have used the same symbol for an element of the Lie algebra $\mathfrak{t}_n$ and the corresponding vector field on $C(S)$ induced by the group action. The abuse of notation should not cause confusion as the meaning should always be clear. The Hamiltonian property means that $\mathbb{T}^n$ acts on constant r surfaces in $C(S)$. This in turn implies that there exists a $\mathbb{T}^n$-invariant moment map

$$\mu : C(S) \to \mathfrak{t}_n^* \ , \qquad \text{where} \quad \langle \mu, \zeta \rangle = \tfrac{1}{2} r^2 \eta(\zeta) \ , \quad \forall \zeta \in \mathfrak{t}_n \ . \tag{5.1}$$

Definition 5.1 is taken from [**74**], and is compatible with the earlier definition of toric contact manifold appearing in [**16**] (up to a factor of $\frac{1}{2}$ in the moment map definition).

The condition on the Reeb vector field, $\xi \in \mathfrak{t}_n$, implies that the image $\mu(C(S)) \cup \{0\}$ is a *strictly convex rational polyhedral cone* $\mathcal{C}^* \subset \mathfrak{t}_n^*$ [**44, 66**]. (Toric symplectic cones with Reeb vector fields not satisfying this condition form a short list and have been classified [**66**].) By definition this means that $\mathcal{C}^*$ may be presented as

$$\mathcal{C}^* = \{ y \in \mathfrak{t}_n^* \mid \langle y, v_a \rangle \geq 0 \ , \quad a = 1, \ldots, d \} \subset \mathfrak{t}_n^* \ . \tag{5.2}$$

Here the rationality condition means that $v_a \in \mathbb{Z}_{\mathbb{T}^n} \equiv \ker\{\exp : \mathfrak{t}_n \to \mathbb{T}^n\}$. On choosing a basis this means that we may think of $v_a \in \mathbb{Z}^n \subset \mathbb{R}^n \cong \mathfrak{t}_n$, and without loss of generality we assume that the $\{v_a\}$ are primitive. We also assume that the set $\{v_a\}$ is minimal, in the sense that removing any v_a from the definition in (5.2) would change the polyhedral cone $\mathcal{C}^*$. The strictly convex condition means that $\mathcal{C}^*$ is a cone over a compact convex polytope of dimension $n-1$. It follows that necessarily the number of bounding hyperplanes is $d \geq n$.

The polyhedral cone $\mathcal{C}^*$ is also *good*, in the sense of [**66**]. This may be defined as follows. Each face $\mathcal{F} \subset \mathcal{C}^*$ may be realized uniquely as the intersection of some number of *facets* $\{\langle y, v_a\rangle = 0\} \cap \mathcal{C}^*$. Denote by $v_{a_1}, \ldots, v_{a_N}$ the corresponding collection of normal vectors in $\{v_a\}$, where N is the codimension of $\mathcal{F}$ – thus $\{a_1, \ldots, a_N\}$ is a subset of $\{1, \ldots, d\}$. Then the cone is good if and only if

$$\left\{\sum_{A=1}^{N} \nu_A v_{a_A} \mid \nu_A \in \mathbb{R}\right\} \cap \mathbb{Z}^n = \left\{\sum_{A=1}^{N} \nu_A v_{a_A} \mid \nu_A \in \mathbb{Z}\right\} ,$$

holds for all faces $\mathcal{F}$.

We denote by $\mathrm{Int}\,\mathcal{C}^*$ the open interior of $\mathcal{C}^*$. The $\mathbb{T}^n$ action on $\mu^{-1}(\mathrm{Int}\,\mathcal{C}^*)$ is free, and moreover the latter is a Lagrangian torus fibration over $\mathrm{Int}\,\mathcal{C}^*$. On the other hand, the bounding facets $\{\langle y, v_a\rangle = 0\} \cap \mathcal{C}^*$ lift to $\mathbb{T}^{n-1}$-invariant complex codimension one submanifolds of $C(S)$ that are fixed point sets of the $U(1) \cong \mathbb{T} \subset \mathbb{T}^n$ subgroup specified by $v_a \in \mathbb{Z}_{\mathbb{T}^n}$.

The image $\mu(S) = \mu(\{1\} \times S \subset C(S))$ is easily seen from (5.1) to be

$$\mu(S) = \{y \in \mathcal{C}^* \mid \langle y, \xi\rangle = \tfrac{1}{2}\} .$$

Here the hyperplane $\{y \in \mathfrak{t}_n^* \mid \langle y, \xi\rangle = \frac{1}{2}\} \subset \mathfrak{t}_n^*$ is called the *characteristic hyperplane* [**16**]. This intersects the moment cone $\mathcal{C}^*$ to form a compact n-dimensional polytope $\Delta(\xi) = \mu(\{r \leq 1\})$, bounded by $\partial\mathcal{C}^*$ and a $(n-1)$-dimensional compact convex polytope $H(\xi)$ which is the image $\mu(S)$ of the Sasakian manifold S in $\mathfrak{t}_n^*$. Since $\mu(\xi) = \frac{1}{2}r^2 > 0$ on $C(S)$ this immediately implies that the Reeb vector field $\xi \in \mathrm{Int}\,\mathcal{C}$ where

$$\mathcal{C} = \{\xi \in \mathfrak{t}_n \mid \langle y, \xi\rangle \geq 0 , \quad \forall y \in \mathcal{C}^*\} \subset \mathfrak{t}_n$$

is the *dual cone* to $\mathcal{C}^*$. This is also a convex rational polyhedral cone by Farkas' Theorem.

Recall that in section 1.4 we explained that the space $X = C(S) \cup \{r = 0\}$ can be made into a complex analytic space in a unique way. For a toric Sasakian manifold in fact X is an affine toric variety; that is, X is an affine variety equipped with an effective holomorphic action of the *complex* torus $\mathbb{T}^n_{\mathbb{C}} \cong (\mathbb{C}^*)^n$ which has a dense open orbit. The affine toric variety X may be constructed rather explicitly as follows. Given the polyhedral cone $\mathcal{C}^*$ one defines a linear map $A : \mathbb{R}^d \to \mathbb{R}^n$ via $A(e_a) = v_a$, where $\{e_a\}$ denotes the standard orthonormal basis of $\mathbb{R}^d$. The strictly convex condition on $\mathcal{C}^*$ implies that A is surjective. This induces a corresponding map of tori $\tilde{A} : \mathbb{T}^d \to \mathbb{T}^n$. The kernel $\ker \tilde{A}$ is a compact abelian subgroup of $\mathbb{T}^d$ of rank $d-n$ and $\pi_0(\ker \tilde{A}) \cong \mathbb{Z}_{\mathbb{T}^n}/\mathrm{span}_{\mathbb{Z}}\{v_a\}$. Then the affine variety X is simply $X = \mathrm{Spec}\,\mathbb{C}[z_1, \ldots, z_d]^{\ker \tilde{A}}$, the ring of invariants. This is a standard construction in toric geometry [**46**], and goes by the name of Delzant's Theorem. The goodness condition on $\mathcal{C}^*$ is necessary and sufficient for $X \setminus \{o\}$ to be a *smooth* manifold, away from the apex $\{o\}$ [**66**]. The fact that X is toric is also clear via this construction: the torus $\mathbb{T}^n_{\mathbb{C}} \cong \mathbb{T}^d_{\mathbb{C}}/\ker \tilde{A}_{\mathbb{C}}$ acts holomorphically on X with a dense open orbit. In this algebro-geometric language the cone $\mathcal{C}$ is precisely the *fan* for the affine toric variety X.

Let ∂_{ϕ_i}, $i = 1, \ldots, n$, be a basis for $\mathfrak{t}_n$, where $\phi_i \in [0, 2\pi)$ are coordinates on the real torus $\mathbb{T}^n$. We then have the following very explicit description of the space of toric Sasakian metrics [**74**]:

PROPOSITION 5.2. *The space of toric Kähler cone metrics on $C(S)$ is a product*

$$\mathrm{Int}\,\mathcal{C} \times \mathcal{H}^1(\mathcal{C}^*)$$

where $\xi \in \mathrm{Int}\,\mathcal{C} \subset \mathfrak{t}_n$ labels the Reeb vector field and $\mathcal{H}^1(\mathcal{C}^*)$ denotes the space of homogeneous degree one functions on $\mathcal{C}^*$ that are smooth up to the boundary (together with the convexity condition below).

Explicitly, on the dense open image of $\mathbb{T}^n_{\mathbb{C}}$ we have

$$\bar{g} = \sum_{i,j=1}^{n} G_{ij} \mathrm{d}y^i \mathrm{d}y^j + G^{ij} \mathrm{d}\phi_i \mathrm{d}\phi_j \ , \tag{5.3}$$

where $G_{ij} = \partial_{y_i}\partial_{y_j} G$ with matrix inverse G^{ij}, and the function

$$G(y) = G_{\mathrm{can}}(y) + G_{\xi}(y) + \psi(y) \tag{5.4}$$

is required to be strictly convex with $\psi(y) \in \mathcal{H}^1(\mathcal{C}^*)$ and

$$\begin{aligned} G_{\mathrm{can}}(y) &= \frac{1}{2}\sum_{a=1}^{d} \langle y, v_a\rangle \log\langle y, v_a\rangle \ , \\ G_{\xi}(y) &= \frac{1}{2}\langle y, \xi\rangle \log\langle y, \xi\rangle - \frac{1}{2}\left(\sum_{a=1}^{d}\langle y, v_a\rangle\right) \log\left(\sum_{a=1}^{d}\langle y, v_a\rangle\right) \ . \end{aligned}$$

The coordinates (y_i, ϕ_i) are called *symplectic toric* coordinates. The y_i are simply the Hamiltonian functions for ∂_{ϕ_i}:

$$y_i = \langle \mu, \partial_{\phi_i}\rangle = \tfrac{1}{2} r^2 \eta(\partial_{\phi_i}) \ , \qquad \omega = \sum_{i=1}^{n} \mathrm{d}y_i \wedge \mathrm{d}\phi_i \ .$$

The function $G(y)$ is called the *symplectic potential.* Setting $G(y) = G_{\mathrm{can}}(y)$ gives precisely the Kähler metric on $C(S)$ induced via Kähler reduction of the flat metric on $\mathbb{C}^d$. That is, $C(S)$ equipped with the metric given by (5.3) and $G(y) = G_{\mathrm{can}}(y)$ is isomorphic to the Kähler quotient $(\mathbb{C}^d, \omega_{\mathrm{flat}})//\ker \tilde{A}$ at level zero. The origin of $\mathbb{C}^d$ here projects to the singular point $\{r = 0\}$ in X. The function $G_{\mathrm{can}}(y)$ has a certain singular behaviour at the boundary $\partial\mathcal{C}^*$ of the polyhedral cone. This is required precisely so that the metric compactifies to a smooth metric on $C(S)$. By construction, the Kähler metric $\bar{g}$ in (5.3) is a cone with respect to $r\partial_r = \sum_{i=1}^n 2y_i\partial_{y_i}$. On the other hand, the complex structure in these coordinates is

$$J = \begin{pmatrix} 0 & -G^{ij} \\ G_{ij} & 0 \end{pmatrix} ,$$

and one easily checks that $J(r\partial_r) = \sum_{i,j=1}^n 2G_{ij}y_j\partial_{\phi_i} = \xi$, with ξ determined by $G_\xi(y)$ in (5.4). Proposition 5.2 extends earlier work of Guillemin [**57**] and Abreu [**1**] from the Kähler case to the Sasakian case.

The following topological result is from [**67**]:

PROPOSITION 5.3. *Let $\mathcal{S}$ be a toric Sasakian manifold. Then* $\pi_1(S) \cong \mathbb{Z}_{\mathbb{T}^n}/\mathrm{span}_{\mathbb{Z}}\{v_a\}$, $\pi_2(S) \cong \mathbb{Z}^{d-n}$.

In particular, S is simply-connected if and only if the vectors $\{v_a\}$ that define the moment polyhedral cone $\mathcal{C}^*$ form a $\mathbb{Z}$-basis of $\mathbb{Z}_{\mathbb{T}^n} \cong \mathbb{Z}^n$. Using the Hurewicz Isomorphism Theorem and Smale's Theorem 2.7 then gives:

COROLLARY 5.4. *Let $\mathcal{S}$ be a simply-connected toric Sasakian 5-manifold. Then S is diffeomorphic to $\#k\left(S^2 \times S^3\right)$ where $k = d - n$.*

Finally, we note that an affine toric variety is ℓ-Gorenstein in the sense of Definition 1.11 if and only if there is a basis for the torus $\mathbb{T}^n$ for which $v_a = (\ell, w_a)$ for each $a = 1, \ldots, d$, and $w_a \in \mathbb{Z}^{n-1}$. In particular, for a simply-connected toric Sasaki-Einstein manifold the affine toric variety X will be Gorenstein, and hence there will exist a basis such that $v_a = (1, w_a)$.

EXAMPLE 5.5. The Sasaki-Einstein manifolds in Theorem 4.8 are toric, and in fact the proof makes it evident that the corresponding affine toric varieties are $X = \mathrm{Spec}\,\mathbb{C}[z_1, z_2, z_3, z_4]^{\mathbb{C}^*(a,b,c)}$, where $\mathbb{C}^*(a, b, c)$ is the 1-dimensional subgroup of $(\mathbb{C}^*)^4$ specified by the lattice vector $(a, b, -c, -a - b + c)$. The fact that the entries in this vector sum to zero is equivalent to X being Gorenstein.

5.2. Sasaki-Einstein metrics. Proposition 5.2 gives a rather explicit description of the space of toric Sasakian metrics on the link of an affine toric singularity. We may then ask which of these are Sasaki-Einstein. In fact a rather more basic question is for which Reeb vector fields $\xi \in \mathrm{Int}\,\mathcal{C}$ is there a Sasaki-Einstein metric. Notice there was no analogous question in the approach of section 3: there we had a fixed affine variety, namely a weighted homogeneous hypersurface singularity, with a *fixed* choice of holomorphic Reeb vector field $\xi - \mathrm{i}J(\xi)$, namely that associated to the weighted $\mathbb{C}^*$ action.

Without any essential loss of generality, we consider simply-connected Sasakian manifolds. Then we know that the corresponding affine toric variety must be Gorenstein if the cone is to admit a Ricci-flat Kähler cone metric, and hence there is a basis such that $v_a = (1, w_a)$. We assume we have chosen such a basis.

The key idea in [**74**] was that an Einstein metric g on S with Ricci curvature $\mathrm{Ric}_g = 2(n-1)g$ is a critical point of the Einstein-Hilbert action

$$I[g] = \int_S \left[\mathrm{Scal}_g + 2(n-1)(3-2n)\right] \mathrm{d}\mu_g \ , \tag{5.5}$$

where $\mathrm{d}\mu_g$ is the Riemannian volume form associated to the metric g and as earlier Scal_g denotes the scalar curvature. We may then restrict this functional to the space of toric Sasakian metrics. The insight in [**74**] was that this functional in fact depends only on the Reeb vector field ξ of the Sasakian structure. Direct calculation gives:

PROPOSITION 5.6. *The Einstein-Hilbert action (5.5), restricted to the space of toric Sasakian metrics on the link of an affine toric Gorenstein singularity, induces a function* $I : \mathrm{Int}\,\mathcal{C} \to \mathbb{R}$ *given by*

$$I(\xi) = 8n(n-1)(2\pi)^n \left[\langle e_1, \xi\rangle - (n-1)\right] \mathrm{vol}(\Delta(\xi)) \ . \tag{5.6}$$

Here $e_1 = (1, 0, 0, \ldots, 0)$ *and* $\mathrm{vol}(\Delta(\xi))$ *denotes the Euclidean volume of the polytope* $\Delta(\xi) = \mu\left(\{r \leq 1\}\right)$.

A toric Sasaki-Einstein metric is a critical point of I defined in (5.6). Of course $\mathcal{C}$ is itself a cone, and one may first take the derivative of I along the Euler vector field of this cone. Using the fact that $\mathrm{vol}(\Delta(\xi))$ is homogeneous degree $-n$ one easily checks that this derivative is zero if and only if $\langle e_1, \xi\rangle = n$. Thus a critical point of I lies on the interior of the intersection of this plane with $\mathcal{C}$. Call the latter compact convex polytope $P \subset \mathfrak{t}_n$. It follows that the Reeb vector field for a Sasaki-Einstein metric is a critical point of

$$I\mid_{\mathrm{Int}\,P} = 8n(n-1)(2\pi)^n \mathrm{vol}(\Delta) \ .$$

This is, up to a constant of proportionality, also just the *Riemannian* volume of (S, g). If we write $\xi = \sum_{i=1}^n \xi_i \partial_{\phi_i}$ then it is simple to compute

$$\frac{\partial \mathrm{vol}(\Delta)}{\partial \xi_i} = \sum_{k=1}^n \frac{1}{2\xi_k\xi_k} \int_{H(\xi)} y^i \, \mathrm{d}\sigma \ , \tag{5.7}$$

$$\frac{\partial^2 \mathrm{vol}(\Delta)}{\partial \xi_i \partial \xi_j} = \sum_{k=1}^n \frac{2(n+1)}{\xi_k\xi_k} \int_{H(\xi)} y^i y^j \, \mathrm{d}\sigma \ . \tag{5.8}$$

Here $\mathrm{d}\sigma$ is the standard measure induced on the $(n-1)$-polytope $H(\xi) = \mu(S) \subset \mathcal{C}^*$. Notice that $\mathrm{vol}(\Delta)$ diverges to $+\infty$ at ∂P. This can be seen rather explicitly from the formula for the volume of the polytope $\mathrm{vol}(\Delta)$, but more conceptually for $\xi \in \partial\mathcal{C}$ the vector field ξ in fact vanishes somewhere on $C(S)$. Specifically, the bounding facets of $\mathcal{C}$ correspond to the generating rays of $\mathcal{C}^*$ under the duality map between cones; ξ being in a bounding facet of $\mathcal{C}$ implies that the corresponding vector field then vanishes on the inverse image, under the moment map, of the dual generating ray of $\mathcal{C}^*$.

Uniqueness and existence of a critical point of I now follows from a standard convexity argument: $I\mid_{\mathrm{Int}\,P} = 8n(n-1)(2\pi)^n\mathrm{vol}(\Delta)$ is a strictly convex (by (5.8)) positive function on the interior of a compact convex polytope P. By strict convexity, a critical point of I is equivalent to a local minimum, and this in turn is then the unique global minimum on P. Since I diverges to $+\infty$ on the boundary of P, such a critical point must occur in the interior of P. Also notice that I is bounded from below by zero on P, so it must have a global minimum somewhere, and hence a critical point. It follows from these comments that I must have precisely one critical point in the interior of P, and we have thus proven:

THEOREM 5.7. *There exists a unique Reeb vector field $\xi \in \mathrm{Int}\,\mathcal{C}$ for which the toric Sasakian structure on the link of an affine toric Gorenstein singularity can be Sasaki-Einstein.*

Having fixed the Reeb vector field, the problem of finding a Sasaki-Einstein metric now reduces to deforming the transverse Kähler metric to a transverse Kähler-Einstein metric. As in the regular and quasi-regular cases, this is a Monge-Ampère problem. To analyze this it is more convenient to introduce complex coordinates. Recall that the complex torus $\mathbb{T}^n_{\mathbb{C}}$ is a dense open subset of $C(S)$. Introducing log complex coordinates $z_i = x_i + \mathrm{i}\phi_i$ on $\mathbb{T}^n_{\mathbb{C}}$, the Kähler structure is

$$\omega = 2\mathrm{i}\partial\bar{\partial}F \ , \qquad \bar{g} = \sum_{i,j=1}^{n} F_{ij}\mathrm{d}x_i\mathrm{d}x_j + F_{ij}\mathrm{d}\phi_i\mathrm{d}\phi_j \ .$$

Here the Kähler potential is $F(x) = \frac{1}{4}r^2$ and $F_{ij} = \partial_{x_i}\partial_{x_j}F$. This is related to the symplectic potential G by Legendre transform

$$F(x) = \left(\sum_{i=1}^{n} y_i\partial_{y_i}G - G\right)(y = \partial_x F) \ .$$

Having fixed the holomorphic structure on $C(S)$ (this being determined uniquely up to equivariant biholomorphism by $\mathcal{C}^*$) and fixing the Reeb vector field to be the unique critical point of I, we may set $\psi = 0$ in (5.4) to obtain an explicit toric Sasakian metric g_0 that is a critical point of I. This is our background metric. We are then in the situation of Proposition 1.4: any other Sasakian metric with the same holomorphic structure on the cone and same Reeb vector field is related to this metric via a smooth basic function $\phi \in C^\infty_B(S)$. Thus, if g is a Sasaki-Einstein metric with this property then

$$\omega^T - \omega^T_0 = \mathrm{i}\partial_B\bar{\partial}_B\phi \ ,$$

where ω^T_0 and ω^T are the transverse Kähler forms associated to g_0 and g, respectively. The holomorphic volume form on $C(S)$ is [**74**]

$$\Omega = \mathrm{e}^{x_1+\mathrm{i}\phi_1}\left(\mathrm{d}x_1 + \mathrm{i}\mathrm{d}\phi_1\right) \wedge \cdots \wedge \left(\mathrm{d}x_n + \mathrm{i}\mathrm{d}\phi_n\right) \ , \tag{5.9}$$

and the critical point condition $\langle e_1, \xi\rangle = n$ is equivalent to $\mathcal{L}_\xi\Omega = in\Omega$. Thus using Proposition 1.10 and the transverse $\partial\bar{\partial}$ lemma again we may also write

$$\rho^T_0 - 2n\omega^T_0 = \mathrm{i}\partial_B\bar{\partial}_B f \ , \tag{5.10}$$

with $f \in C^\infty_B(S)$ smooth and basic. Then Proposition 2.8 goes through in exactly the same way in the transverse sense, with resulting transverse Monge-Ampère equation

$$\frac{\det\left(g^T_{0\,i\bar{j}} + \frac{\partial^2\phi}{\partial z_i \partial \bar{z}_j}\right)}{\det g^T_{0\,i\bar{j}}} = \mathrm{e}^{f-2n\phi} \ , \tag{5.11}$$

where now $z_1, \ldots, z_{n-1}$ are local complex coordinates on the leaf space of the Reeb foliation $\mathcal{F}_\xi$.

This problem was recently studied in detail in [**48**]. In fact the Monge-Ampère problem is almost identical to the case of toric Kähler-Einstein manifolds studied in [**108**]. The moment polytope in the latter case is essentially replaced by the polytope $H(\xi)$ in the Sasakian case. The continuity method is used to prove existence, as in section 3.3, and crucially the work of [**43**] on extending Yau's estimates [**110**] to transverse Monge-Ampère equations is appealed to to show that the C^0 estimate for the basic function ϕ is sufficient to solve the equation. Thus the main step is to prove the C^0 estimate for ϕ, and this closely follows the proof in the Kähler-Einstein case [**108**]. The result of [**48, 35**] is the following:

Theorem 5.8. *There exists a unique toric Sasaki-Einstein metric on the link of any affine toric Gorenstein singularity.*

Here uniqueness was proven in [**35**], and is understood up to automorphisms of the transverse holomorphic structure. Thus the existence and uniqueness problem for toric Sasaki-Einstein manifolds is completely solved. We note that in [**35**] the authors stated this theorem with the weaker requirement that the affine toric singularity is ℓ-Gorenstein. For $\ell > 1$ the links of such singularities will not be simply-connected, although the converse is not true. However, more importantly the existence of a Killing spinor implies the Gorenstein condition, as discussed in section 1.6, which is why we have presented Theorem 5.8 this way. If one does not care for the existence of a Killing spinor, one can weaken the Gorenstein condition to $\mathbb{Q}$-Gorenstein (ℓ-Gorenstein for some ℓ).

Combining Theorem 5.8 with the topological result of Corollary 5.4 leads to:

Corollary 5.9. *There exist infinitely many toric Sasaki-Einstein structures on* $\#k\left(S^2 \times S^3\right)$, *for every* $k \geq 1$.

In fact we have now done enough to see that the explicit metrics in Theorem 4.8 are all of the toric Sasaki-Einstein metrics on $S^2 \times S^3$.

Example 5.10. We comment on two particularly interesting examples. The Sasaki-Einstein structure on $S^2 \times S^3$ with $p = 2$, $q = 1$ in Theorem 4.1 has corresponding affine variety $X = \operatorname{Spec}\mathbb{C}[z_1, z_2, z_3, z_4]^{\mathbb{C}^*(2,2,-1,-3)}$ [**71**]. Using standard toric geometry methods it is straightforward to see that $X \setminus \{o\}$ is the total space of the canonical line bundle over the 1-point blow up of $\mathbb{CP}^2$, minus the zero section. Equivalently, X is obtained from this canonical line bundle by contracting the zero section. The latter is a Fano surface which doesn't admit a Kähler-Einstein metric, as discussed in section 2.4. It is equivalent to say that the canonical choice of holomorphic Reeb vector field $\xi - \mathrm{i}J(\xi)$ that rotates the $\mathbb{C}^*$ fibre over the del Pezzo surface cannot be the Reeb vector field for a Sasaki-Einstein metric. Indeed, in this example one can easily compute the function I in Proposition 5.6 and show that this choice of ξ is indeed not a critical point. Instead the critical ξ gives an irregular Sasaki-Einstein structure of rank 2 [**74**].

In the latter case this irregular Sasaki-Einstein structure associated to the 1-point blow-up of $\mathbb{CP}^2$ is completely explicit. For the 2-point blow-up there is no known explicit metric, but Theorem 5.8 implies there exists a unique toric Sasaki-Einstein metric on the total space of the principal $U(1)$ bundle associated to the canonical line bundle over the surface. In [**74**] the critical Reeb vector field was computed explicitly, showing that this is again irregular of rank 2.

Finally, although Theorem 5.8 settles the existence and uniqueness of toric Sasaki-Einstein manifolds in general, we point out that prior to this result van Coevering [**105**] proved the existence of infinite families of distinct toric quasi-regular Sasaki-Einstein structures on $\#k(S^2 \times S^3)$, for each odd $k > 1$, using a completely different method. He finds certain quasi-regular toric Sasakian submanifolds of 3-Sasakian manifolds obtained via the quotient construction mentioned in section 1.5, and then applies an orbifold generalization of a result of Batyrev-Selivanova [**8**] to deform the corresponding Kähler orbifold to a Kähler-Einstein orbifold. Thus, although 3-Sasakian geometry plays a role in this construction, the Sasaki-Einstein metrics are not induced from the 3-Sasakian structure.

6. Obstructions

6.1. The transverse Futaki invariant. A toric Sasaki-Einstein metric has a Reeb vector field ξ which is a critical point of the function I in Proposition 5.6. The derivative of I is of course a linear map on a space of holomorphic vector fields, and its vanishing is a necessary (and in the toric case also sufficient) condition for existence of a Sasaki-Einstein metric, or equivalently a transverse Kähler-Einstein metric for the foliation $\mathcal{F}_\xi$. Given the discussion in section 2.4 it is then not surprising that the derivative of the function I is essentially a *transverse Futaki invariant.* This was demonstrated in [**75**], although our discussion here follows more closely the subsequent treatment in [**48**].

Throughout this section we suppose that we have Sasakian structure $\mathcal{S}$ with Reeb foliation $\mathcal{F}_\xi$ satisfying $0 < c_1^B \in H_B^{1,1}(\mathcal{F}_\xi)$ and $0 = c_1(D) \in H^2(S, \mathbb{R})$. Via Proposition 1.10 it is equivalent to say that, after a possible D-homothetic transformation, we have $2\pi c_1^B = n[\mathrm{d}\eta] \in H_B^{1,1}(\mathcal{F}_\xi)$. Assuming also that S is simply-connected, then by Proposition 1.10 it is also equivalent to say that the corresponding Stein space $X = C(S) \cup \{r = 0\}$ is Gorenstein with $\mathcal{L}_\xi \Omega = \mathrm{i} n \Omega$, where Ω is a nowhere zero holomorphic $(n, 0)$-form on $C(S)$.

Following [**48**] we begin with:

DEFINITION 6.1. A complex vector field ζ on a Sasakian manifold $\mathcal{S}$ is said to be *Hamiltonian holomorphic* if

(1) its projection to each leaf space is a holomorphic vector field; and
(2) the complex-valued function $u_\zeta \equiv \frac{1}{2}\eta(\zeta)$ satisfies

$$\bar{\partial}_B u_\zeta = -\tfrac{1}{4} i_\zeta \mathrm{d}\eta \ .$$

Such a function u_ζ is called a *Hamiltonian function.*

If $(x, z_1, \ldots, z_{n-1})$ are coordinates for a local foliation chart U_α then one may write

$$\zeta = \eta(\zeta)\partial_x + \sum_{i=1}^{n-1} \zeta^i \partial_{z_i} - \eta \left(\sum_{i=1}^{n-1} \zeta^i \partial_{z_i} \right) \partial_x \ ,$$

where ζ^i are local basic holomorphic functions.

It is straightforward to see that $\zeta + \mathrm{i}\eta(\zeta) r\partial_r$ is then a holomorphic vector field on $C(S)$. A Hamiltonian holomorphic vector field in the sense of Definition 6.1 is precisely the orthogonal projection to $S = \{r = 1\}$ of a Hamiltonian holomorphic vector field on the Kähler cone $(C(S), \bar{g}, \omega)$ whose Hamiltonian function is basic and homogeneous degree zero under $r\partial_r$. As pointed out in [**48**], the set of all Hamiltonian holomorphic vector fields is a Lie algebra $\mathfrak{h}$. Moreover, if the transverse Kähler metric has constant scalar curvature then $\mathfrak{h}$ is necessarily reductive [**82**]; this is a transverse generalization of the Matsushima result mentioned in section 2.4. Thus the nilpotent radical of $\mathfrak{h}$ acts as an obstruction to the existence of a transverse constant scalar curvature Kähler metric, and in particular a transverse Kähler-Einstein metric.

Since $2\pi c_1^B = n[\mathrm{d}\eta]$, by the transverse $\partial\bar{\partial}$ lemma there exists a discrepancy potential $f \in C_B^\infty(S)$ such that

$$\rho^T - n\mathrm{d}\eta = \mathrm{i}\partial_B \bar{\partial}_B f \ .$$

We may then define

$$\mathscr{F}(\zeta) = \int_S \zeta(f)\, \mathrm{d}\mu_g \ . \tag{6.1}$$

Here the Riemannian measure is

$$\mathrm{d}\mu_g = \frac{1}{2^{n-1}(n-1)!}\, \eta \wedge (\mathrm{d}\eta)^{n-1} \ . \tag{6.2}$$

Compare this to the Futaki invariant (2.5): in the regular, or quasi-regular, case (6.1) precisely reduces to (2.5) by integration over the $U(1)$ Reeb fibre, up to an overall proportionality constant. By following Futaki's original computation [**47**] it is not difficult to show that $\mathscr{F}(\zeta)$ is independent of the transverse Kähler metric in the Kähler class $[\omega^T] = [\frac{1}{2}\mathrm{d}\eta] \in H^{1,1}_B(\mathcal{F}_\xi)$. Thus $\mathscr{F} : \mathfrak{h} \to \mathbb{C}$ is a linear function on $\mathfrak{h}$ whose non-vanishing obstructs the existence of a transverse Kähler-Einstein metric in the fixed basic Kähler class. This result was extended [**48, 28**] to obstructions to the existence of Sasakian metrics with harmonic basic kth Chern form, again generalizing the Kähler result to the transverse setting.

6.2. The relation to Kähler cones. The results of section 5 motivated the following set-up in [**75**]. Fix a complex manifold $(C(S) \cong \mathbb{R}_{>0} \times S, J)$ where S is compact, with maximal torus $\mathbb{T}^s \subset \mathrm{Aut}(C(S), J)$. Then let $\mathrm{KCM}(C(S), J)$ be the space of Kähler cone metrics on $(C(S), J)$ which are compatible with the complex structure J and such that $\mathbb{T}^s$ acts Hamiltonianly (preserving constant r surfaces) with the Reeb vector field $\xi \in \mathfrak{t}_s =$ Lie algebra of $\mathbb{T}^s$. For each metric in $\mathrm{KCM}(C(S), J)$ there is then an associated moment map given by

$$\mu : C(S) \to \mathfrak{t}_s^* , \qquad \langle \mu, \zeta \rangle = \tfrac{1}{2} r^2 \eta(\zeta) .$$

The image is a strictly convex rational polyhedral cone [**44**], and moreover all these cones are isomorphic for any metric in $\mathrm{KCM}(C(S), J)$. The toric case is when the rank of the torus is maximal, $s = n$.

For any metric $\bar{g} \in \mathrm{KCM}(C(S), J)$ we may consider the volume functional

$$\mathrm{Vol} : \mathrm{KCM}(C(S), J) \to \mathbb{R}_{>0} , \qquad \mathrm{Vol}(\bar{g}) = \int_S \mathrm{d}\mu_g = \mathrm{vol}(S, g) . \tag{6.3}$$

Here g is the Sasakian metric on S induced from the Kähler cone metric $\bar{g}$. Alternatively, it is simple to see that

$$2^{n-1}(n-1)!\,\mathrm{Vol}(\bar{g}) = \int_{C(S)} \mathrm{e}^{-r^2/2}\,\mathrm{d}\mu_{\bar{g}} = \int_{C(S)} \mathrm{e}^{-r^2/2} \frac{\omega^n}{n!} , \tag{6.4}$$

where ω is the Kähler form for $\bar{g}$. Of course, the volume of the cone itself is divergent, and the factor of $\exp(-r^2/2)$ here acts as a convergence factor. Since $\frac{1}{2}r^2$ is the Hamiltonian function for the Reeb vector field ξ, the second formula in (6.4) takes the form of a Duistermaat-Heckman integral [**41, 42**]. Corresponding localization formulae were discussed in [**75**], but we shall not discuss this here. We then have the following from [**75**]:

Proposition 6.2. *The functional* Vol *depends only on the Reeb vector field* ξ.

This perhaps needs some clarification. Via Proposition 1.4, any two Kähler cone metrics $\bar{g}$, $\bar{g}' \in \mathrm{KCM}(C(S), J)$ with the same Reeb vector field ξ have corresponding contact forms related by

$$\eta' = \eta + \mathrm{d}^{\mathrm{c}}_B \phi ,$$

where $\phi \in C^\infty_B(S)$ is basic with respect to $\mathcal{F}_\xi$. Notice that the Kähler cone metrics will have Kähler potentials given by smooth functions r, r', and that these then give different embeddings of S into $C(S)$. Proposition 6.2

is proven by writing $r' = r\exp(t\phi/2)$ and showing that the derivative of Vol with respect to t at $t=0$ is zero, independently of the choice of $\phi \in C^\infty_B(S)$.

We may next consider the derivatives of Vol [**75**]:

PROPOSITION 6.3. *The first and second derivatives of* Vol *are given by*

$$\mathrm{dVol}(\chi) = -n\int_S \eta(\chi)\,\mathrm{d}\mu_g\ , \tag{6.5}$$

$$\mathrm{d}^2\mathrm{Vol}(\chi_1,\chi_2) = n(n+1)\int_S \eta(\chi_1)\eta(\chi_2)\,\mathrm{d}\mu_g\ . \tag{6.6}$$

More formally, what we mean here by (6.5) is that we have a 1-parameter family $\{\bar{g}(t)\}_{-\epsilon<t<\epsilon}$ of Kähler cone metrics in $\mathrm{KCM}(C(S),J)$ with $\bar{g}(0)=\bar{g}$ and $\chi = \mathrm{d}\xi/\mathrm{d}t\mid_{t=0}$. Then (6.5) is the derivative of Vol with respect to t at $t=0$. Similarly for the second derivative (6.6). Of course, these formulae reduce to (5.7), (5.8), respectively, in the toric case. Also as in that case, the second derivative formula implies that Vol is formally a strictly convex function. Notice that setting $\chi=\xi$ in (6.5) implies that Vol is homogeneous degree $-n$ under a D-homothetic transformation with $\xi\mapsto\lambda\xi$. Indeed, (1.13) implies that $\eta\mapsto\lambda^{-1}\eta$ and hence the volume element (6.2) scales as $\mathrm{d}\mu_g\mapsto\lambda^{-n}\mathrm{d}\mu_g$.

Of course our interest is Ricci-flat Kähler cones. If S is simply-connected, which we assume for simplicity, then a necessary condition for $(C(S),J)$ to admit a compatible such metric is that there is a compatible nowhere zero holomorphic $(n,0)$-form Ω. The analytic space $X = C(S)\cup\{r=0\}$ is hence Gorenstein. We then introduce the space $\mathrm{KCM}(C(S),\Omega)$ as those $\bar{g}\in\mathrm{KCM}(C(S),J)$ such that $\mathcal{L}_\xi\Omega = in\Omega$. By Proposition 1.10, this is equivalent to those $\bar{g}$ for which $[\rho^T] = 2n[\omega^T]\in H^{1,1}_B(\mathcal{F}_\xi)$, where ξ is the associated Reeb vector field for the Kähler cone metric $\bar{g}$. As discussed around Proposition 1.10, this is a necessary condition for a Sasaki-Einstein metric. We then have the following from [**75**]:

PROPOSITION 6.4. *The Einstein-Hilbert action (5.5), interpreted as a functional on the space* $\mathrm{KCM}(C(S),\Omega)$*, depends only on the Reeb vector field (in the same sense as Proposition 6.2). Moreover,*

$$I(\bar{g}) = 4(n-1)\mathrm{Vol}(\bar{g})\ , \tag{6.7}$$

for $\bar{g}\in\mathrm{KCM}(C(S),\Omega)$.

Again, the proof is by direct calculation. Since a Ricci-flat Kähler cone metric is a critical point of I, the derivative of I is an obstruction. More precisely, combining the last two Propositions gives:

COROLLARY 6.5. *Let* $\bar{g}\in\mathrm{KCM}(C(S),\Omega)$*. Then a necessary condition to be able to deform the corresponding Sasakian metric* g *on* S *via a transverse Kähler deformation to a Sasaki-Einstein metric is that*

$$\int_S \eta(\chi)\,\mathrm{d}\mu_g = 0 \tag{6.8}$$

holds for all $\chi \in \mathfrak{t}_s$ *satisfying* $\mathcal{L}_\chi \Omega = 0$.

By assumption, a vector field $\chi \in \mathfrak{t}_s$ is holomorphic on the cone and preserves the Kähler structure. The condition that it also preserves Ω guarantees we are varying within $\mathrm{KCM}(C(S), \Omega)$, in accordance with Proposition 6.4. Since χ preserves r there is a corresponding Hamiltonian function μ_χ given by $\mu_\chi = \langle \mu, \chi \rangle = \frac{1}{2} r^2 \eta(\chi)$. Since also $\xi \in \mathfrak{t}_s$, in particular $[\xi, \chi] = 0$ and it follows that μ_χ is a basic function of homogeneous degree zero under $r\partial_r$. Thus by the comments after Definition 6.1, the orthogonal projection to S of χ is a Hamiltonian holomorphic vector field on S. Let us denote this by $p_* \chi$ where $p : C(S) \to S$ is the projection.

In [**75**] it was shown that for χ as in Corollary 6.5

$$\mathrm{dVol}(\chi) = -n \int_S \eta(\chi) \, \mathrm{d}\mu_g = -\frac{1}{2} \int_S p_* J(\chi)(f) \, \mathrm{d}\mu_g = -\frac{1}{2} \mathscr{F}(p_* J(\chi)) \ ,$$

where $\mathscr{F}$ was defined in (6.1). The proof in [**75**] used spin geometry, while in [**48**] the same relation was obtained without using spinors. Both are quite elementary computations and we refer the reader to these references for details.

Thus the Futaki invariant attains an interesting new interpretation when lifted to Sasakian geometry: it is essentially the derivative of the Einstein-Hilbert action for Sasakian metrics.

6.3. The Bishop and Lichnerowicz obstructions. We turn now to two further simple obstructions [**53**] to the existence of a Ricci-flat Kähler cone metric. As in the previous section, we fix a complex manifold $(C(S) \cong \mathbb{R}_{>0} \times S, \Omega)$, with S compact and Ω a nowhere zero holomorphic $(n, 0)$-form on $C(S)$. For example, we could take this to be the smooth part $X_F \setminus \{o\}$ of a weighted homogeneous hypersurface singularity, as in section 3.4. We will suppose that $\bar{g} \in \mathrm{KCM}(C(S), \Omega)$ is a Ricci-flat Kähler cone metric on $C(S)$ compatible with this structure, with some Reeb vector field ξ, and aim to show that under certain conditions we are led to a contradiction.

Recall from Proposition 6.2 that the Riemannian volume of a Sasakian manifold induced from a choice of Kähler cone metric $\bar{g} \in \mathrm{KCM}(C(S), \Omega)$ depends only on the Reeb vector field ξ. Here the underlying complex manifold is regarded as fixed. We may also understand this as follows:

PROPOSITION 6.6. *Suppose that* $\bar{g} \in \mathrm{KCM}(C(S), \Omega)$ *induces a regular or quasi-regular Sasakian structure on S (simply-connected) with Kähler leaf space Z. Then*

$$\frac{\mathrm{vol}(S, g)}{\mathrm{vol}(S^{2n-1}, g_{\mathrm{standard}})} = \frac{I(Z)}{n^n} \int_Z c_1(Z)^{n-1} \ , \tag{6.9}$$

where $I(Z) \in \mathbb{Z}_{>0}$ *is the orbifold Fano index of* Z *and* $(S^{2n-1}, g_{\mathrm{standard}})$ *is the standard round sphere.*

The condition $\bar{g} \in \mathrm{KCM}(C(S), \Omega)$ implies that, in the regular or quasi-regular case, $[\rho_Z] = 2n[\omega]_Z \in H^{1,1}(Z, \mathbb{R})$. The above result then follows since $c_1(Z)$ is represented by $\rho_Z/2\pi$. Notice also that we have integrated over the Reeb $U(1)$ fibre. Here the simply-connected condition means that the associated complex line orbibundle has first Chern class $-c_1(Z)/I(Z)$, as in Theorem 3.1, which determines the length of the generic Reeb S^1 fibre. Note that $\mathrm{vol}(S^{2n-1}, g_{\mathrm{standard}}) = 2\pi^n/(n-1)!$.

In the regular or quasi-regular case, the independence of the volume of transverse Kähler transformations follows simply because the volume of the Kähler leaf space depends only on the Kähler class. On the other hand, this is sufficient to prove the more general statement in Proposition 6.2 since the space of quasi-regular Reeb vector fields in $\mathrm{KCM}(C(S), \Omega)$ is dense in the space of all possible Reeb vector fields. This is simply the statement that an irregular Reeb vector field corresponds to an irrational slope vector in $\mathfrak{t}_s$, while regular or quasi-regular Reeb vector fields are rational vectors in this Lie algebra. The latter are dense of course.

Example 6.7. Our main class of examples will be the weighted homogeneous hypersurface singularities of section 3.4. Thus $C(S)$ is the smooth locus $X_F \setminus \{o\}$, with Ω given by (3.10). If $\bar{g} \in \mathrm{KCM}(C(S), \Omega)$ with Reeb vector field generating the canonical $U(1) \subset \mathbb{C}^*$ action given by the corresponding weighted $\mathbb{C}^*$ action, then one can compute

$$\mathrm{vol}(S, g) = \frac{2d}{w(n-1)!}\left(\frac{\pi(|\mathbf{w}| - d)}{n}\right)^n . \tag{6.10}$$

The reader should consult section 3.4 for a reminder of the definitions here. One can prove (6.10) either by directly using (6.9), which was done in [**11**] for the case of well-formed orbifolds (where all orbifold singularities have complex codimension at least 2), or [**53**] using the methods developed in [**75**].

We turn now to the related obstruction. Bishop's theorem [**13**] implies that for any $(2n-1)$-dimensional compact Einstein manifold (S, g) with $\mathrm{Ric}_g = 2(n-1)g$ we have

$$\mathrm{vol}(S, g) \leq \mathrm{vol}(S^{2n-1}, g_{\mathrm{standard}}) . \tag{6.11}$$

In the current set-up the left hand side depends only on the holomorphic structure of the cone $C(S)$ and Reeb vector field ξ. If one picks a Reeb vector field and computes this using, for example, (6.9), and the result violates (6.11), then there cannot be a Ricci-flat Kähler cone metric on $(C(S), \Omega)$ with ξ as Reeb vector field.

Of course, it is not clear *a priori* that this condition can ever obstruct existence in this way. However, Example 6.7 shows that the condition is not vacuous. Combining (6.10) with (6.11) leads immediately to:

Theorem 6.8. *The link of a weighted homogeneous hypersurface singularity admits a compatible Sasaki-Einstein structure only if*

$$d(|\mathbf{w}| - d)^n \leq wn^n .$$

It is simple to write down infinitely many examples of weighted homogeneous hypersurface singularities that violate this inequality. For example, take $F = z_1^2+z_2^2+z_3^2+z_4^k$. For k odd the link is diffeomorphic to S^5, while for k even it is diffeomorphic to $S^2 \times S^3$. The Bishop inequality then obstructs compatible Sasaki-Einstein structures on these links for all $k > 20$.

On the other hand, in [**53**] we conjectured more generally that for *regular* Reeb vector fields the Bishop inequality never obstructs. This is equivalent to the following conjecture about smooth Fano manifolds:

Conjecture 6.9. *Let Z be a smooth Fano manifold of complex dimension $n-1$ with Fano index $I(Z) \in \mathbb{Z}_{>0}$. Then*

$$I(Z)\int_Z c_1(Z)^{n-1} \leq n \int_{\mathbb{CP}^{n-1}} c_1(\mathbb{CP}^{n-1})^{n-1} = n^n \ ,$$

with equality if and only if $Z = \mathbb{CP}^{n-1}$.

This is related to, although slightly different from, a standard conjecture about Fano manifolds. For further details, see [**53**].

We turn next to another obstruction. To state this, fix a Kähler cone metric $\bar{g} \in \mathrm{KCM}(C(S), \Omega)$ with Reeb vector field ξ and consider the eigenvalue equation

$$\xi(f) = \mathrm{i}\lambda f \ . \tag{6.12}$$

Here $f : C(S) \to \mathbb{C}$ is a holomorphic function and we consider $\lambda > 0$. The holomorphicity of f implies that $f = r^\lambda u$ where u is a complex-valued homogeneous degree zero function under $r\partial_r$, or in other words a complex-valued function on S. Now on a Kähler manifold a holomorphic function is in fact *harmonic*. That is $\Delta_{\bar{g}} f = 0$, where $\Delta_{\bar{g}}$ denotes the Laplacian on $(C(S), \bar{g})$ acting on functions. On the other hand, since $\bar{g} = \mathrm{d}r^2 + r^2 g$ is a cone we have

$$\Delta_{\bar{g}} = r^{-2}\Delta_g - r^{-2n+1}\partial_r\left(r^{2n-1}\partial_r\right) \ ,$$

and so (6.12) implies that

$$\Delta_g u = \nu u \ ,$$

where

$$\nu = \lambda(\lambda + 2(n-1)) \ . \tag{6.13}$$

In other words, a holomorphic function f on $C(S)$ with definite weight λ, as in (6.12), leads automatically to an eigenfunction of the Laplacian $\Delta_g = \mathrm{d}^*\mathrm{d}$ on (S, g) acting on functions.

We again appeal to a classical estimate in Riemmanian geometry. Suppose that (S, g) is a compact Einstein manifold with $\mathrm{Ric}_g = 2(n-1)g$. The first non-zero eigenvalue $\nu_1 > 0$ of Δ_g is bounded from below

$$\nu_1 \geq 2n - 1 \ .$$

This is Lichnerowicz's theorem [**68**]. Moreover, equality holds if and only if (S, g) is isometric to the round sphere $(S^{2n-1}, g_{\mathrm{standard}})$ [**83**]. From (6.13) we immediately see that for holomorphic functions f on $C(S)$ of weight λ

under ξ, Lichnerowicz's bound becomes $\lambda \geq 1$. This leads to another potential holomorphic obstruction to the existence of Sasaki-Einstein structures. Again, *a priori* it is not clear whether or not this will ever serve as an obstruction. In fact for *regular* Sasakian structures one can prove [**53**] this condition is always trivial. This follows from the fact that $I(Z) \leq n$ for any smooth Fano Z of complex dimension $n-1$.

However, there exist plenty of obstructed quasi-regular examples:

THEOREM 6.10. *The link of a weighted homogeneous hypersurface singularity admits a compatible Sasaki-Einstein structure only if*

$$|\mathbf{w}| - d \leq n w_{\min} \ .$$

Here $w_{\min}$ is the smallest weight. Moreover, this bound can be saturated if and only if $(C(S), \bar{g})$ is $\mathbb{C}^n \setminus \{0\}$ with its flat metric. Notice this result is precisely the necessary direction in Theorem 3.10. As another example, consider again the case $F = z_1^2 + z_2^2 + z_3^2 + z_4^k$. The coordinate z_4 has Reeb weight $\lambda = 6/(k+2)$, which obstructs for all $k > 4$. For $k = 4$ we have $\lambda = 1$, but since in this case the link is diffeomorphic to $S^2 \times S^3$ the Obata result [**83**] obstructs this marginal case also. In fact these examples are interesting because the compact Lie group $SO(3) \times U(1)$ is an automorphism group of the complex cone and acts with cohomogeneity one on the link. The Matsushima result implies this will be the isometry group of any Sasaki-Einstein metric, and then Theorem 4.2 in fact also rules out all $k \geq 3$. Indeed, notice that $k = 1$ corresponds to the round S^5 while $k = 2$ is the homogeneous Sasaki-Einstein structure on $S^2 \times S^3$. The reader will find further interesting examples in [**53**].

Very recently it has been shown in [**33**] that for weighted homogeneous hypersurface singularities the Lichnerowicz condition obstructs if the Bishop condition obstructs. More precisely:

THEOREM 6.11. *Let $w_0, \ldots, w_n, d$ be positive real numbers such that*

$$d\,(|\mathbf{w}| - d)^n > w n^n \ ,$$

and $d < |\mathbf{w}|$, where $|\mathbf{w}| = \sum_{i=0}^n w_i$, $w = \prod_{i=0}^n w_i$. Then $|\mathbf{w}| - d > n w_{\min}$.

In particular, this shows that Conjecture 6.9 is true for smooth Fanos realized as hypersurfaces in weighted projective spaces.

As a final comment, we note that more generally the Lichnerowicz obstruction involves holomorphic functions on $(C(S), \Omega)$ of small weight with respect to ξ, whereas the Bishop obstruction is a statement about the volume of (S, g), which is determined by the asymptotic growth of holomorphic functions on $C(S)$, analogously to Weyl's asymptotic formula [**11, 53**].

7. Outlook

We conclude with some brief comments on open problems in Sasaki-Einstein geometry. Clearly, one could describe many more. We list the problems in decreasing order of importance (and difficulty).

In general, the existence of a Kähler-Einstein metric on a Fano manifold is expected to be equivalent to an appropriate notion of stability, in the sense of geometric invariant theory. We discussed this briefly in section 2.4 and described K-stability. The relation between the Futaki invariant and such notions of stability is well-understood. Very recently, in the preprint [**90**] the Lichnerowicz obstruction of section 6 was related to stability of Fano orbifolds. The stability condition comes from a Kodaira-type embedding into a weighted projective space, as opposed to the Kodaira embedding into projective space used for Fano manifolds. This then leads to a notion of stability under the automorphisms of this weighted projective space. In particular, slope stability leads to some fairly explicit obstructions to the existence of Kähler-Einstein metrics (or more generally constant scalar curvature Kähler metrics) on Fano orbifolds. This includes the Lichnerowicz obstruction as a special case, although the role of the Bishop obstruction is currently rather more mysterious from this point of view.

A natural question is how to extend these ideas to Sasaki-Einstein geometry in general. In particular, how should one understand stability for irregular Sasakian structures?

Problem 7.1. Develop a theory of stability for Sasakian manifolds that is related to necessary and sufficient conditions for existence of a Sasaki-Einstein metric.

An obvious approach here is to use an idea we have already alluded to in the previous section: one might approximate an irregular Sasakian structure using a quasi-regular one, or perhaps more precisely a sequence of quasi-regular Sasakian structures that converge to an irregular Sasakian structure in an appropriate sense. Then one can use the notions of stability for orbifolds developed in [**90**]. A key difference between the Kähler and Sasakian cases, though, is that one is free to move the Reeb vector field, which in fact changes the Kähler leaf space. Given that the Sasakian description of obstructions to the existence of Kähler-Einstein orbifold metrics led to rather simple differentio-geometric descriptions of these obstructions in section 6, one might also anticipate that embedding Sasakian manifolds into spheres might be a beneficial viewpoint. That is, one takes the Sasakian lift of the Kodaira-type embeddings encountered in stability theory.

It is just about conceivable that one could classify Sasaki-Einstein manifolds in dimension 5:

Problem 7.2. Classify simply-connected Sasaki-Einstein 5-manifolds.

We remind the reader that this has been done for *regular* Sasaki-Einstein 5-manifolds (Theorem 2.6). This includes the homogeneous Sasaki-Einstein 5-manifolds, and moreover cohomogeneity one Sasaki-Einstein manifolds are also classified by Theorem 4.2. The latter two classes are subsets of the toric Sasaki-Einstein manifolds, which are classified by Theorem 5.8. Indeed, recall that the rank of a Sasakian structure is the dimension of the closure of the 1-parameter subgroup of the isometry group generated by the Reeb vector field. If a Sasaki-Einstein 5-manifold has rank 3 then it is toric, and so classified in terms of polytopes by Theorem 5.8. On the other hand, rank 1 are regular and quasi-regular. Is it possible to state necessary and sufficient conditions for the orbifold leaf space of a transversely Fano Sasakian 5-manifold to admit a Kähler-Einstein metric? We remind the reader that those simply-connected spin 5-manifolds that can possibly admit Sasaki-Einstein structures are listed as a subset of the Smale-Barden classification of such 5-manifolds in [**21**]; many, but apparently not all, of these can be realized as transversely Fano links of weighted homogeneous hypersurface singularities. The most complete discussion of what is known about existence of quasi-regular Sasaki-Einstein structures in this case appears in [**29**]. It is rank 2 that is perhaps most problematic. In fact, all known rank 2 Sasaki-Einstein 5-manifolds are toric. This leads to the simpler problem of whether or not there exist rank 2 Sasaki-Einstein 5-manifolds that are not toric. This problem is interesting for the reason that none of the methods for constructing Sasaki-Einstein manifolds described in this paper are capable of producing such an example.

Theorem 4.6 gives a fairly explicit local description of Kähler manifolds admiting a Hamiltonian 2-form. Using also Proposition 4.7 one thereby obtains a *local* classification of Sasaki-Einstein manifolds with a transverse Kähler structure that admits a transverse Hamiltonian 2-form. Via the comments in section 4.3, this implies the existence of a transverse Killing tensor. However, only in real dimension 5 have the global properties been investigated in detail, leading to Theorem 4.1 and Theorem 4.8.

Problem 7.3. Classify all complete Sasaki-Einstein manifolds admitting a transverse Hamiltonian 2-form.

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