GEOMETRIC QUANTIZATION I: SYMPLECTIC GEOMETRY
AND MECHANICS

A useful reference is Simms and Woodhouse, Lectures on Geometric Quantization, available online.

1. SYMPLECTIC GEOMETRY.

1.1. Linear Algebra. A pre-symplectic form $\omega$ on a (real) vector space $V$ is a skew bilinear form $\omega: V \otimes V \to \mathbb{R}$. For any $W \subset V$ write $W^\perp = \{ v \in V \mid \omega(v, w) = 0 \forall w \in W \}$. $(V, \omega)$ is symplectic if $\omega$ is non-degenerate: $\ker \omega := V^\perp = 0$.

**Theorem 1.2.** If $(V, \omega)$ is symplectic, there is a “canonical” basis $P_1, \ldots, P_n, Q_1, \ldots, Q_n$ such that $\omega(P_i, P_j) = 0 = \omega(Q_i, Q_j)$ and $\omega(P_i, Q_j) = \delta_{ij}$.

$P_i$ and $Q_i$ are said to be conjugate. The theorem shows that all symplectic vector spaces of the same (always even) dimension are isomorphic.

1.3. Geometry. A symplectic manifold is one with a non-degenerate 2-form $\omega$; this makes each tangent space $T_m M$ into a symplectic vector space with form $\omega_m$. We should also assume that $\omega$ is closed: $d\omega = 0$.

We can also consider pre-symplectic manifolds, i.e. ones with a closed 2-form $\omega$, which may be degenerate. In this case we also assume that $\dim \ker \omega_m$ is constant; this means that the various $\ker \omega_m$ fit together into a sub-bundle $\ker \omega$ of $TM$.

**Example 1.1.** If $V$ is a symplectic vector space, then each $T_m V = V$. Thus the symplectic form on $V$ (as a vector space) determines a symplectic form on $V$ (as a manifold). This is locally the only example:

**Theorem 1.4.** Every symplectic manifold $M$ is locally isomorphic to a symplectic vector space $V$. (In particular, for any $m \in M$ there are local ‘canonical’ coordinates $P_1, \ldots, Q_n$ corresponding to a canonical basis for $V$).

**Example 1.2.** If $M$ is any manifold, then $T^* M$ is a symplectic manifold. In fact, there is a unique one-form $\theta$ on $T^* M$ such that for any section $\alpha: M \to T^* M$ one has $\alpha^*(\theta) = \alpha$. Then $\omega := d\theta$.

In general, if $\omega = d\theta$ then $\theta$ is called a symplectic potential. One always exists locally (because $\omega$ is closed) but not always globally.

**Example 1.3.** If $M$ is a complex manifold with a Hermitian metric $\eta$ then define $\omega_m(\xi_1, \xi_2) = \text{Re} \eta_m(i\xi_1, \xi_2)$ for all $\xi_1, \xi_2 \in T_m M$. Then $\omega$ is a non-degenerate 2-form; if it is symplectic (i.e. closed) then $M$ is Kähler. For example: $M = \mathbb{P}^n(\mathbb{C})$ with $\eta$ the unique $U_{n+1}(\mathbb{C})$-invariant metric. Note this is compact (unlike a cotangent bundle) and there is no symplectic potential.

1.5. Hamiltonian Reduction. Other examples are found using Hamiltonian reduction. Suppose first that $W$ is a pre-symplectic vector space; then $W/\ker \omega$ is symplectic. In particular, if $V$ is symplectic, and $W^\perp \subset W \subset V$ (i.e. $W$ is ‘coisotropic’) then $W$ is pre-symplectic and $W/W^\perp$ is symplectic. This construction globalises in the following way:

Suppose $(N, \omega)$ is pre-symplectic. Then $\ker \omega$ is (by assumption) a subbundle of $TN$. The fact that $\omega$ is closed means that $\ker \omega$ is integrable. This means there
exists a ‘foliation’, i.e. a family $N_\alpha$ of submanifolds of $N$, such that $N = \sqcup_{\alpha \in I} N_\alpha$, and if $m \in N_\alpha$ then $T_m N_\alpha = \ker \omega_m$. In good cases, the set of ‘leaves’ $N/\ker \omega := \mathcal{I}$ is a manifold, and then it is symplectic. Indeed, $T_m(N/\ker \omega) = (T_m N)/\ker \omega_m$.

In particular, if $(M, \omega)$ is symplectic, we can choose some $N \subset M$ that is coisotropic: this means that $T_m N$ is a coisotropic subspace of $T_m M$, for any $m \in N$. In good cases, $(N, \omega|_N)$ will then be presymplectic, and $N/\ker \omega|_N$ will be symplectic.

**Example 1.4.** $V = \mathbb{R}^n$, so $M := TV \cong \mathbb{R}^{2n}$ is symplectic. Let $N \subset M$ be the unit sphere. Then the Hamiltonian reduction $N/\ker \omega|_N$ is naturally $\mathbb{P}^{n-1}(\mathbb{C})$. In this case the leaves of the foliation are circles on the sphere.

1.6. **Poisson Brackets.** The fact that $M$ is symplectic endows $C^\infty(M)$ with the structure of a Lie algebra, under the Poisson bracket, defined as follows.

Given $f \in C^\infty(M)$, there exists a unique vector field $X_f$ on $M$ such that $\omega(X_f, -) = df$. Such a vector field is called globally Hamiltonian. For $f, g \in C^\infty(M)$, define $[f, g] = 2\omega(X_f, X_g)$. This makes $C^\infty(M)$ into a Lie algebra (the Jacobi identity is equivalent to the fact that $\omega$ is closed). This gives a short exact sequence of Lie algebras:

$$0 \to \mathbb{R} \to C^\infty(M) \to \{(\text{Glob. Ham. VFs})\} \to 0.$$ 

The Lie bracket on globally Hamiltonian vector fields is the usual bracket of vector fields.

1.7. **A Naive Idea About Quantization.** The idea of quantization is to associate to a symplectic manifold $M$ a Hilbert space $H$, and to each ‘observable’ $f \in C^\infty(M)$, and operator $O_f$ on $H$, such that the Poisson bracket on $C^\infty(M)$ becomes the commutator of operators. Naively, we can take $H = L^2(M)$ and $O_f = X_f$, acting by derivations. But life is more complicated.

## 2. Mechanics

The kind of situation we want to describe is that of a particle moving in space $S$. How can one describe the trajectories? We sketch three methods.

2.1. **Hamiltonian mechanics.** General setup: $M$ a symplectic manifold; $H \in C^\infty(M)$ ‘the Hamiltonian.’ Points of $M$ label instantaneous states; time evolution is given by flow along the vector field $X_H$. In other words, $\gamma: \mathbb{R} \to M$ is a trajectory if it satisfies the differential equation $d\gamma/dt = X_H$ (maybe with a minus sign).

**Example 2.1.** $S$ space. Suppose that $S$ has a Riemannian metric $\eta$ (e.g. $S = \mathbb{R}^3$ with the usual inner product). We can use $\eta$ to identify each tangent space $T_m S$ with the cotangent space $T_m^* S$, thus making $M = TS$ into a symplectic manifold. $H(s, \xi) = \frac{1}{2} \eta(\xi, \xi)$ for $s \in S$ and $\xi \in T_s S$. The trajectories $R \to TS$ project to geodesics on $S$. (Note that any curve $\gamma: \mathbb{R} \to S$ lifts naturally to a curve $(\gamma, \gamma') : \mathbb{R} \to TS$)

2.2. **Presymplectic mechanics.** Now $N$ is pre-symplectic. Thus we have a foliation of $N$ with tangent spaces $\ker \omega$. The trajectories are leaves of this foliation, i.e. unparameterized integral surfaces for $\ker \omega$. (The simplest case is when $\ker \omega_m$ is one-dimensional, so the leaves are curves in $N$).

**Example 2.2.** With $S$ as before, and $H$ a function on $TM$, consider ‘spacetime’ $S \times \mathbb{R}$. Let $M = T^*(S \times \mathbb{R})$ and define $N \subset M$ to be those $(s, t; \alpha_s, \alpha_t) \in M$ such that $H(s, t) + \alpha_t = 0$. Then $N$ is coisotropic and presymplectic, and the trajectories project to graphs of geodesics $\mathbb{R} \to S$. 


This setup makes sense when $H$ depends on time, and also in relativistic setting where there is no canonical decomposition of spacetime into space $S$ and time $\mathbb{R}$.

Usually the situation is described in a slightly different way: one is given a ‘relativistic Hamiltonian’ $H_r \in C^\infty(M)$ (in the example, $H(s, t) + \alpha t$), and the trajectories are again integral curves for $X_{H_r}$, but subject to the constraint $H_r = 0$.

2.3. **Nice Thing I Forgot to Say.** The Hamiltonian reduction $N/\ker\omega$ of the presymplectic manifold $N$ is (by definition) the space of trajectories, which is therefore a symplectic manifold. In the example, we can, for each time $t \in \mathbb{R}$, identify $N/\ker\omega$ with the space $T^*S$ of instantaneous states. In quantum theory, $N/\ker\omega$ is called the ‘Heisenberg picture’ (the ‘states’ are entire trajectories) whereas $T^*S$ is called the ‘Schrodinger picture’ (the ‘states’ are instantaneous). As remarked before, the Schrodinger picture is not very natural in relativistic settings. In general, $N/\ker\omega$ might not even be isomorphic to a cotangent bundle.

2.4. **Lagrangian Mechanics.** Instead of characterising trajectories by differential equations, we use a ‘variational principle’ – like the one that says that geodesics minimize length. Lagrangian mechanics takes place in the tangent space $T S$. We are given a function $L \in C^\infty(TS)$. A trajectory $\gamma : \mathbb{R} \to S$ is one that extremizes the action

$$A(\gamma) = \int_{[0,1]} L(\gamma(t), \gamma'(t)) \, dt.$$ 

The relation to Hamiltonian mechanics is roughly as follows: one can use $L$ to define a (non usually linear) map $T_mS \to T^*_mS$, called the Legendre transform. In good cases, the Legendre transform is a local diffeomorphism, making $TS$ into a symplectic manifold. Flows are determined by an appropriate Hamiltonian.

**Example 2.3.** Take $L(s, \xi) = \frac{1}{2} \eta(\xi, \xi)$. Then the action is the length\(^1\) of $\gamma$, and the trajectories are geodesics. More generally, if we had a Hamiltonian $H(s, \xi) = \frac{1}{2} \eta(\xi, \xi) + U(s)$, then the corresponding Lagrangian is $L(s, \xi) = \frac{1}{2} \eta(\xi, \xi) - U(s).

The Lagrangian formalism is ubiquitous in physics, but it’s not yet clear to me how important it will be for our immediate aims.

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\(^1\)Actually not quite the length, which would be the integral of the square-root of the Lagrangian; but it turns out not to matter.