1.1. Construction of Representations. The unique-up-to-isomorphism irreducible representation of the Heisenberg group is constructed in the following way. First one chooses a Lagrangian subspace $L \subset V$: this means that $L^\perp = L$. Then $L \oplus U(1) \subset H(V)$ is a maximal abelian subgroup, and it has a one-dimensional representation, the projection $\psi: L \oplus U(1) \to U(1)$. Finally, $(\mathcal{H}_L, \rho_L)$ is the induced representation $\text{Ind}_{L \oplus U(1)}^{H(V)} \psi$.

More concretely, this amounts to the following. The space $\delta_{1/2}(V/L)$ of half-densities is a one-dimensional complex vector space with an isomorphism

$$\delta_{1/2}(V/L) \otimes \delta_{1/2}(V/L) = \delta_1(V/L),$$

where $\delta_1(V/L)$ is the space of translation-invariant measures on $V/L$. The Hilbert space $\mathcal{H}_L$ is the completion of the space of smooth functions $\phi: V \to \delta_{1/2}(V/L)$ satisfying the condition

$$\phi(x + a) = \phi(x)e^{i\omega(x,a)/2}$$

for all $x \in V, a \in L$, and finite with respect to the norm

$$\|\phi\|^2 := \int_{V/L} \bar{\phi}\phi.$$

(Here note that $\bar{\phi}: V \to \delta_1(V/L)$ is constant along $L$, so it defines a function $V/L \to \delta_1(V/L)$, i.e. a measure on $V/L$.) The action of $(v,t) \in H(V)$ is given by

$$\rho_L(v,t)\phi(x) = \phi(x - v)e^{i\omega(v,x)/2}t.$$

1.2. Change of Lagrangian. The main thing to take away is that although the isomorphism class of $(\mathcal{H}_L, \rho_L)$ is independent of $L$, there is no canonical isomorphism $\mathcal{H}_L \to \mathcal{H}_L'$ for different Lagrangians $L, L'$. (Of course such isomorphisms exist, and, because the representations are irreducible, they are unique up to phase.)

The dependence on the choice of Lagrangian is neatly summarised by considering the action of the symplectic group $\text{Sp}(V) = \text{Aut}(V, \omega)$. $\text{Sp}(V)$ acts on $H(V)$ by group automorphisms $g \cdot (v,t) = (gv, t)$. It’s then natural to ask whether the representation $(\mathcal{H}_L, \rho_L)$ extends to a representation of the semi-direct product $\text{Sp}(V) \ltimes H(V)$.

The answer is no. However, there is a natural representation of $\text{Mp}(V) \ltimes H(V)$ on $\mathcal{H}_L$, where $\text{Mp}(V)$ is the unique non-trivial double cover of $\text{Sp}(V)$.

More precisely: Let $\Lambda$ be the manifold of all Lagrangian subspaces of $V$. $\text{Sp}(V)$ acts transitively on $\Lambda$, and each $g \in \text{Sp}(V)$ naturally determines an isomorphism $\mathcal{H}_L \to \mathcal{H}_{gL}$ of Hilbert spaces. As mentioned above, there exists an isomorphism of representations $\mathcal{H}_{gL} \to \mathcal{H}_L$. The composition of these two maps defines an operator $\rho_L(g)$ on $\mathcal{H}_L$. The question is whether one can choose these operators in such a way that $\rho_L(g)\rho_L(g') = \rho_L(gg')$. It is automatic that this will work, if not for $g, g' \in \text{Sp}(V)$, then for a covering group of $\text{Sp}(V)$. The claim is that the correct covering group is $\text{Mp}(V)$.

\footnote{\textsuperscript{1}I always consider complex-valued (so, not necessarily positive) measures. Thus $\delta_1(V/L)$ is a one-dimensional complex vector space. A smooth function $V/L \to \delta_1(V/L)$ is the same thing as a smooth measure on $V/L$.}
1.3. **Weyl Quantization.** So far we found a representation of the Lie subalgebra of $C^\infty(V)$ generated by linear functions on $V$. Can one use this construction to represent other functions as well? 

There is a very natural way to try, but it does not satisfy our original (somewhat arbitrary) criterion for quantization. For a sufficiently nice (e.g. Schwartz) function $f$ on $V$, define the *Weyl transform* $W(f)$ to be the operator on $H_L$ defined by 

$$W(f)\phi(x) = \int_{v\in V} f(v)\rho_L(v,0)\phi(x) \, dv.$$ 

It turns out that $W$ defines an isomorphism of vector spaces from $L^2(V, dv)$ to the algebra of ‘Hilbert-Schmidt’ operators on $H_L$ (i.e. those represented by $L^2$-integral kernels). We can pull back the multiplication of operators to define a product $\star$ on $L^2(V, dv)$ making $W$ into an isomorphism of algebras. This $\star$ is the ‘star’ or ‘Moyal’ product.

So we have associated operators $W(f)$ on $H_L$ to a large class of functions $f$ on $V$. But for this to count as quantization, we must have $[W(f_1), W(f_2)] = iW([f_1, f_2])$, or, in terms of $\star$,

$$f_1 \star f_2 - f_2 \star f_1 = i[f_1, f_2] \quad \text{(Poisson bracket)}.$$ 

This does not hold. However, something else, possibly just as good, is true.

Instead of a single symplectic form $\omega$, introduce a real parameter $\hbar \in \mathbb{R}$ and for each $\hbar$ a symplectic form $\omega_\hbar = \hbar \omega$. Let $\star_\hbar$ be the corresponding star product on $L^2(V, dv)$. As $\hbar \to 0$, $\star_\hbar$ becomes the standard, commutative, multiplication of functions on $V$ – this is ‘the classical limit’. Moreover, 

$$f_1 \star_\hbar f_2 - f_2 \star_\hbar f_1 = i\hbar[f_1, f_2] + o(\hbar^2).$$

So we do have a quantization to first order in $\hbar$. It is not immediately clear whether the higher-order terms are problematic – maybe our initial criterion for ‘quantization’ was too naive.

**Remark 1.1.** Iain points out that the star product generalises to arbitrary symplectic manifolds, which mathematically, at least, suggests it is a good object to consider. Also, $\star$ is independent of the choice of Lagrangian $L$. (One can see this by first making $L^2(H(V))$ into an algebra, the group algebra of $H(V)$. Then $L^2(V, dv)$ is just the largest quotient of $L^2(H(V))$ on which $U(1)$ acts by scalars.)

### 2. Half-forms and the metaplectic gerbe

The idea of a gerbe is not used in the references I have given, but it is a beautiful and (in some sense) elementary thing, so let’s plunge in.

#### 2.1. **Gerbes.**

Recall that any group $G$ can understood as a category $BG$ with one object whose automorphism group is $G$. A ‘connected $G$-groupoid’ is any category equivalent to $BG$. An analogous idea: a $G$-torsor is any set with a simply transitive action of $G$. (In what follows, I am thinking of $G$ as a discrete group, but there are analogues for Lie groups.)

One can with reasonable clarity say that a covering space of a manifold $M$ is a locally trivial family of sets parameterized by $M$. Similarly, a vector bundle is a locally trivial family of vector spaces. Similarly, a principal $G$-bundle is a locally trivial family of $G$-torsors.

A $G$-gerbe is nothing but a locally trivial family of connected $G$-groupoids. Note that covering spaces, vector bundles, principal bundles, etc., are families of sets, and are classified by first cohomology groups $H^1(M, -)$. A gerbe on the other hand is a family of categories, and we will see that $G$-gerbes are classified by $H^2(M, G)$. 

2.2. Square-roots of line bundles. Suppose we have a complex line bundle $\mathcal{B}$ over a manifold $M$. We would like to find a line bundle $\mathcal{A}$ and an isomorphism $\mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{B}$. In general, $\mathcal{A}$ does not exist. That is: it does not exist globally. The situation is very similar to the familiar one in which the square-root of a given function cannot be defined globally on $M$, but is defined on a double-cover. In our case, $\mathcal{A}$ is defined not on $M$ but on a $\mu_2$-gerbe $\tilde{M}$ over $M$, where $\mu_2 = \{\pm 1\}$. For each $m \in M$, the fibre, the connected $\mu_2$-groupoid $\tilde{M}|_m$, is the category of pairs $(\mathcal{A}_m, \gamma_m)$, where $\mathcal{A}_m$ is a one-dimensional vector space and $\gamma_m$ is an isomorphism $\gamma_m: \mathcal{A}_m \otimes \mathcal{A}_m \rightarrow \mathcal{B}_m$.

Thus a point of $\tilde{M}$ is a triple $(m, \mathcal{A}_m, \gamma_m)$. For each such triple, we have a line $\mathcal{A}_m$, and these form a line bundle $\mathcal{A}$ on $\tilde{M}$. By construction, $\mathcal{A} \otimes \mathcal{A}$ is the pullback of $\mathcal{B}$ to $\tilde{M}$.

Of course, it may happen that there is such an $\mathcal{A}$ on $M$ itself. This happens when $\tilde{M}$ is the trivial $\mu_2$-gerbe, meaning that there is a section $M \rightarrow \tilde{M}$. For example, on $\mathbb{P}^1$, the canonical bundle $O(-2)$ has a square-root $O(-1)$.

2.3. Half forms. In discussing representations of $H(V)$, we used the space $\delta_{1/2}(V/L)$ of half-densities to define the Hilbert space. In general, it works out better to use a space $\Delta_{1/2}(V/L)$ of half-forms. This is a one-dimensional complex vector space with an isomorphism

$$\Delta_{1/2}(V/L) \otimes \Delta_{1/2}(V/L) = \Delta_1(V/L)$$

where now $\Delta_1(V/L) = \wedge^{\dim V/L} \text{Hom}_R(V/L, \mathbb{C})$ is the space of (complex) translation-invariant volume forms on $V/L$. The difficulty is that while $\delta_{1/2}(V/L)$ can be defined canonically in terms of $L$, $\Delta_{1/2}(V/L)$ cannot. Rather, following the above gerbey discussion, one has the following situation.

There is a line bundle $\mathcal{B} = \Delta_1$ on $\Lambda$ with fibre $\Delta_1(V/L)$ over $L$. There is a $\mu_2$-gerbe $\tilde{\Lambda}$ over $\Lambda$ with a line-bundle $\mathcal{A} = \Delta_{1/2}$ such that $\Delta_{1/2} \otimes \Delta_{1/2}$ is the pull-back of $\Delta_1$. By definition, ‘the’ space $\Delta_{1/2}(V/L)$ of half-forms is the fibre $\mathcal{A}_\Lambda$, for some choice of $\tilde{L} \in \tilde{\Lambda}$ over $L \in \Lambda$.

2.4. Metaplectic again. The symplectic group $\text{Sp}(V)$ acts on $\Lambda$ and indeed on the total space of the line-bundle $\Delta_1$. But it does not naturally act on $\tilde{\Lambda}$ or on $\Delta_{1/2}$. Rather, the metaplectic group $\text{Mp}(V)$ acts. In fact, this gives a kind of construction of $\text{Mp}(V)$. We’ll come back to this next time when we talk about half-forms on manifolds.