QFT I: WIGHTMAN AXIOMS

1. Quantum Mechanics and Generalisations

Classic quantum *mechanics* has the following features:

(a) The backdrop is Newtonian spacetime $M \cong \mathbb{R}^3 \times \mathbb{R}$.
(b) The corresponding classical theories describe the motions of a fixed number of particles. Thus a typical classical phase space is T^*S.
(c) Typical observables are the positions/momenta/energies of the particles.

Correspondingly, there are three seemingly independent directions in which one could generalize the ideas of quantum mechanics.

(a) Instead of Newtonian spacetime $S \times \mathbb{R}$, let M be Minkowski space.
(b) Instead of particles, consider *fields* (like the electromagnetic field). Now the phase space is infinite-dimensional.
(c) Instead of global observables like energy, consider local ones like *energy density* or *the energy in this room*.

It turns out that (up to some fudge) all three directions lead to similar ideas, which constitute *quantum field theory*. Thus QFT is supposed to be about (1) relativistic particle physics; (2) quantization of classical field theories; (3) algebras of local observables.

This talk will mainly be about aspect (2), as codified by the Wightman axioms (1950s). A future talk will be about (1), a point of view which is emphasized (for example) by Weinberg's textbook. The 'algebraic' approach to QFT, (3), is perhaps the most ambitious in trying to lay foundations; see Haag, *Local Quantum Physics*.

Most textbooks are at least nominally interested in some amalgam of (1) and (2). However, there is a disconnect between their pragmatic 'Lagrangian' methods and the kind of axioms we give here, as evidenced by the lack of interesting models on the axiomatic side. A new approach (I have seen but don't yet know much about) is described in a book by Kevin Costello, giving an axiomatic treatment of some aspects of Lagrangian QFT, especially renormalization.

2. Wightman Axioms, Part I

2.1. **Hilbert Space.** We are going to describe axioms for a quantum theory. A general part of the structure is

Axiom 1 We are given a Hilbert space $\mathcal{H}$, a dense subspace $\mathcal{D} \subset \mathcal{H}$, and a 'vacuum' vector $\Omega \in \mathcal{D}$.

Specifying $\mathcal{D}$ is a purely technical crutch. For example, in quantum mechanics, if $\mathcal{H} = L^2(\mathbb{R})$, then we want to consider an operator $i\partial$ (the momentum operator). But you can't take derivatives of arbitrary L^2 functions. Nonetheless, ∂ is defined on a dense subspace of $\mathcal{H}$ (e.g. the Schwartz space), and that's good enough. We want to have available a single domain $\mathcal{D}$ on which all our operators are defined.

2.2. **Fields.** A *field* is just a function $\phi\colon M \to V$, where M is our spacetime manifold and V a vector space. Of course, we may wish to think about smooth functions, etc; or allow for distributions instead of functions.

Axiom 2 We are given a 'spacetime' manifold M and a vector space V.

2.3. **Observables.** A typical observable is 'the value of ϕ at $x \in M$'. However, everything will work out better if we look instead at average values of ϕ in small regions. (The usual heuristic is that measuring the value of a field at a single point would require infinitely much energy.)

Thus let $\mathcal{S}$ be the space of Schwartz functions $f\colon M \to V^*$. The observables we consider are ones of the form

$$\mathcal{O}_f(\phi) = \int_M \langle f, \phi \rangle \,.$$

Here $\langle f, \phi \rangle$ is the measure on M one obtains by contracting V^* (the values of f) with V (the values of ϕ).[1]

Axiom 3 We are given a 'quantization' map $Q\colon \mathcal{S} \to \mathrm{End}(\mathcal{D})$. The resulting operators on $\mathcal{D}$ should be Hermitian. Q should be continuous in some appropriate topology (more on this later).

2.4. **Locality.** The quantized observables $Q(f)$ have one more property, which is a kind of 'locality'. To understand this, let us (finally) take M to be Minkowski space. Then we know what it means for two small regions in M to be *causally separated* (i.e. spacelike separated). The naive version of the locality axiom is that if $f_1, f_2 \in \mathcal{S}$ have causally separated supports, then $[Q(f_1), Q(f_2)] = 0$. This means we can simultaneously diagonalize $Q(f_1)$ and $Q(f_2)$; roughly speaking, it means that the field can have well-defined values over an entire spacelike surface, and spatially separated observers can make independent measurements of these values. Thus interpreting $Q(f_1)$ and $Q(f_2)$ as *observables* is compatible with the causal structure of spacetime. However, the real axiom looks like this.

Axiom 4 We are given a decomposition $V = V_0 \oplus V_1$. If $f_1, f_2 \in \mathcal{S}$ have causally separated supports, and f_1 has values in V_m and f_2 in V_n, then

$$Q(f_1)Q(f_2) = (-1)^{mn} Q(f_2)Q(f_1).$$

Mathematically, this is not so awful; it is the natural '$\mathbb{Z}_2$-graded' version of our naive locality axiom. However, physically, it means that if f_1, f_2 have values in V_1 then the 'observables' $Q(f_i)$ must not be truly observable, at the risk of allowing interference between causally separated observations. However, certain *combinations* of these 'observables' may still be directly measurable.

The fields with values in V_0 are called 'bosonic' and those with values in V_1 are 'fermionic'. For example, the electromagnetic field is bosonic (and observable) while the field associated to electrons is fermionic (and unobservable). However, electrons obviously have physical effects even if the fields themselves are not observable.

Part of the ambition of Haag's algebraic approach to QFT is to eliminate fermionic fields as fundamental objects, so that only the naive locality axiom is really in play.

2.5. **Quantum fields and Correlation functions.** The condition that Q is continuous is designed to have the following implications. First, Q should be given by a distribution $\Psi\colon M \to V \otimes \mathrm{End}(\mathcal{D})$, i.e.

$$Q(f) = \int_M \langle f, \Psi \rangle \,.$$

Ψ is 'the quantum field' one finds in textbooks. Second, and more importantly, for each $n \geq 0$ the functional on $\mathcal{S}^n$ defined by

$$(f_1, \ldots, f_n) \mapsto \langle \Omega, Q(f_1) \cdots Q(f_n)\Omega \rangle$$

should be represented by a distribution $W_n\colon M^n \to V^n$.

These distributions are called *Wightman* or *correlation* functions. They determine the QFT $(\mathcal{H}, \mathcal{D}, \Omega, Q)$ (for fixed M and V). The idea is to consider the vector

[1] Thus to make make the units work out, we should assume that fields ϕ are measures rather than functions; or we could just choose a reference measure on M.

space spanned by symbols "$Q(f_1)\cdots Q(f_n)\Omega$". The correlation functions specify the inner products between such symbols, and from this we can reconstruct the Hilbert space.

The calculation of these correlation functions (and other closely-related quantities) is essentially what textbook QFT is all about. Even when working in our axiomatic framework, one typically proves theorems by dealing with the correlation functions rather than the quantization map.

3. Symmetry

The remaining Wightman axioms will state that QFTs are invariant under the Poincaré group. Here I lay down some motivating ideas.

A symmetry is just an automorphism of the data $(\mathcal{H}, M, V, \mathcal{D}, \Omega, Q)$, in other words, a triple of maps $(u\colon H \to H, u_M\colon M \to M, u_V\colon V \to V)$ such that u preserves Ω and $\mathcal{D}$, and such that

$$uQ(f)u^{-1} = Q(u_V \circ f \circ u_M^{-1}).$$

Remark 3.1. It would alternatively be natural to consider automorphisms of the vector bundle $V \times M \to M$, rather than separate mapst u_M and u_V. This would lead to a more general notion of 'symmetry'.

What kind of map should u be? Naively, it should be unitary. But physical reasoning demands only that it should preserve the set of rays ('states') in $\mathcal{H}$, and it should preserve the amplitudes

$$\|v, w\| = \frac{\langle v, w\rangle^2}{\langle v, v\rangle\,\langle w, w\rangle} \in [0, 1]$$

for $v, w \in \mathcal{H}$, which are interpreted as transition probabilities. (Note that $\|v, w\|$ depends only on the rays through v and w). A theorem of Wigner says that a map of rays preserving probabilities can be realized as a real-linear map that is *either* unitary (and complex-linear) or anti-unitary (and anti-linear); moreover, this realisation is unique up to a phase. Thus in the definition of a 'symmetry' we allow that u is either unitary or anti-unitary.

4. Wightman Axioms, continued

Now we take M to be Minkowski space. Its isometry group is the Poincaré group P. It has a normal subgroup M (acting by translations) and the quotient is the Lorentz group L. In this lecture we take P to be connected (i.e. we only consider the connected component of the full isometry group).

P is not simply connected; its universal cover $\tilde{P} = M \rtimes \tilde{L}$ is a double cover. We are going to consider QFTs with $\tilde{P}$ acting by symmetries.

Remark 4.1. Suppose we had an action of P on the set of rays in $\mathcal{H}$, preserving the probability amplitudes. Then Wigner's theorem gives an operator $u(p)$ for each $p \in P$, unique *up to phase.* This phase ambiguity means that we get a representation of $\tilde{P}$ (not P in general) on $\mathcal{H}$. This is one reason for considering $\tilde{P}$ instead of P. Note that the representation of $\tilde{P}$ would be unitary, because $1 \in \tilde{P}$ acts unitarily (by the identity) and $\tilde{P}$ is connected.

Note that a representation of $\tilde{L}$ (hence of $\tilde{P}$) on V makes $\tilde{P}$ act on $\mathcal{S}$ by $(gf)(x) = g(f(g^{-1}x))$ for $g \in \tilde{P}$, $f \in \mathcal{S}$, $x \in M$.

Axiom 5 We have a representation of $\tilde{L}$ on V and a unitary representation of $\tilde{P}$ on $\mathcal{H}$ such that Q is $\tilde{P}$-equivariant. Moreover, $\tilde{P}$ fixes Ω and preserves $\mathcal{D} \subset \mathcal{H}$.

Sometimes additional things are assumed, e.g. that Ω is the unique $\tilde{P}$-fixed vector.

4.1. **Positivity.** There is one last condition on the type of representation of $\tilde{P}$ on H, corresponding to the idea that particles have future-pointing trajectories and positive energies. $\tilde{P}$ contains the additive group M as a normal subgroup. An irreducible representation of M is the same as a element of the dual M^* (that is, $\lambda \in M^*$ determines a representation $x \mapsto e^{i\lambda(x)}$). And we can identify M^* with M using the inner product on M. Thus, the *spectrum* of M acting on $\mathcal{H}$ is a subset of M – the set of irreducible representations of M occuring as subquotients of $\mathcal{H}$.

Axiom 6 The spectrum of M acting on $\mathcal{H}$ is contained in the future-pointing light cone.

(Note this axiom requires not only the inner product on M, but also a temporal orientation.)