CHARACTERISTIC CYCLES OF THETA SHEAVES

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Abstract. We extend from characteristic p to characteristic zero S. Lysenko's theory of theta sheaves on the moduli stack of metaplectic bundles. The main tool is a 'Fourier transform' for semi-homogeneous sheaves on vector bundles. We then calculate the characteristic cycles of the theta sheaves, showing that they lie in a small part of the global nilpotent cone.

1. Introduction

1.1. Let K be a global field, $\mathfrak{A}$ its ring of adèles, and $\mathfrak{O} \subset \mathfrak{A}$ the ring of integral adèles. The adelic symplectic group $\mathrm{Sp}_{2n}(\mathfrak{A})$ has a double cover, the adelic metaplectic group $\mathrm{Mp}_{2n}(\mathfrak{A})$. Following Sergey Lysenko [Ly], we study a geometric analogue of the classical theta series, which can be understood as a function on $\mathrm{Mp}_{2n}(\mathfrak{A})$. This function descends to a double coset space

$$\mathrm{Sp}_{2n}(K)\backslash \mathrm{Mp}_{2n}(\mathfrak{A})/\mathrm{Sp}_{2n}(\mathfrak{O}).$$

When K is the function field of an algebraic curve X over a finite field $\mathbb{F}_q$, Lysenko interpreted such a double coset space as the $\mathbb{F}_q$ points of an algebraic stack $\mathrm{Bun}_{\mathrm{Mp}}$ of "metaplectic bundles over X," and explained that the theta function arises "geometrically," i.e. as the function corresponding to a certain perverse sheaf Θ on $\mathrm{Bun}_{\mathrm{Mp}}$ under Grothendieck's sheaf-function dictionary (see, for example, [La2]). More precisely, $\Theta = \Theta_+ \oplus \Theta_-$ is the direct sum of two irreducible perverse sheaves.

1.2. The first purpose of this paper is to extend Lysenko's study of $\Theta_\pm$ to the case of curves X in characteristic zero. The main work here is to replace the Fourier-Deligne transform, which he uses, by another construction, the "semi-homogeneous Fourier-transform," which coincides (up to a twist) with Fourier-Deligne in positive characteristic. The theory of this semi-homogeneous Fourier transform is developed in §§3–4; Theorem 4.3 states its connection to Fourier-Deligne. It is also closely related to the homogeneous Fourier transform of Laumon [La3].

The definition of $\Theta_\pm$ is finally given in §5, and in §6 we give an alternative description of $\Theta_\pm$ as a Goresky-MacPherson extension.

1.3. The second goal is to compute the characteristic cycle of $\Theta_\pm$, a problem that makes sense only in characteristic zero. Such a characteristic cycle is a conic Lagrangian cycle in the cotangent stack $T^*\mathrm{Bun}_{\mathrm{Mp}}$. In §7, we very briefly recall the theory of characteristic cycles, and then describe how they transform under semi-homogeneous Fourier (following [Br] and and the appendix to [La1]).

1.3.1. In order to state the main result, let us recall the definition of the stack $\mathrm{Bun}_{\mathrm{Mp}}$, and describe its cotangent stack. For any test scheme S, let p_{X_S} be the projection $p_{X_S}\colon X_S := X \times S \to S$, and set $\Omega := \Omega^1_{X_S/S}$, $\mathscr{O} := \mathscr{O}_{X_S}$. If $\mathscr{L}$ is a vector bundle on X_S set $\mathscr{L}^\vee = \mathscr{L}^* \otimes \Omega$.

Define $\mathrm{Bun}_{\mathrm{Sp}}$, the moduli stack of rank-$2n$ symplectic bundles on X, in such a way that $\mathrm{Bun}_{\mathrm{Sp}}(S)$ is the groupoid of pairs $(\mathscr{V}, b_{\mathscr{V}})$, with $\mathscr{V}$ a rank-$2n$ vector bundle on X_S, and $b_{\mathscr{V}}\colon \mathscr{V} \to \mathscr{V}^\vee$ a skew-symmetric isomorphism. We often omit $b_{\mathscr{V}}$ from the notation.

The cotangent stack $T^*\mathrm{Bun}_{\mathrm{Sp}}$ is the stack whose S-points are pairs $(\mathscr{V}, \eta)$ with $\mathscr{V} \in \mathrm{Bun}_{\mathrm{Sp}}(S)$, and η a symmetric homomorphism $\eta\colon \mathscr{V}^* \to \mathscr{V}$ (see e.g. [Gi2]).

The stack $\mathrm{Bun}_{\mathrm{Mp}}$ of metaplectic bundles is defined so that $\mathrm{Bun}_{\mathrm{Mp}}(S)$ is the groupoid of data $(\mathscr{V}, \mathscr{D}, \delta)$ with $\mathscr{V} \in \mathrm{Bun}_{\mathrm{Sp}}(S)$, $\mathscr{D}$ a line bundle on S, and $\delta\colon \mathscr{D} \otimes \mathscr{D} \to \det Rp_{X_S,*}\mathscr{V}$ an isomorphism. In other words, writing $\tilde{\mathscr{V}}$ for the universal bundle over $X_{\mathrm{Bun}_{\mathrm{Sp}}}$, $\mathrm{Bun}_{\mathrm{Mp}}$ is the gerbe over $\mathrm{Bun}_{\mathrm{Sp}}$ parameterising square-roots of the line bundle $\det Rp_{X_{\mathrm{Bun}_{\mathrm{Sp}}},*}\tilde{\mathscr{V}}$.

The cotangent stack $T^*\mathrm{Bun}_{\mathrm{Mp}}$ therefore parameterises data $(\mathscr{V}, \mathscr{D}, \delta, \eta)$ with η a symmetric homomorphism $\eta\colon \mathscr{V}^* \to \mathscr{V}$.

1.3.2. Let $\mathbb{M}$ be the stack such that $\mathbb{M}(S)$ is the groupoid of data $(\mathscr{V}, \mathscr{D}, \delta, \phi)$, consisting of an object of $(\mathscr{V}, \mathscr{D}, \delta) \in \mathrm{Bun}_{\mathrm{Mp}}(S)$ and a homomorphism $\phi \in \mathrm{Hom}(\mathscr{V}, \Omega)$, with $\phi \neq 0$. Define a morphism $s\colon \mathbb{M} \to T^*\mathrm{Bun}_{\mathrm{Mp}}$ by

$$s(\mathscr{V}, \mathscr{D}, \delta, \phi) = (\mathscr{V}, \mathscr{D}, \delta, \eta := (b_{\mathscr{V}})^{-1} \circ \phi^* \circ \phi \circ (b_{\mathscr{V}})^{-1}).$$

Theorem 1.1. *The characteristic cycles of Θ_- and Θ_+ are given by*

$$\mathrm{CC}(\Theta_-) = [\overline{\mathrm{Im}\, s}] \qquad \mathrm{CC}(\Theta_+) = [\overline{\mathrm{Im}\, s}] + [\mathrm{Bun}_{\mathrm{Mp}}].$$

Here $[\mathrm{Bun}_{\mathrm{Mp}}]$ is the cycle corresponding to the zero section of $T^*\mathrm{Bun}_{\mathrm{Mp}}$, counted with multiplicity one. Similarly, $\overline{\mathrm{Im}\, s}$ is the closure of the image of s in $T^*\mathrm{Bun}_{\mathrm{Mp}}$, and $[\overline{\mathrm{Im}\, s}]$ is the corresponding cycle.

Remark 1.1. It follows that $\mathrm{CC}(\Theta_\epsilon)$ is supported inside the global nilpotent cone, which, according to a theorem of Faltings and Laumon, is a Lagrangian substack of $T^*\mathrm{Bun}_{\mathrm{Mp}}$ (see [Gi2]); moreover, the component $[\overline{\mathrm{Im}\, s}]$ corresponds to the minimal coadjoint orbit of Sp_{2n}.

1.3.3. In §8 we describe certain cotangent stacks used in the proof of Theorem 1.1, and then complete the proof in §9.

1.3.4. *Remark.* It would be possible to define theta sheaves in characteristic zero using Laumon's homogeneous Fourier transform instead of our semi-homogeneous one. To do this, one has first to use the homogenous sheaf f_0 in (6) instead of f. Second, we note that in the definition of Θ_ϵ we consider Lagrangian subbundles whose Euler characteristic has parity $-\epsilon$ (see §5.1.1); in the homogeneous case, one should use parity ϵ instead. With these changes, the resulting 'theta sheaves' would differ from ours only by a 'constant factor,' that is, by a local system on $\mathrm{Spec}(k)$.

The reasons that we have chosen the current emphasis on semi-homogeneous sheaves are, first, that the semi-homogeneous Fourier transform is technically simpler; and second, that we are interested in the following example. Suppose that $\mathscr{A}, \mathscr{B}, \mathscr{C}$ are rank-2 vector bundles on X, such that $\det \mathscr{A} \otimes \det \mathscr{B} \otimes \det \mathscr{C} = \Omega^1_X$. Then $\mathscr{V} := \mathscr{A} \otimes \mathscr{B} \otimes \mathscr{C}$ is naturally a rank-8 symplectic, in fact, metaplectic bundle. For any line subbundle $\ell \subset \mathscr{A}$, the Lagrangian subbundle $\ell \otimes \mathscr{B} \otimes \mathscr{C} \subset \mathscr{V}$ has even Euler characteristic. Thus, to study the restriction of Θ_- to metaplectic bundles of this form, it is most convenient to use the semi-homogeneous construction.

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2. General Notation

We collect here some general notation to be used throughout.

2.1. All stacks are Artin stacks, locally of finite type, over a fixed ground field k of characteristic not 2. For any such stack $\mathbb{X}$ let $D(\mathbb{X})$ be the bounded derived category of complexes of ℓ-adic sheaves on $\mathbb{X}$ with constructible cohomology; here ℓ is a prime different from the characteristic of k. Let $\mathfrak{D}$ be the Verdier duality operator, $\mathfrak{D}\colon D(\mathbb{X}) \to D(\mathbb{X})^{\mathrm{op}}$. Choose a square-root $\bar{\mathbb{Q}}_\ell(\frac{1}{2})$ of $\bar{\mathbb{Q}}_\ell(1)$, and for any $f \in D(\mathbb{X})$ and integer r, define $f\langle r\rangle := f(\frac{r}{2})[r]$.

2.2. If $\mathscr{L}$ is (the coherent sheaf of sections of) a vector bundle and similarly $\mathscr{A}$ a line bundle, write

$$\mathscr{H}om(\mathscr{L}, \mathscr{L}^* \otimes \mathscr{A}) = \mathscr{H}om_{\mathrm{sym}}(\mathscr{L}, \mathscr{L}^* \otimes \mathscr{A}) \oplus \mathscr{H}om_{\mathrm{sk}}(\mathscr{L}, \mathscr{L}^* \otimes \mathscr{A})$$

for the decomposition into sheaves of symmetric and skew-symmetric homomorphisms, and similarly for the global sections $\mathrm{Hom}_{\mathrm{sym}}(\mathscr{L}, \mathscr{L}^* \otimes \mathscr{A})$, etc.

2.3. For $\mathbb{V}$ (the total space of) a vector bundle over a stack $\mathbb{B}$, set $\mathbb{V}^\circ := \mathbb{V} - \mathbb{B}$, where $\mathbb{B}$ is identified with the zero section; $\bar{\mathbb{V}} := \mathbb{V}/\mathbb{G}_m$, the quotient stack ($\mathbb{G}_m$ acts by homotheties); and denote the canonical maps as in the following commutative diagram.

$$\begin{tikzcd}
\mathbb{B} \arrow[r, "i_{\mathbb{V}}"'] \arrow[dd, "\rho_{\mathbb{B}}"'] & \mathbb{V} \arrow[dr, "\pi_{\mathbb{V}}"'] \arrow[dd, "\rho_{\mathbb{V}}"'] & & \mathbb{V}^\circ \arrow[ll, "j_{\mathbb{V}}"'] \arrow[dl, "\pi^\circ_{\mathbb{V}}"] \arrow[dd, "\rho^\circ_{\mathbb{V}}"] \\
& & \mathbb{B} & \\
\bar{\mathbb{B}} \arrow[r, "\bar{i}_{\mathbb{V}}"'] & \bar{\mathbb{V}} \arrow[ur, "\bar{\pi}_{\mathbb{V}}"'] & & \mathbb{P}\mathbb{V} \arrow[ll, "\bar{j}_{\mathbb{V}}"] \arrow[ul, "\bar{\pi}^\circ_{\mathbb{V}}"']
\end{tikzcd}$$

Let $\mathbb{V}^*$ be the vector bundle dual to $\mathbb{V}$, $p_{\mathbb{V}}$, $p_{\mathbb{V}^*}$ the projections of $\mathbb{V} \times_{\mathbb{B}} \mathbb{V}^*$ to $\mathbb{V}$ and $\mathbb{V}^*$. Let $\mathrm{ev}_{\mathbb{V}}\colon \mathbb{V} \times_{\mathbb{B}} \mathbb{V}^* \to \mathbb{A}^1$ be the evaluation map.

3. The Semi-Homogeneous Fourier Transform

The semi-homogeneous Fourier transform is a functor between certain categories of semi-homogeneous complexes, defined in §3.1 (the terminology is explained in Remark 3.1.

I define the Fourier functor in §3.3.1 and will explain its basic properties in §4.

3.1. Semi-Homogeneous Complexes. Let μ_N be the etale sheaf on $\mathrm{Spec}(k)$ of N-th roots of unity (we will chiefly be interested in the case $N = 2$). Suppose given a character $\chi\colon \mu_N \to \bar{\mathbb{Q}}_\ell^\times = \mathrm{Aut}(\bar{\mathbb{Q}}_\ell)$, that is, a homomorphism of sheaves of groups. Let χ^* denote the dual character: $\chi^*(g) = \chi(g^{-1})$.

Let $a_{\mathbb{X}}\colon \mathbb{G}_m \times \mathbb{X} \to \mathbb{X}$ be an action of the multiplicative group on a stack $\mathbb{X}$. (Recall that stacks form a 2-category, and so the action of $\mathbb{G}_m$ on $\mathbb{X}$ actually involves, besides $a_{\mathbb{X}}$, specified 2-morphisms giving the associativity and unit structure; see [Ly, Appendix A].) Let $a_{\mathbb{X}}^{(N)}$ be the action

$$a_{\mathbb{X}}^{(N)}(g, x) := a_{\mathbb{X}}(g^N, x)$$

of $\mathbb{G}_m$ on $\mathbb{X}$ and let $\mathbb{X}_{(N)}$ be the corresponding stack quotient. Since, under $a_{\mathbb{X}}^{(N)}$, $\mu_N \subset \mathbb{G}_m$ acts trivially, μ_N acts on each object of $D(\mathbb{X}_{(N)})$.

Definition 3.1. Denote by $D(\mathbb{X})_\chi$ the full subcategory of $D(\mathbb{X}_{(N)})$ made up of those objects on which μ_N acts through χ. I call any object of $D(\mathbb{X})_\chi$ a "semi-homogeneous complex on $\mathbb{X}$," or "χ-homogeneous" when it is important to remember χ.

Remark 3.1. For a *function* f on a vector space V to be "homogeneous" would be for it to transform by some character χ_0 of the multiplicative group: $f(ax) = \chi_0(a)f(x)$. Letting f stand for an ℓ-adic complex on V instead of a function, the obvious analogy would have χ_0 as a character sheaf (an ℓ-adic local system on $\mathbb{G}_m$ with some additional structure). The character χ associates to the Kummer sequence

$$1 \longrightarrow \mu_N \longrightarrow \mathbb{G}_m \xrightarrow{x \mapsto x^N} \mathbb{G}_m \longrightarrow 1$$

such a local system χ_0 (see [De, §1.3, 4.7]). Conversely, any character sheaf χ_0 of finite order arises from some χ. We call our complexes "semi-homogeneous" because Laumon in [La3] uses "homogeneous" in the case when χ is trivial.

Remark 3.2. Suppose that χ_1 is a character of μ_{N_1}, and N_1 divides N_2; then we obtain a character $\chi_2(x) := \chi_1(x^{N_2/N_1})$ of μ_{N_2}. There is, moreover, a canonical equivalence of categories $D(\mathbb{X})_{\chi_1} \to D(\mathbb{X})_{\chi_2}$.

Remark 3.3. Suppose that $\mathbb{X} \to \mathbb{B}$ is a vector bundle, with $\mathbb{G}_m$ acting by homotheties (this is the main case of interest). Then $\mathbb{X}_{(1)}(S)$ is the groupoid of triples $(\mathscr{A}, s, a)$ with $\mathscr{A}$ a line-bundle on S, s an S-point of $\mathbb{B}$, and $a\colon \mathscr{A} \to \mathscr{O}(s^*\mathbb{X})$ a map of locally free sheaves (i.e. here $\mathscr{O}(s^*\mathbb{X})$ means the sheaf of sections of the pulled-back vector bundle $s^*\mathbb{X}$ over S). Thus there is a universal line-bundle $\tilde{\mathscr{A}}$ over $\mathbb{X}_{(1)}$. In these terms, $\mathbb{X}_{(n)}$ is the μ_n-gerbe over $\mathbb{X}_{(1)}$ parametrising nth roots of $\tilde{\mathscr{A}}$.

3.2. Duality and Functoriality. We will often write as if a χ-homogeneous sheaf $f \in D(\mathbb{X})_\chi$ is an object on $\mathbb{X}$ rather than $\mathbb{X}_{(N)}$. For example, if $F\colon \mathbb{X} \to \mathbb{Y}$ is a $\mathbb{G}_m$-equivariant map, then it induces also a map $F_{(N)}\colon \mathbb{X}_{(N)} \to \mathbb{Y}_{(N)}$, and we write things like $F_!(f)$ when we mean $F_{(N),!}(f)$. Bearing this in mind, the functors $F_!, F_*, F^*, F^!$ take χ-homogeneous complexes to naturally χ-homogeneous complexes, and if f is χ-homogeneous, then $\mathfrak{D}f$ is naturally χ^*-homogeneous.

We will use the following lemma repeatedly in calculations:

Lemma 3.1. *Suppose given $f \in D(\mathbb{X})_\chi$, with χ non-trivial. Let $\mathbb{G}_m$ act trivially on $\mathbb{X}'$, and suppose given a $\mathbb{G}_m$-equivariant map $F\colon \mathbb{X}' \to \mathbb{X}$. Then $F^*f = F^!f = 0$. Similarly, given equivariant $F\colon \mathbb{X} \to \mathbb{X}'$, we have $F_*f = F_!f = 0$.*

Proof. It suffices to show that a χ-homogeneous complex on $\mathbb{X}'$ is zero. Since the actions $a_{\mathbb{X}'}$ and $a^{(N)}_{\mathbb{X}'}$ are equal, μ_N acts trivially on every object of $D(\mathbb{X}'_{(N)})$. But we assumed that χ was non-trivial. □

3.3. The Fourier Transform. Let $a_{\mathbb{V}}\colon \mathbb{G}_m \times \mathbb{V} \to \mathbb{V}$ be the action of $\mathbb{G}_m$ on a vector bundle $\mathbb{V}$ by homotheties, and define

$$\mathbb{I} = \mathbb{I}_{\mathbb{V}} := \mathrm{ev}_{\mathbb{V}}^{-1}(1) \subset \mathbb{V} \times_{\mathbb{B}} \mathbb{V}^*.$$

(See §2.3 for notation.) The action of $\mathbb{G}_m$ on $\mathbb{V}^*$ by $(g, x) \mapsto g^{-1}x$ makes $p^{\mathbb{I}}_{\mathbb{V}^*}$ and $p^{\mathbb{I}}_{\mathbb{V}}$ into $\mathbb{G}_m$-equivariant maps.

3.3.1. *Definition.* Suppose that χ is non-trivial. Define a functor $\mathfrak{F}_{\mathbb{V}}\colon D(\mathbb{V})_\chi \to D(\mathbb{V}^*)_{\chi^*}$ by

$$\mathfrak{F}_{\mathbb{V}} = (p^{\mathbb{I}}_{\mathbb{V}^*})_! \circ (p^{\mathbb{I}}_{\mathbb{V}})^* \circ \langle r-1 \rangle$$

where $p^{\mathbb{I}}_{\mathbb{V}^*}, p^{\mathbb{I}}_{\mathbb{V}}$ are the projections of $\mathbb{I}$ to $\mathbb{V}^*$ and $\mathbb{V}$. We call $\mathfrak{F}_{\mathbb{V}}$ the χ-homogeneous Fourier transform.

Remark 3.4. Laumon [La3] considered a similar construction when χ is trivial, i.e. he considered $\mathbb{G}_m$-equivariant sheaves on $\mathbb{V}$. In that case our definition is not appropriate (the resulting functor does not agree with Fourier-Deligne, and is not even an equivalence).

4. Properties of the Fourier Transform

Theorem 4.1.

(a) Involutivity. *There is a canonical isomorphism of functors*

$$\mathfrak{F}_{\mathbb{V}^*} \circ \mathfrak{F}_{\mathbb{V}} \cong \mathrm{id}\colon D(\mathbb{V})_\chi \to D(\mathbb{V})_\chi .$$

(b) Duality. *There is a canonical isomorphism of functors*

$$\mathfrak{F}_{\mathbb{V}} \circ \mathfrak{D} \cong \mathfrak{D} \circ \mathfrak{F}_{\mathbb{V}}\colon D(\mathbb{V})_\chi \to D(\mathbb{V}^*)_{\chi^*} .$$

(c) Exactness. *If $f \in D(\mathbb{V})$ is perverse, then so is $\mathfrak{F}_{\mathbb{V}} f$.*

Proof. **(a)** By Lemma 3.1, we have $i_{\mathbb{V}}^* f = 0 = i_{\mathbb{V}^*}^* \mathfrak{F}_{\mathbb{V}} f$ in $D(\mathbb{B})$, so it suffices to compare $f' := j_{\mathbb{V}}^* f$ with $j_{\mathbb{V}}^* \mathfrak{F}_{\mathbb{V}^*} \mathfrak{F}_{\mathbb{V}} f$. Let

$$h\colon \mathbb{I} \times_{\mathbb{V}^*} \mathbb{I} \to \mathbb{V}^\circ \times_{\mathbb{B}} \mathbb{V}^\circ$$

be the natural map, and p_1, p_2 the projections of $\mathbb{V}^\circ \times_{\mathbb{B}} \mathbb{V}^\circ$ to $\mathbb{V}^\circ$. By base change and the projection formula we have canonical isomorphisms

$$j_{\mathbb{V}}^* \mathfrak{F}_{\mathbb{V}^*} \mathfrak{F}_{\mathbb{V}} f = (p_2 \circ h)_!(p_1 \circ h)^* f' \langle 2r-2 \rangle = p_{2,!}(h_! \bar{\mathbb{Q}}_\ell \otimes p_1^* f') \langle 2r-2 \rangle .$$

To calculate the right-hand side, we consider the closed embedding $i\colon \mathbb{V}^\circ \times_{\mathbb{P}\mathbb{V}} \mathbb{V}^\circ \to \mathbb{V}^\circ \times_{\mathbb{B}} \mathbb{V}^\circ$ and the complementary open embedding j. It suffices to describe canonical isomorphisms

$$p_{2,!} i_! i^* (h_! \bar{\mathbb{Q}}_\ell \otimes p_1^* f') = f' \langle 2-2r \rangle \tag{1}$$

$$p_{2,!} j_! j^* (h_! \bar{\mathbb{Q}}_\ell \otimes p_1^* f') = 0. \tag{2}$$

To prove (1), consider the cartesian square

$$\begin{tikzcd} \mathbb{I} \arrow[r, "\tilde{\imath}"] \arrow[d, "\Delta \circ p^{\mathbb{I}}_{\mathbb{V}^\circ}"] & \mathbb{I} \times_{\mathbb{V}^*} \mathbb{I} \arrow[d, "h"] \\ \mathbb{V}^\circ \times_{\mathbb{P}\mathbb{V}} \mathbb{V}^\circ \arrow[r, "i"] & \mathbb{V}^\circ \times_{\mathbb{B}} \mathbb{V}^\circ \end{tikzcd} .$$

where Δ and $\tilde{\imath}$ are diagonals. Since $p^{\mathbb{I}}_{\mathbb{V}^\circ}$ makes $\mathbb{I}$ into an affine $\mathbb{A}^{r-1}$-bundle over $\mathbb{V}^\circ$, we find by base change

$$i^* h_! \bar{\mathbb{Q}}_\ell = \Delta_! (p^{\mathbb{I}}_{\mathbb{V}^\circ})_! \bar{\mathbb{Q}}_\ell = \Delta_! \bar{\mathbb{Q}}_\ell \langle 2-2r \rangle .$$

Therefore the projection formula gives

$$p_{2!} i_! i^* (h_! \bar{\mathbb{Q}}_\ell \otimes p_1^* f') = p_{2!}(\Delta_! \bar{\mathbb{Q}}_\ell \otimes i^* p_1^* f') \langle 2-2r \rangle = f' \langle 2-2r \rangle$$

as desired.

For (2), let $\tilde{\jmath}$ be the open complement to $\tilde{\imath}$, so that

$$p_{2!} j_! j^* (h_! \bar{\mathbb{Q}}_\ell \otimes p_1^* f') = p_{2!} \left(h_! \tilde{\jmath}_! \tilde{\jmath}^* \bar{\mathbb{Q}}_\ell \otimes j_! j^* p_1^* f' \right) .$$

Now $h \circ \tilde{\jmath}$ is an affine $\mathbb{A}^{r-2}$-bundle over the image of j. Thus $h_! \tilde{\jmath}_! \tilde{\jmath}^* \bar{\mathbb{Q}}_\ell = j_! \bar{\mathbb{Q}}_\ell \langle 4-2r \rangle$ and

$$p_{2!} \left(h_! \tilde{\jmath}_! \tilde{\jmath}^* \bar{\mathbb{Q}}_\ell \otimes j_! j^* p_1^* f' \right) = p_{2!} j_! j^* p_1^* f' \langle 4-2r \rangle .$$

Finally, $j_! j^* p_1^* f'$ is χ-homogeneous with respect to the action $(\mathrm{id} \times a_{\mathbb{V}})$ on $\mathbb{V}^\circ \times_{\mathbb{B}} \mathbb{V}^\circ$. We conclude from Lemma 3.1 that $p_{2!} j_! j^* p_1^* f' = 0$, as desired.

(b) The isomorphism $\mathfrak{D} \circ \mathfrak{F}_{\mathbb{V}} \circ \mathfrak{D} = \mathfrak{F}_{\mathbb{V}}$ arises from the fact that both $\mathfrak{D} \circ \mathfrak{F}_{\mathbb{V}} \circ \mathfrak{D}$ and $\mathfrak{F}_{\mathbb{V}}$ are right-adjoint to $\mathfrak{F}_{\mathbb{V}^*}$ (cf. the proof of Laumon's Theorem 4.1 [La3]).

(c) $\mathbb{I}$ is an affine $\mathbb{A}^{r-1}$-bundle over both $\mathbb{V}^\circ$ and $\mathbb{V}^{*\circ}$. Thus if f is perverse then $(p^{\mathbb{I}}_{\mathbb{V}})^* f \langle r-1 \rangle$ is perverse, and by [BBD] 4.1.2, it follows that $\mathfrak{F}_{\mathbb{V}} f$ lies in positive perverse degrees. But we also have $\mathfrak{D}\mathfrak{F}_{\mathbb{V}} f = \mathfrak{F}_{\mathbb{V}} \mathfrak{D} f$, so $\mathfrak{D}\mathfrak{F}_{\mathbb{V}} f$ lies in positive degrees as well. Therefore $\mathfrak{F}_{\mathbb{V}} f$ is perverse, as desired. $\square$

Theorem 4.2. *Suppose $F\colon \mathbb{W} \to \mathbb{V}$ is a linear map of vector bundles. There are canonical isomorphisms of functors*

$$\mathfrak{F}_{\mathbb{V}} \circ F_! \cong G^* \circ \mathfrak{F}_{\mathbb{W}} \quad \mathfrak{F}_{\mathbb{V}} \circ F_* \cong G^! \circ \mathfrak{F}_{\mathbb{W}} \quad \mathfrak{F}_{\mathbb{W}} \circ F^! \cong G_* \circ \mathfrak{F}_{\mathbb{V}} \quad \mathfrak{F}_{\mathbb{W}} \circ F^* \cong G_! \circ \mathfrak{F}_{\mathbb{V}}$$

where $G\colon \mathbb{V}^* \to \mathbb{W}^*$ is the adjoint of F.

Proof. The four isomorphisms are related by applications of $\mathfrak{D}$ and $\mathfrak{F}$, so it suffices to explain the first one. By base change, $\mathfrak{F}_{\mathbb{V}} \circ F_! \cong A_! \circ B^*$ where A, B are the projections of $\mathbb{I}_{\mathbb{V}} \times_{\mathbb{V}} \mathbb{W}$ to $\mathbb{V}^*$ and $\mathbb{W}$. On the other hand $G^* \circ \mathfrak{F}_{\mathbb{W}} = C_! \circ D^*$ where C, D are the projections of $\mathbb{I}_{\mathbb{W}} \times_{\mathbb{W}^*} \mathbb{V}^*$ to $\mathbb{V}^*$ and $\mathbb{W}$. But the identities

$$\begin{aligned}\mathbb{I}_{\mathbb{W}} \times_{\mathbb{W}^*} \mathbb{V}^* &= \{(w,\lambda) \in \mathbb{W} \times \mathbb{V}^* \mid \mathrm{ev}_{\mathbb{V}}(F(w),\lambda) = 1\} \\ &= \{(\lambda, w) \in \mathbb{V}^* \times \mathbb{W} \mid \mathrm{ev}_{\mathbb{W}}(w, G(\lambda)) = 1\} = \mathbb{I}_{\mathbb{W}} \times_{\mathbb{W}^*} \mathbb{V}^*\end{aligned}$$

give a canonical isomorphism $A_! \circ B^* \cong C_! \circ D^*$. □

4.1. Fourier-Deligne. For the remainder of this section, suppose that the ground field k has characteristic $p > 0$; choose a non-trivial additive character $\psi\colon \mathbb{F}_p \to \bar{\mathbb{Q}}_\ell^\times$ and let $\mathfrak{L}_\psi$ be the corresponding Artin-Schreier sheaf on $\mathbb{A}^1$ (see [La2]). Then one has the Fourier-Deligne transform

$$\mathfrak{F}^D_{\mathbb{V}}\colon D(\mathbb{V}) \to D(\mathbb{V}^*), \qquad \mathfrak{F}^D_{\mathbb{V}} f = (p_{\mathbb{V}^*})_!(\mathrm{ev}^*_{\mathbb{V}}\mathfrak{L}_\psi \otimes p^*_{\mathbb{V}} f)\,\langle r\rangle\,.$$

Finally, let $\rho^{(N)}_{\mathbb{V}}\colon \mathbb{V} \to \mathbb{V}_{(N)}$ be the quotient map (for $\mathbb{V}_{(N)}$ defined as in §3.1).

Theorem 4.3. *There exists a rank-1 local system γ_χ on* $\mathrm{Spec}(k)$*, depending only on χ and ψ, and a canonical isomorphism of functors $D(\mathbb{V})_\chi \to D(\mathbb{V}^*)_{\chi^*}$:*

$$\mathfrak{F}^D_{\mathbb{V}}((\rho^{(N)}_{\mathbb{V}})^*(-)) \cong \gamma_\chi \otimes (\rho^{(N)}_{\mathbb{V}^*})^*(\mathfrak{F}_{\mathbb{V}}(-)).$$

In fact, we can define $\gamma_\chi := R\Gamma_c(\mathbb{G}_m, \chi \otimes \mathfrak{L}_\psi)\,\langle 1\rangle$ (compare to [De, §4]*).*

Proof. Consider the closed embedding $i\colon \mathbb{I}_0 := \mathrm{ev}^{-1}_{\mathbb{V}}(0) \to \mathbb{V} \times_{\mathbb{B}} \mathbb{V}^*$. Since the restriction $i^*(\mathrm{ev}^*_{\mathbb{V}}\mathfrak{L}_\psi \otimes p^*_{\mathbb{V}} f) = i^* p^*_{\mathbb{V}} f$ is χ-homogeneous on $\mathbb{I}_0$ with respect to $(a_{\mathbb{V}} \times \mathrm{id})$, Lemma 3.1 implies that

$$(p_{\mathbb{V}^*})_! i_! i^*(\mathrm{ev}^*_{\mathbb{V}}\mathfrak{L}_\psi \otimes p^*_{\mathbb{V}} f) = 0.$$

Therefore, if j is the open complement to i,

$$\mathfrak{F}^D_{\mathbb{V}} f = (p_{\mathbb{V}^*})_! j_! j^*(\mathrm{ev}^*_{\mathbb{V}}\mathfrak{L}_\psi \otimes p^*_{\mathbb{V}} f)\,\langle r\rangle\,. \tag{3}$$

Now, we have a commuting diagram

$$\begin{tikzcd} \mathbb{G}_m \times \mathbb{I} \arrow[r, "\alpha"] \arrow[d, "\mathrm{pr}"] & \mathbb{V} \times_{\mathbb{B}} \mathbb{V}^* - \mathbb{I}_0 \arrow[d, "p_{\mathbb{V}^*}"] \\ \mathbb{I} \arrow[r, "p^{\mathbb{I}}_{\mathbb{V}^*}"] & \mathbb{V}^* \end{tikzcd} \tag{4}$$

where α is given by the action of $\mathbb{G}_m$ on the first factor $\mathbb{V}$. Moreover, by the χ-homogeneity of f,

$$\alpha^* j^*(\mathrm{ev}^*_{\mathbb{V}}\mathfrak{L}_\psi \otimes p^*_{\mathbb{V}} f) = (\chi \otimes \mathfrak{L}_\psi) \boxtimes (p^{\mathbb{I}}_{\mathbb{V}})^* f.$$

Therefore (3) and the commutativity of (4) give

$$\mathfrak{F}^D_{\mathbb{V}} f = pr_!(\chi \otimes \mathfrak{L}_\psi) \boxtimes (p^{\mathbb{I}}_{\mathbb{V}^*})_!(p^{\mathbb{I}}_{\mathbb{V}})^* f\,\langle r\rangle$$

and thence the first statement of the theorem.

For the second, choose $\mathbb{V} := \mathbb{A}^1 = \mathbb{V}^*$ over $\mathbb{B} := \mathrm{Spec}(k)$, and $f := (j_{\mathbb{A}^1})_*\chi\,\langle 1\rangle$. By direct calculation, $\mathfrak{F}_{\mathbb{V}} f = (j_{\mathbb{A}^1})_*\chi^*\,\langle 1\rangle$. Thus by the first part, $j^*_{\mathbb{A}^1}\mathfrak{F}^D_{\mathbb{V}} = \gamma_\chi \otimes \chi^*\,\langle 1\rangle$ is clearly the sum of rank-1 local systems, in possibly various degrees; but by t-exactness of $\mathfrak{F}^D_{\mathbb{V}}$, it must also be an irreducible perverse sheaf. This can only happen if γ_χ has rank one and sits in degree zero. □

5. The Theta Sheaf

We continue with the notation from §1.3.1. In this section we define the sheaf Θ_ϵ on $\mathrm{Bun}_{\mathrm{Mp}}$, for any choice of sign $\epsilon = \pm 1$. The format of the definition is illustrated by the following diagram:

$$\begin{tikzcd}
\mathbb{V}^* \arrow[d] \arrow[dr] & & \mathbb{V} \arrow[dl] \\
\mathrm{Bun}_{\mathrm{Mp}} & \mathbb{B} &
\end{tikzcd}$$

There is a stack $\mathbb{V}^*$, smooth and surjective over $\mathrm{Bun}_{\mathrm{Mp}}$, and $\mathbb{V}^*$ is itself a vector bundle over a base $\mathbb{B}$. Let ζ be the unique non-trivial character of μ_2. The pullback of Θ_ϵ to $\mathbb{V}^*$ is ζ-homogeneous under homotheties, and we describe its Fourier transform, a ζ-homogeneous sheaf f on the dual bundle $\mathbb{V}$.

5.1. The Stacks $\mathbb{B}, \mathbb{V}, \mathbb{V}^*$.

5.1.1. Let Bun_n be the moduli stack of rank-n vector bundles on X. So $\mathrm{Bun}_n(S)$ is the groupoid of vector bundles on X_S. Let $\tilde{\mathscr{L}}$ be the universal bundle on X_{Bun_n}, and in general we use tildes to mark the universal families over various moduli stacks.

Let $\mathbb{B} = \mathbb{B}_n^\epsilon$ be the open substack of Bun_n parameterising the bundles $\mathscr{L}$ such that $H^0(\mathrm{Sym}^2(\mathscr{L}^\vee)) = 0$, and such that $(-1)^{h^0(\mathscr{L})} = -\epsilon$.

Remark 5.1. The condition that $H^0(\mathrm{Sym}^2(\mathscr{L}^\vee)) = 0$ implies that $H^0(\mathscr{L}^\vee) = 0$ (since otherwise $\phi \mapsto \phi \otimes \phi$ defines a non-zero function $H^0(\mathscr{L}^\vee) \to H^0(\mathrm{Sym}^2(\mathscr{L}^\vee))$). Thus $-\epsilon$ is just the parity of the Euler characteristic of $\mathscr{L}$.

5.1.2. Let $\mathbb{V}$ be the vector bundle over $\mathbb{B}$, such that $\mathbb{V}(S)$ is the groupoid of pairs $(\mathscr{L}, v)$ with $\mathscr{L} \in \mathbb{B}(S)$ and $v \in \mathrm{Hom}_{\mathrm{sym}}(\mathscr{L}^*, \mathscr{L})$.

5.1.3. Let $\mathbb{V}^*$ be the dual bundle; $\mathbb{V}^*(S)$ is the groupoid of *symmetric extensions*, i.e. data $(\mathscr{L}, \gamma)$ with $\mathscr{L} \in \mathbb{B}(S)$, and γ an extension

$$\gamma = [\qquad 0 \longrightarrow \mathscr{L}^\vee \xrightarrow{\gamma_1} \mathscr{V} \xrightarrow{\gamma_2} \mathscr{L} \longrightarrow 0 \qquad] \tag{5}$$

with $\mathscr{V} \in \mathrm{Bun}_{\mathrm{Sp}}(S)$ and $\gamma_2 = \gamma_1^\vee \circ b_{\mathscr{V}}$. Thus $\mathscr{L}^\vee$ is a Lagrangian subbundle of $\mathscr{V}$.

5.2. The map $\mathbb{V}^* \to \mathrm{Bun}_{\mathrm{Mp}}$.

There is a canonical lift $p^{\mathbb{V}^*}_{\mathrm{Bun}_{\mathrm{Mp}}} \colon \mathbb{V}^* \to \mathrm{Bun}_{\mathrm{Mp}}$ of the natural map

$$p^{\mathbb{V}^*}_{\mathrm{Bun}_{\mathrm{Sp}}} \colon \mathbb{V}^* \to \mathrm{Bun}_{\mathrm{Sp}} \qquad (\mathscr{L}, \gamma) \mapsto \mathscr{V}.$$

Namely, $p^{\mathbb{V}^*}_{\mathrm{Bun}_{\mathrm{Mp}}}(\mathscr{L}, \gamma) = (\mathscr{V}, \mathscr{D}, \delta)$ where $\mathscr{D} := (\det p_{X_S,*}\mathscr{L})$ and δ is the composition

$$\delta \colon (\det p_{X_S,*}\mathscr{L})^{\otimes 2} \to (\det p_{X_S,*}\mathscr{L}) \otimes (\det p_{X_S,*}\mathscr{L}^\vee) \to \det p_{X_S,*}\mathscr{V}.$$

We will describe the sheaf Θ_ϵ in terms of its pullback to $\mathbb{V}^*$. The following Lemma shows that it is determined by this pullback; it can be proved in the same way as [Ly, Lemma 6].

Lemma 5.1. *The natural map* $p^{\mathbb{V}^*}_{\mathrm{Bun}_{\mathrm{Mp}}} \colon \mathbb{V}^{*\circ} \to \mathrm{Bun}_{\mathrm{Sp}}$ *is smooth and surjective.*

5.3. Definition. Let $\mathbb{W}$ be the vector bundle over $\mathbb{B}$ that parameterises data $(\mathscr{L}, w)$ with $w \in \mathrm{Hom}(\mathscr{O}, \mathscr{L})$. Define

$$\sigma \colon \mathbb{W} \to \mathbb{V}$$

by $\sigma(\mathscr{L}, w) = (\mathscr{L}, v := w \circ w^*)$. Let

$$\sigma_* \bar{\mathbb{Q}}_\ell \langle \dim \mathbb{W} \rangle = f_0 \oplus f \tag{6}$$

be the decomposition into irreducible perverse sheaves, in such a way that f_0 is $\mathbb{G}_m$-equivariant, and f is ζ-homogeneous.

Let $a \colon \mathrm{Spec}(k[x]/(x^2+1)) \to \mathrm{Spec}(k)$ be the natural map. Since a is a two-fold cover, $a_* \bar{\mathbb{Q}}_\ell$ is $\mathbb{Z}/2\mathbb{Z}$-graded; let L be the odd part (thus L is trivial if and only if -1 is a square in k). Finally, let L_χ be the following local system on $\mathbb{V}^*$: on the component of $\mathbb{V}^*$ with $h^0(\mathscr{L}) = m$, L_χ equals the pullback of $L^{\otimes \lfloor (m-1)/2 \rfloor}$ from $\mathrm{Spec}(k)$. Here $\lfloor (m-1)/2 \rfloor$ denotes the integral part of $(m-1)/2$.

Definition-Theorem 5.2. *There is an irreducible perverse sheaf Θ_ϵ on $\mathrm{Bun}_{\mathrm{Mp}}$ and an isomorphism $(p^{\mathbb{V}^*}_{\mathrm{Mp}})^* \Theta_\epsilon \langle \dim \mathbb{V}^* - \dim \mathrm{Bun}_{\mathrm{Mp}} \rangle \cong L_\chi \otimes \mathfrak{F}_{\mathbb{V}} f$.*

Theorem 5.2 was proved in [Ly] in positive characteristic (using $\mathfrak{F}^D_{\mathbb{V}}$ instead of $\mathfrak{F}_{\mathbb{V}}$), by giving an alternative construction of Θ_ϵ, which we explain in this generalized setting in §6.

Remark 5.2. The appearance of the twist L_χ in Theorem 5.2 corresponds to the choice, in the original [Ly, Proposition 7], of a square-root of -1 in k. [1]

6. Direct description of Θ_ϵ.

Following [Ly], stratify

$$\mathrm{Bun}_{\mathrm{Sp}} = \coprod_{m \geq 0} {}_m\mathrm{Bun}_{\mathrm{Sp}}$$

so that ${}_m\mathrm{Bun}_{\mathrm{Sp}}$ parameterises symplectic bundles $\mathscr{V}$ with $h^0(\mathscr{V}) = m$. The corresponding strata ${}_m\mathrm{Bun}_{\mathrm{Mp}}$ are trivial μ_2-gerbes over each ${}_m\mathrm{Bun}_{\mathrm{Sp}}$. Let c_m be the 2-fold cover $c_m \colon {}_m\mathrm{Bun}_{\mathrm{Sp}} \to {}_m\mathrm{Bun}_{\mathrm{Mp}}$. To be precise, let $c_m(\mathscr{V}) = (\mathscr{V}, \mathscr{D}, \delta)$ where $\mathscr{D} := \det R^0 p_{X_S,*}$ and $\delta \colon \mathscr{D} \otimes \mathscr{D} \to \det R p_{X_S,*} \mathscr{V}$ is the isomorphism

$$\mathscr{D} \otimes \mathscr{D} \to \mathscr{D} \otimes (\det R^1 p_{X_S,*} \mathscr{V}^\vee)^* \to \mathscr{D} \otimes (\det R^1 p_{X_S,*} \mathscr{V})^* = \det R p_{X_S,*} \mathscr{V}$$

induced by Serre duality and the symplectic form on $\mathscr{V}$.

Then $c_{m,*} \bar{\mathbb{Q}}_\ell$ is naturally $\mathbb{Z}/2\mathbb{Z}$-graded; let Aut_m be the odd part. Finally, let Aut_g and Aut_s be the Goresky-MacPherson extensions of $\mathrm{Aut}_0 \langle \dim \mathrm{Bun}_{\mathrm{Mp}} \rangle$ and $\mathrm{Aut}_1 \langle \dim \mathrm{Bun}_{\mathrm{Mp}} - 1 \rangle$ respectively to $\mathrm{Bun}_{\mathrm{Mp}}$.

Proposition 6.1. *If one defines*

$$\Theta_+ = \mathrm{Aut}_g \qquad \Theta_- = \mathrm{Aut}_s$$

then there exist isomorphisms as in Theorem 5.2, canonical up to sign.

The proof of this proposition is essentially the same as that of Lysenko's Proposition 7, so we won't give all the details. The main step is to calculate the stalks of the 'finite dimensional model' of $\mathfrak{F}^D_{\mathbb{V}} f$ (Lysenko's Propositions 1 and 2). We now carry out the analogous calculation for $\mathfrak{F}_{\mathbb{V}} f$.

[1] Having said that, it still puzzles me a bit. Should there be another sign that cancels it somehow?

6.1. Finite Dimensional Model. Let W be a vector space of dimension r over k, i.e. a vector bundle over $B := \mathrm{Spec}(k)$, and let $V = \mathrm{Sym}^2 W$. Define $\sigma\colon W \to V$ by $w \mapsto w \otimes w$.

We can decompose $\sigma_! \bar{\mathbb{Q}}_\ell \langle r \rangle$ into irreducible perverse sheaves

$$\sigma_! \bar{\mathbb{Q}}_\ell \langle r \rangle = f_0 \oplus f$$

where f_0 is constant on the image of σ, and f is ζ-homogeneous(recall that ζ is the non-trivial character of μ_2).

Alternatively, allowing $\mathbb{Z}/2\mathbb{Z}$ to act on W by the sign representation, and on V trivially, $\sigma_! \bar{\mathbb{Q}}_\ell \langle r \rangle$ is naturally $\mathbb{Z}/2\mathbb{Z}$-graded, and f is the odd part.

Definition 6.1. Set $S_r := \mathfrak{F}_V f$. It is an irreducible perverse sheaf on V^*.

6.2. Description of Stalks. V^* is stratified by substacks $i_m\colon Q_m \hookrightarrow V^*$, $0 \le m \le r$, where Q_m consists of the symmetric forms on W with kernel of dimension m. Let $c_m\colon \tilde{Q}_m \to Q_m$ be the 2-fold cover whose fiber over q consists of the square-roots of the discriminant $\det q|_{W/\ker q}$.

Let Aut_m be the odd-$\mathbb{Z}/2\mathbb{Z}$-graded part $(c_{m,*}\bar{\mathbb{Q}}_\ell)_-$ of $c_{m,*}\bar{\mathbb{Q}}_\ell$; it is a $\mathrm{GL}(W)$-equivariant local system on Q_m.

Whenever $r - m$ is odd, set

$$C_m := \mathrm{Spec}(k[x]/(x^2 - (-1)^{(r-m-1)/2})).$$

Thus C_m is a $\mathbb{Z}/2\mathbb{Z}$-torsor over $\mathrm{Spec}(k)$; let $\ell_m \in D(\mathrm{Spec}(k))$ be the corresponding sheaf, i.e. the odd-graded part of the push-forward of $\bar{\mathbb{Q}}_\ell$ from C_m to $\mathrm{Spec}(k)$.

Here is the analogue of [Ly, Propositions 1]:

Proposition 6.2. *There are isomorphisms $i_m^* S_r \cong \ell_m \otimes \mathrm{Aut}_m \langle \dim V - m \rangle$ whenever $r - m$ is odd, and $i_m^* S_r = 0$ whenever $r - m > 0$ is even. These isomorphisms are canonical up to a sign. In particular, S_r is the Goresky-MacPherson extension of $\ell_0 \otimes \mathrm{Aut}_0$ when r is odd, and of $\ell_1 \otimes \mathrm{Aut}_1$ when r is even.*

Proof. Since S_r is a geometrically non-constant, $\mathrm{GL}(W)$-equivariant complex, and $\mathrm{GL}(W)$ acts transitively on Q_m, it suffices to calculate $i^* S_r$ for any single point $i\colon \mathrm{Spec}(k) \to Q_m$. Let q be the corresponding quadratic form on W. Let N be the closed subscheme of W given by $q(w) = 1$. The action of $\mathbb{Z}/2\mathbb{Z}$ on W preserves N, and $i^* S_r$ is the odd part $R\Gamma_c(N, \bar{\mathbb{Q}}_\ell \langle r + \dim V - 1 \rangle)_-$.

Consider a q-orthogonal decomposition $W = K \oplus H \oplus P$, where $K = \ker q$ and $q|_H$ is hyperbolic; that is, there is a decomposition $H = L \oplus L^*$ for which $q(a \oplus a') = a'(a)$.

Lemma 6.3. *Let $N' := N \cap P$. There is a canonical isomorphism between $i^* S_r$ and the odd part*

$$R\Gamma_c(N', \bar{\mathbb{Q}}_\ell \langle r + \dim V - 1 - 2\dim K - \dim H \rangle)_-.$$

Proof. Let $U \subset W$ be the complement of $U' := K \oplus L \oplus P$ in W. Then $N \cap U$ is an $\mathbb{A}^{h-1+1}$-bundle over $(L - B) \oplus P$, where $h = \dim L$, $m = \dim K$. Thus $R\Gamma_c(N \cap U, \bar{\mathbb{Q}}_\ell) = R\Gamma_c((L - B) \oplus P, \bar{\mathbb{Q}}_\ell \langle 2 - 2h - 2m \rangle)$. This last complex is acyclic, because of the distinguished triangle

$$R\Gamma_c((L - B) \oplus P, \bar{\mathbb{Q}}_\ell) \longrightarrow R\Gamma_c(L \oplus P, \bar{\mathbb{Q}}_\ell) \xrightarrow{\cong} R\Gamma_c(P, \bar{\mathbb{Q}}_\ell).$$

On the other hand, $N \cap U'$ is a $\mathbb{A}^{m+h}$-bundle over N'. Therefore we have

$$i^* S_r \langle 1 - r - \dim V \rangle = R\Gamma_c(N \cap U', \bar{\mathbb{Q}}_\ell)_- = R\Gamma_c(N', \bar{\mathbb{Q}}_\ell \langle -2m - 2h \rangle)_-$$

as stated in the Lemma. (We remark that this isomorphism is independent of the decomposition $H = L \oplus L^*$.) $\square$

Now, to prove the Theorem, we specialize q. If $r-m$ is even, we can assume that q is hyperbolic, i.e. $P=0$, whence $N'=\emptyset$, so the lemma gives $i^*S_r=0$ as desired. If $r-m$ is odd, we can assume that P is one-dimensional, and even that $q|_P$ is isometric to the quadratic form $x\mapsto x^2$. Thus $N'=\mathrm{Spec}(k)\sqcup\mathrm{Spec}(k)$. On the other hand, since $(-1)^{(r-m-1)/2}\det q$ is a square in k, there is an isomorphism of $\mathbb{Z}/2\mathbb{Z}$-torsors $i^*\tilde{Q}_m\cong C_m$. The isomorphism demanded by the theorem is induced by a choice of isomorphism $N'\cong C_m\otimes_{\mathbb{Z}/2\mathbb{Z}} i^*\tilde{Q}_m$. □

7. Characteristic Cycles

In this section the ground field k is assumed to have characteristic zero. The material in §7.1 goes back to MacPherson [Ma], and was developed in the sources cited below.

7.1. Characteristic Cycles and Characteristic Functions. Given a stack $\mathbb{S}$, let $\mathrm{C}(\mathbb{S})$ be the abelian group of $\mathbb{Z}$-valued locally constructible functions on $\mathbb{S}$ (that is, they are certain functions on the set of geometric points of $\mathbb{S}$; see [Jo]). To each locally closed, reduced substack $\mathbb{T}$ of $\mathbb{S}$, we associate its characteristic function $1_{\mathbb{T}}\in\mathrm{C}(\mathbb{S})$; these locally generate $\mathrm{C}(\mathbb{S})$.

For any map $F\colon\mathbb{S}\to\mathbb{S}'$ there is a functorial pull-back $F^*=F^!\colon\mathrm{C}(\mathbb{S}')\to\mathrm{C}(\mathbb{S})$ defined in an obvious way: $F^*1_{\mathbb{T}}=1_{F^{-1}(\mathbb{T})}$. If F is representable of finite type (but see [Jo] for generalisations), there are functorial push-forwards $F_!=F_*\colon\mathrm{C}(\mathbb{S})\to\mathrm{C}(\mathbb{S}')$, characterised by the following property (see [G-S], Proposition 1.1):

$$(F_*1_{\mathbb{T}})(p)=\chi(\mathbb{T}\times_{\mathbb{S}'}\mathrm{Spec}(\bar{k}))$$

for every geometric point $p\colon\mathrm{Spec}(\bar{k})\to\mathbb{S}'$. Here χ is the (ℓ-adic, compact-support) Euler characteristic.

There is a group homomorphism $\mathrm{CF}\colon K_0(D(\mathbb{S}))\to\mathrm{C}(\mathbb{S})$ that assigns to a constructible sheaf f the function giving the rank of f at each point, and CF commutes with $F_*,F_!,F^!,F^*$ in the obvious sense. We also have $\mathrm{CF}(\mathfrak{D}f)=\mathrm{CF}(f)$ and $\mathrm{CF}(f)\langle 1\rangle:=\mathrm{CF}(f\langle 1\rangle)=-\mathrm{CF}(f)$.

7.1.1. Let $\mathrm{CC}(\mathbb{S})$ be the abelian group of conic Lagrangian cycles in the cotangent stack $T^*\mathbb{S}$ (i.e. locally finite formal sums of conic Lagrangian substacks). There is an isomorphism $\mathrm{cyc}\colon\mathrm{C}(\mathbb{S})\to\mathrm{CC}(\mathbb{S})$, see [Ke], and therefore associated to each complex f a cycle $\mathrm{CC}(f):=\mathrm{cyc}(\mathrm{C}(f))$. The properties and functoriality of the 'characteristic cycle' map CC have been studied in detail (see e.g. [Gi1]), but we only need the following simple cases.

(a) If f is a local system of rank n on $\mathbb{S}$, then $\mathrm{CC}(f)=n[\mathbb{S}]$ (that is, the class of the zero section $\mathbb{S}\subset T^*\mathbb{S}$, with multiplicity n).

Given $F\colon\mathbb{X}\to\mathbb{Y}$ with $\mathbb{X}$ and $\mathbb{Y}$ smooth, consider the natural diagram

$$T^*\mathbb{X}\xleftarrow{\ a\ }\mathbb{X}\times_{\mathbb{Y}}T^*\mathbb{Y}\xrightarrow{\ b\ }T^*\mathbb{Y}.$$

(b) Suppose F is smooth of relative dimension d. Then

$$\mathrm{CC}(F^*f\langle d\rangle)=a_*b^*\,\mathrm{CC}(f).$$

(c) Suppose $F=F_2\circ F_1$, where F_1 is an étale epimorphism, and F_2 is the closed embedding of a smooth substack. Then

$$\mathrm{CC}(F_*f)=b_*a^*\,\mathrm{CC}(f).$$

7.2. The Effect of Fourier. Suppose given $f\in D(\mathbb{V})_\chi$, with χ a non-trivial character. We wish to describe $\mathrm{CC}(\mathfrak{F}_{\mathbb{V}}f)$ in terms of $\mathrm{CC}(f)$. We deduce our result (Theorem 7.2) from analogous work by Brylinski [Br] and Laumon [La1] on the Radon transform.

7.2.1. *The Radon Transform.* Let us first recall that theory. Let $\mathbb{I}_0 \subset \mathbb{PV}^* \times_{\mathbb{B}} \mathbb{PV}$ be the incidence variety (i.e. the image of $\mathrm{ev}_{\mathbb{V}}^{-1}(0)$), and $p^{\mathbb{I}_0}_{\mathbb{V}^*}, p^{\mathbb{I}_0}_{\mathbb{V}}$ the projections to $\mathbb{V}^*$ and $\mathbb{V}$. The Radon transform $\mathfrak{R}\colon D(\mathbb{PV}) \to D(\mathbb{PV}^*)$ is defined by

$$\mathfrak{R} = (p^{\mathbb{I}_0}_{\mathbb{V}^*})_! \circ (p^{\mathbb{I}_0}_{\mathbb{V}})^* \circ \langle r-2\rangle\,. \tag{7}$$

We now describe how characteristic cycles transform under Radon. First, identify $T^*(\mathbb{PV} \times \mathbb{PV}^*) \cong T^*\mathbb{PV} \times T^*\mathbb{PV}^*$; here we take the standard isomorphism twisted by the homothety -1 on the second factor. The conormal bundle $T^*_{\mathbb{I}_0}(\mathbb{PV} \times \mathbb{PV}^*)$ then gives a correspondence Υ between $T^*\mathbb{PV}$ and $T^*\mathbb{PV}^*$.

Define

$$U_{\mathbb{V}} := (T^*\mathbb{PV} - \mathbb{PV} \times_{\mathbb{B}} T^*\mathbb{B}) \qquad U_{\mathbb{V}^*} := (T^*\mathbb{PV}^* - \mathbb{PV}^* \times_{\mathbb{B}} T^*\mathbb{B}).$$

The following theorem is stated in [La1, Appendix] and essentially proved by Brylinski [Br].

Theorem 7.1.

(a) *The correspondence* Υ *restricts to an isomorphism*

$$\Upsilon\colon U_{\mathbb{V}} \to U_{\mathbb{V}^*}.$$

(b) *Suppose* $f \in D(\mathbb{PV}))$ *is such that* $\mathrm{CC}(f) \cap (\mathbb{PV} \times_{\mathbb{B}} T^*\mathbb{B}) \subset \mathbb{PV}$. *Then similarly* $\mathrm{CC}(\mathfrak{R}f) \cap (\mathbb{PV}^* \times_{\mathbb{B}} T^*\mathbb{B}) \subset \mathbb{PV}^*$ *and*

$$\mathrm{CC}(\mathfrak{R}(f))|_{U_{\mathbb{V}^*}} = \Upsilon_*(\mathrm{CC}(f)|_{U_{\mathbb{V}}}).$$

7.2.2. Now suppose f is a χ-homogeneous sheaf. The characteristic function $\mathrm{CF}(f)$ is $\mathbb{G}_m$-invariant, and $i^*_{\mathbb{V}} f = 0$, so there exists a constructible function $\mathrm{cf}(f)$ on $\mathbb{PV}$ such that

$$\mathrm{CF}(f) = (j_{\mathbb{V}})_!(\rho^\circ_{\mathbb{V}})^*\,\mathrm{cf}(f).$$

Similarly put $\mathrm{cc}(f) = \mathrm{cyc}(\mathrm{cf}(f))$. Thus $\mathrm{cc}(f)$ and $\mathrm{CC}(f)|_{T^*\mathbb{V}^{*\circ}}$ are related by 7.1.1(b) applied to $F := \rho^\circ_{\mathbb{V}}$.

Theorem 7.2. *Suppose* $f \in D(\mathbb{V})_\chi$ *is a* χ*-homogeneous sheaf such that* $\mathrm{cc}(f) \cap (\mathbb{PV} \times_{\mathbb{B}} T^*\mathbb{B}) \subset \mathbb{PV}$, *and such that* $(\bar{\pi}^\circ_{\mathbb{V}})_!\,\mathrm{cf}(f)$ *is locally constant. Then* $\mathrm{cc}(\mathfrak{F}_{\mathbb{V}} f) \cap (\mathbb{PV}^* \times_{\mathbb{B}} T^*\mathbb{B}) \subset \mathbb{PV}^*$ *and*

$$\mathrm{cc}(\mathfrak{F}_{\mathbb{V}} f)|_{U_{\mathbb{V}^*}} = \Upsilon_*(\mathrm{cc}(f)|_{U_{\mathbb{V}}}).$$

Proof. First we relate the Fourier and Radon transforms.

Lemma 7.3. *For any* $f \in D(\mathbb{V})_\chi$ *we have*

$$\mathrm{cf}(\mathfrak{F}_{\mathbb{V}} f) = \mathfrak{R}(\mathrm{cf}(f)) + (-1)^{r-1}(\bar{\pi}^\circ_{\mathbb{V}^*})^*(\bar{\pi}^\circ_{\mathbb{V}})_!\,\mathrm{cf}(f).$$

Proof. The three terms in this equation are (up to sign) the transforms of $\mathrm{cf}(f)$ under the correspondences between $\mathbb{PV}$ and $\mathbb{PV}^*$ given by $\mathbb{I}$, $\mathbb{I}_0$, and $\mathbb{PV}^* \times_{\mathbb{B}} \mathbb{PV}$, respectively. The equation follows from the partition $\mathbb{PV}^* \times_{\mathbb{B}} \mathbb{PV} = \mathbb{I} \sqcup \mathbb{I}_0$. (The signs are explained by the fact that $\mathfrak{F}_{\mathbb{V}}$ includes a shift $\langle r-1\rangle$ while $\mathfrak{R}$ includes a shift $\langle r-2\rangle$.) ☐

Now, the condition on $\mathrm{cf}(f)$, combined with Lemma 7.3, shows that $\mathrm{cf}(\mathfrak{F}_{\mathbb{V}} f) = \mathfrak{R}(\mathrm{cf}(f))$ up to a locally constant function. So $\mathrm{cc}(\mathfrak{F}_{\mathbb{V}} f) = \mathrm{cyc}(\mathfrak{R}\,\mathrm{cf}(f))$ up to a multiple of the zero section. Thus Theorem 7.2 follows from Theorem 7.1(b). ☐

8. The Cotangent Stacks of $\mathbb{V}$ and $\mathbb{V}^*$

We continue to consider the stacks $\mathbb{B}$, $\mathbb{W}$, $\mathbb{V}$, and $\mathbb{V}^*$ defined in §§5.1 and 5.3, and the notation of §2.2.

8.1. The object of this section is to prove the following results.

Theorem 8.1. *The cotangent stacks $T^*\mathbb{V}$ and $T^*\mathbb{V}^*$ are both canonically isomorphic to the stack of commuting diagrams*

$$\gamma = \left[\begin{tikzcd} 0 \arrow[r] & \mathscr{L}^\vee \arrow[r] \arrow[d, "v"] & \mathscr{V} \arrow[r] \arrow[dl, "v'"] & \mathscr{L} \arrow[r] & 0 \\ & \mathscr{L}\otimes\Omega & & & \end{tikzcd}\right] \tag{8}$$

with $(\mathscr{L}, v) \in \mathbb{V}$ and $(\mathscr{L}, \gamma) \in \mathbb{V}^*$.

Theorem 8.2. *The cotangent stack $T^*\mathbb{W}$ parameterises commutative diagrams*

$$\beta = \left[\begin{tikzcd} 0 \arrow[r] & \Omega \arrow[r] \arrow[d, "w"] & \mathscr{M} \arrow[r] \arrow[dl, "w'"] & \mathscr{L} \arrow[r] & 0 \\ & \mathscr{L}\otimes\Omega & & & \end{tikzcd}\right] \tag{9}$$

with $(\mathscr{L}, w) \in \mathbb{W}$ and $\beta \in \operatorname{Ext}(\mathscr{L}, \Omega)$.

Let Υ' denote the isomorphism $\Upsilon' \colon T^*\mathbb{V} \to T^*\mathbb{V}^*$ induced by Theorem 8.1.

Proposition 8.3. *The isomorphism Υ' induces the isomorphism Υ in Theorem 7.1. That is, Υ' maps $\mathbb{V}^\circ \times_{\mathbb{PV}} U_\mathbb{V} \subset T^*\mathbb{V}$ isomorphically onto $\mathbb{V}^{*\circ} \times_{\mathbb{PV}^*} U_{\mathbb{V}^*} \subset T^*\mathbb{V}^*$, and the following diagram commutes:*

$$\begin{tikzcd} \mathbb{V}^\circ \times_{\mathbb{PV}} U_\mathbb{V} \arrow[r, "\Upsilon'"] \arrow[d] & \mathbb{V}^{*\circ} \times_{\mathbb{PV}^*} U_{\mathbb{V}^*} \arrow[d] \\ U_\mathbb{V} \arrow[r, "\Upsilon"] & U_{\mathbb{V}^*} \end{tikzcd}$$

Remark 8.1. For *any* vector bundle $\mathbb{V} \to \mathbb{B}$ there is a canonical isomorphism $\Upsilon' \colon T^*\mathbb{V} \to T^*\mathbb{V}^*$. Namely, the graph of Υ' inside $T^*\mathbb{V} \times_\mathbb{B} T^*\mathbb{V}^* \cong T^*(\mathbb{V} \times_\mathbb{B} \mathbb{V}^*)$ (see 7.2.1) equals the graph of the differential of the function $\mathrm{ev}_\mathbb{V} \colon \mathbb{V} \times_\mathbb{B} \mathbb{V}^* \to \mathbb{A}^1$. Moreover, Proposition 8.3 always holds for this canonical Υ'.

8.2. Proof of Theorem 8.1. Define a complex on $X_\mathbb{V}$:

$$\mathscr{C}_\mathbb{V} := \left[\mathscr{H}\!om_{\mathrm{sym}}(\tilde{\mathscr{L}}, \tilde{\mathscr{L}}^\vee) \xrightarrow{\tilde{v}\circ} \mathscr{H}\!om(\tilde{\mathscr{L}}, \tilde{\mathscr{L}} \otimes \Omega)\right]$$

in degrees $[0,1]$. (Recall from §5.1.1 that $(\tilde{\mathscr{L}}, \tilde{v})$ denotes the universal family over $X_\mathbb{V} := X \times \mathbb{V}$, so $\tilde{v} \colon \tilde{\mathscr{L}}^* \to \tilde{\mathscr{L}}$.) The statement to be proved about $T^*\mathbb{V}$ amounts to the identity

$$T^*\mathbb{V} = \operatorname{Vect}(\tau^{[0,1]} \circ Rp_{X_\mathbb{V},*}(\mathscr{C}_\mathbb{V}))^* \tag{10}$$

of vector bundles over $\mathbb{V}$ (see [LMB], example 14.4.10 and Theorem 17.16, in which our notation Vect stands for their notation $\mathbb{V}$).

Now recall that the tangent stack $T\mathbb{B}$ classifies extensions

$$\alpha = \left[0 \longrightarrow \mathscr{L} \longrightarrow \mathscr{W} \longrightarrow \mathscr{L} \longrightarrow 0\right]$$

over $\mathscr{L}$ an object of $\mathbb{B}$ (see §6.5 in [FG]). In the same way, $T\mathbb{V}$ classifies diagrams

$$\begin{tikzcd} \alpha^* = [& 0 \arrow[r] & \mathscr{L}^* \arrow[r] \arrow[d, "v"] & \mathscr{W}^* \arrow[r] \arrow[d, "v_0"] & \mathscr{L}^* \arrow[r] \arrow[d, "v"] & 0 &] \\ \alpha = [& 0 \arrow[r] & \mathscr{L} \arrow[r] & \mathscr{W} \arrow[r] & \mathscr{L} \arrow[r] & 0 &] \end{tikzcd} \tag{11}$$

with $v_0^* = v_0$. (The canonical projection to $\mathbb{V}$ takes value $(\mathscr{L}, v)$, and that to $T\mathbb{B}$ takes value α.) This description (11) of $T\mathbb{V}$ shows that as a vector bundle over $\mathbb{V}$,

$$T\mathbb{V} = \mathrm{Vect}(\tau^{[0,1]} \circ Rp_{X_{\mathbb{V}},*}(\mathscr{C}_{\mathbb{V}}^{\vee}[-1]))^*$$

where

$$\mathscr{C}_{\mathbb{V}}^{\vee}[-1] := [\qquad \mathscr{H}om(\tilde{\mathscr{L}}, \tilde{\mathscr{L}}) \xrightarrow{\circ \tilde{v}} \mathscr{H}om_{\mathrm{sym}}(\tilde{\mathscr{L}}^*, \tilde{\mathscr{L}}) \qquad]$$

in degrees $[0,1]$. Therefore (10) follows by duality.

Similarly, we wish to show that

$$T^*\mathbb{V}^* = \mathrm{Vect}(R^{[0,1]}p_{X_{\mathbb{V}^*},*}(\mathscr{C}_{\mathbb{V}^*}))^* \tag{12}$$

where $\mathscr{C}_{\mathbb{V}^*}$ is the complex of coherent sheaves on $\mathbb{V}^*$

$$\mathscr{C}_{\mathbb{V}^*} := [\qquad \mathscr{H}om(\tilde{\mathscr{V}}, \tilde{\mathscr{L}} \otimes \Omega) \xrightarrow{\circ \tilde{\gamma}_1} \mathscr{H}om_{\mathrm{sk}}(\tilde{\mathscr{L}}^{\vee}, \tilde{\mathscr{L}} \otimes \Omega) \qquad]$$

in degrees $[1,2]$. Dually, we wish to show

$$T\mathbb{V}^* = \mathrm{Vect}(R^{[0,1]}p_{X_{T\mathbb{B}},*}\mathscr{C}_{\mathbb{V}^*}^{\vee}[-1])^* \tag{13}$$

where, in degrees $[-1,0]$,

$$\mathscr{C}_{\mathbb{V}^*}^{\vee}[-1] = [\qquad \mathscr{H}om_{\mathrm{sk}}(\tilde{\mathscr{L}}, \tilde{\mathscr{L}}^{\vee}) \longrightarrow \mathscr{H}om(\tilde{\mathscr{L}}, \tilde{\mathscr{V}}) \qquad]. \tag{14}$$

Now, $T\mathbb{V}^*$ classifies diagrams

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
\alpha^{\vee} = [& 0 \arrow[r] & \mathscr{L}^{\vee} \arrow[r] \arrow[d] & \mathscr{W}^{\vee} \arrow[r] \arrow[d] & \mathscr{L}^{\vee} \arrow[r] \arrow[d] & 0 &] \\
 & 0 \arrow[r] & \mathscr{V} \arrow[r] \arrow[d] & \mathscr{V}' \arrow[r] \arrow[d] & \mathscr{V} \arrow[r] \arrow[d] & 0 & \\
\alpha = [& 0 \arrow[r] & \mathscr{L} \arrow[r] \arrow[d] & \mathscr{W} \arrow[r] \arrow[d] & \mathscr{L} \arrow[r] \arrow[d] & 0 &] \\
 & & 0 & 0 & 0 & &
\end{tikzcd} \tag{15}$$

where the columns are symmetric extensions $\gamma, \tilde{\gamma}, \gamma$. We find that as vector bundles over $\mathbb{V}^*$, we have $T\mathbb{V}^* = \mathrm{Vect}(R^{[0,1]}p_{X_{T\mathbb{B}},*}\mathscr{K})^*$ where

$$\mathscr{K} := [\qquad \mathscr{H}om_{\mathrm{sym}}(\tilde{\mathscr{V}}, \tilde{\mathscr{V}}) \longrightarrow \mathscr{H}om_{\mathrm{sym}}(\tilde{\mathscr{L}}^{\vee}, \tilde{\mathscr{L}}) \qquad]$$

in degrees $[0,1]$. On the other hand, there is a quasi-isomorphism $\mathscr{C}_{\mathbb{V}^*}^{\vee}[-1] \to \mathscr{K}$,

$$\begin{tikzcd}
\mathscr{H}om_{\mathrm{sk}}(\tilde{\mathscr{L}}, \tilde{\mathscr{L}}^{\vee}) \arrow[r] \arrow[d] & \mathscr{H}om(\tilde{\mathscr{L}}, \tilde{\mathscr{V}}) \arrow[r] \arrow[d, "\phi"] & 0 \arrow[d] \\
0 \arrow[r] & \mathscr{H}om_{\mathrm{sym}}(\tilde{\mathscr{V}}, \tilde{\mathscr{V}}) \arrow[r] & \mathscr{H}om_{\mathrm{sym}}(\tilde{\mathscr{L}}^{\vee}, \tilde{\mathscr{L}})
\end{tikzcd}$$

given by $\phi(a) = a \circ \tilde{\gamma}_2 + \tilde{\gamma}_1 \circ a^{\vee}$. Hence (13).

8.3. Proof of Theorem 8.2. The first part of Theorem (8.2) states that

$$T^*\mathbb{W} = \mathrm{Vect}(R^{[0,1]}p_{X_{\mathbb{W}},*}(\mathscr{C}_{\mathscr{W}}))^* \tag{16}$$

where

$$\mathscr{C}_{\mathbb{W}} := [\qquad \mathscr{H}om(\tilde{\mathscr{L}},\Omega) \xrightarrow{w\circ} \mathscr{H}om(\tilde{\mathscr{L}},\tilde{\mathscr{L}}\otimes\Omega) \qquad]$$

in degrees $[0,1]$. Now, $T\mathbb{W}$ classifies maps of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{O} \arrow[r] \arrow[d, "w"] & \mathscr{O}\oplus\mathscr{O} \arrow[r] \arrow[d] & \mathscr{O} \arrow[r] \arrow[d, "w"] & 0 \\ 0 \arrow[r] & \mathscr{L} \arrow[r] & \mathscr{W} \arrow[r] & \mathscr{L} \arrow[r] & 0 \end{tikzcd}$$

Therefore, as vector bundles over $\mathbb{W}$, we have $T\mathbb{W} = \mathrm{Vect}(R^{[0,1]}p_{X_{\mathbb{W}},*}\mathscr{C}_{\mathbb{W}}^\vee[-1])^*$, with

$$\mathscr{C}_{\mathbb{W}}^\vee[-1] := [\qquad \mathscr{H}om(\tilde{\mathscr{L}},\tilde{\mathscr{L}}) \xrightarrow{\circ w} \mathscr{H}om(\tilde{\mathscr{O}},\tilde{\mathscr{L}}) \qquad]$$

in degrees $[0,1]$. Dualize to obtain (16).

Finally, the last part of Theorem 8.2 is the observation that $T^*_{(\mathscr{L},w\circ w^*)}\sigma$ is induced by the map of complexes

$$\begin{tikzcd} \sigma^*\mathscr{C}_{\mathbb{V}} = [& \mathscr{H}om_{\mathrm{sym}}(\tilde{\mathscr{L}},\tilde{\mathscr{L}}^\vee) \arrow[r, "w\circ w^\vee\circ"] \arrow[d, "w^\vee\circ"] & \mathscr{H}om(\tilde{\mathscr{L}},\tilde{\mathscr{L}}\otimes\Omega) \arrow[d, "\mathrm{id}"] &] \\ \mathscr{C}_{\mathbb{W}} = [& \mathscr{H}om(\tilde{\mathscr{L}},\Omega) \arrow[r, "w\circ"] & \mathscr{H}om(\tilde{\mathscr{L}},\tilde{\mathscr{L}}\otimes\Omega) &]. \end{tikzcd}$$

8.4. Proof of Proposition 8.3. To show that the diagram commutes, it suffices to show that the graph $\Gamma(\Upsilon) \subset T^*\mathbb{V}\times T^*\mathbb{V}^* \cong T^*(\mathbb{V}\times\mathbb{V}^*)$ of Υ lies in the conormal bundle $T^*_{\mathbb{I}_0}(\mathbb{V}\times_{\mathbb{B}}\mathbb{V}^*)$, where $\mathbb{I}_0 := \mathrm{ev}_{\mathbb{V}}^{-1}(0)$ as in §7.2.1.

8.4.1. To understand the conormal stack to $\mathbb{I}_0$, we first describe the pairing

$$T\mathbb{V}\times_{T\mathbb{B}} T\mathbb{V}^* \to T\mathbb{A}^1 = \mathbb{A}^1\times\mathbb{A}^1 \to \mathbb{A}^1$$

(projection to the fibre of $T\mathbb{A}^1$ over $\mathbb{A}^1$). From (11) we see that as vector bundles over $T\mathbb{B}$

$$T\mathbb{V} = \mathrm{Vect}(\tau^{[0,1]}\circ Rp_{X_{T\mathbb{B}},*}(\mathscr{C}_{T\mathbb{B}}))^*$$

where $\mathscr{C}_{T\mathbb{B}}$ is the complex

$$\mathscr{C}_{T\mathbb{B}} := [\quad \mathscr{H}om_{\mathrm{sk}}(\tilde{\mathscr{L}}^*,\tilde{\mathscr{L}}) \xrightarrow{\circ\tilde{\alpha}_1} \mathscr{H}om(\tilde{\mathscr{L}}^*,\tilde{\mathscr{W}}) \xrightarrow{\tilde{\alpha}_2\circ} \mathscr{H}om_{\mathrm{sk}}(\tilde{\mathscr{L}}^*,\tilde{\mathscr{L}}) \quad]$$

in degrees $[0,2]$. In fact, $\mathscr{C}_{T\mathbb{B}}$ is exact in degrees 0 and 2; let $\mathscr{C}'_{T\mathbb{B}}$ be the sheaf $H^1(\mathscr{C}_{T\mathbb{B}})$ on $X_{T\mathbb{B}}$. Thus $T\mathbb{V}$ parameterizes data $(\mathscr{L},\alpha,v'')$ with $(\mathscr{L},\alpha)\in T\mathbb{B}$ and

$$v'' \in H^0\left(\ker[\mathscr{H}om(\mathscr{L}^*,\mathscr{W}) \to \mathscr{H}om_{\mathrm{sk}}(\mathscr{L}^*,\mathscr{L})]/\operatorname{Im}\mathscr{H}om_{\mathrm{sk}}(\mathscr{L}^*,\mathscr{L})\right). \tag{17}$$

Similarly,

$$T\mathbb{V}^* = \mathrm{Vect}(\tau^{[0,1]}\circ Rp_{X_{T\mathbb{B}},*}(\mathscr{C}_{T\mathbb{B}}^\vee[-1]))^*$$

where $\mathscr{C}_{T\mathbb{B}}^\vee[-1]$ is the complex

$$\mathscr{H}om_{\mathrm{sk}}(\tilde{\mathscr{L}},\tilde{\mathscr{L}}^\vee) \xrightarrow{\circ\tilde{\alpha}_2} \mathscr{H}om(\tilde{\mathscr{W}},\tilde{\mathscr{L}}^\vee) \xrightarrow{\tilde{\alpha}_1\circ} \mathscr{H}om_{\mathrm{sk}}(\tilde{\mathscr{L}},\tilde{\mathscr{L}}^\vee)$$

in degrees $[-1,1]$. The pairing $T\mathbb{V}\times_{T\mathbb{B}}T\mathbb{V}^* \to \mathbb{A}^1$ is induced by the duality between $\mathscr{C}_{T\mathbb{B}}$ and $\mathscr{C}_{T\mathbb{B}}^\vee[-1]$.

8.4.2. To prove Proposition 8.3, consider the (non-commuting) diagram

$$\begin{tikzcd}
& \mathbb{X} := (T\mathbb{V}^* \times_{T\mathbb{B}} T\mathbb{V}) \times_{(\mathbb{V}^* \times_{\mathbb{B}} \mathbb{V})} (T^*\mathbb{V}^* \times_{\mathbb{B}} T^*\mathbb{V}) \arrow[dl] \arrow[d] \arrow[dr] & \\
T\mathbb{V}^* \times_{\mathbb{V}^*} T^*\mathbb{V}^* \arrow[dr] & T\mathbb{V}^* \times_{T\mathbb{B}} T\mathbb{V} \arrow[d] & T\mathbb{V} \times_{T\mathbb{V}} T^*\mathbb{V} \arrow[dl] \\
& \mathbb{A}^1 &
\end{tikzcd}$$

Let f_1, f_2, f_3 be the three compositions. It is clear that

$$\Gamma(\Upsilon) \subset \mathbb{I}_0 \times_{\mathbb{V}^* \times_{\mathbb{B}} \mathbb{V}} (T^*\mathbb{V}^* \times_{\mathbb{B}} T^*\mathbb{V}).$$

We want to show that $f_1 = -f_3$ when restricted to $T\mathbb{I}_0 \times_{\mathbb{I}_0} \Gamma(\Upsilon) \subset \mathbb{X}$. Let $\mathbb{E}$ be the stack of diagrams

$$(\mathscr{L}, \alpha, \alpha', \gamma, \gamma') : \qquad \begin{tikzcd}
\mathscr{L}^\vee \arrow[d, "\gamma_1"] \arrow[dr, "\gamma_1'"] & & \\
\mathscr{V} \arrow[d, "\gamma_2"] \arrow[r, "\alpha_1'"] & \mathscr{W}' \arrow[d, "\gamma_2'"] \arrow[dr, "\alpha_2'"] & \\
\mathscr{L} \arrow[r, "\alpha_1"'] & \mathscr{W} \arrow[r, "\alpha_2"'] & \mathscr{L}
\end{tikzcd}$$

where each line is a short exact sequence, and γ is a symmetric extension; thus we have natural projections to $(\mathscr{L}, \alpha) \in T\mathbb{B}$ and $(\mathscr{L}, \gamma) \in \mathbb{V}^*$.

The projection $\mathbb{E} \to \mathbb{V}^*$ factors through an epimorphism $\mathbb{E} \to T\mathbb{V}^*$. Indeed, as a stack over $\mathbb{V}^*$ we have

$$\mathbb{E} = \mathrm{Vect}(\tau^{[0,1]} \circ Rp_{X_{\mathbb{V}^*},*}\mathscr{H}om(\tilde{\mathscr{L}}, \tilde{\mathscr{V}}))^*$$

and the map $\mathbb{E} \to T\mathbb{V}^*$ is induced by the inclusion $\mathscr{H}om(\tilde{\mathscr{L}}, \tilde{\mathscr{V}}) \to \mathscr{C}^\vee_{\mathbb{V}^*}[-1]$ (see (13), (14)).

Let $\tilde{\mathbb{X}} := \mathbb{E} \times_{T\mathbb{V}^*} (T\mathbb{I}_0 \times_{\mathbb{I}_0} \Gamma(\Upsilon))$, and let $f \colon \tilde{\mathbb{X}} \to \mathbb{X}$ be the natural map. We wish to show

$$f_1 \circ f = -f_3 \circ f. \tag{18}$$

Suppose, then, given data $(\mathscr{L}, \alpha, \alpha', \gamma, \gamma', v, v'', v')$ defining a point of $\tilde{\mathbb{X}}$. We will write

$$(-,-) \mapsto \mathrm{Tr}(- \cup -)$$

for various pairings given by Serre duality.

First, $f_1 \circ f$ factors through $\mathbb{E} \times_{\mathbb{V}^*} T^*\mathbb{V}^* \to T\mathbb{V}^* \times_{\mathbb{V}^*} T^*\mathbb{V}^* \to \mathbb{A}^1$, which is

$$(\mathscr{L}, \alpha, \alpha', \gamma, \gamma'; v') \mapsto \mathrm{Tr}(v' \cup \alpha'). \tag{19}$$

Similarly, $f_3 \circ f$ factors through $\mathbb{E} \times_{T\mathbb{B} \times_{\mathbb{B}} \mathbb{V}^*} (T\mathbb{V} \times_{\mathbb{V}} T^*\mathbb{V}) \longrightarrow T\mathbb{V} \times_{\mathbb{V}} T^*\mathbb{V} \longrightarrow \mathbb{A}^1$,

$$(\mathscr{L}, \alpha, \alpha', \gamma, \gamma'; v_0, v') \mapsto \mathrm{Tr}(\gamma' \cup v'') - \mathrm{Tr}(v' \cup \alpha'). \tag{20}$$

Finally, $f_2 \circ f$ factors through $\mathbb{E} \times_{T\mathbb{B}} T\mathbb{V} \to T\mathbb{V}^* \times_{T\mathbb{B}} T\mathbb{V} \to \mathbb{A}^1$, which is

$$(\mathscr{L}, \alpha, \alpha', \gamma, \gamma'; v'') \mapsto \mathrm{Tr}(\gamma' \cup v''). \tag{21}$$

Since, however, $f_2 \circ f = 0$, we deduce from (19) and (20) that $f_1 \circ f = -f_3 \circ f$, as desired.

9. Proof of Theorem 1.1

9.1. We calculate $\mathrm{CC}(\Theta_\epsilon)$, following the definition in §5.3 step-by-step.

9.1.1. By 7.1.1(a), the characteristic cycle of the local system $\bar{\mathbb{Q}}_\ell \langle \dim \mathbb{W} \rangle$ on $\mathbb{W}$ is just the zero section: $\mathrm{CC}(\bar{\mathbb{Q}}_\ell \langle \dim \mathbb{W} \rangle) = [\mathbb{W}]$.

9.1.2. Now we calculate $\mathrm{CC}(\sigma_* \bar{\mathbb{Q}}_\ell \langle \dim \mathbb{W} \rangle)|_{T^*\mathbb{V}^\circ}$. The restricted map $\sigma^\circ \colon \mathbb{W}^\circ \to \mathbb{V}^\circ$ satisfies the premises of 7.1.1(c). Therefore, if we define $\mathbb{Y}$ and r by the cartesian square

$$\begin{tikzcd} \mathbb{Y} \arrow[r, "r"] \arrow[d] & \mathbb{W}^\circ \times_{\mathbb{V}^\circ} T^*\mathbb{V}^\circ \arrow[d, "b"] \\ \mathbb{W}^\circ \arrow[r] & T^*\mathbb{V}^\circ \end{tikzcd}$$

we have $\mathrm{CC}(\sigma_* \bar{\mathbb{Q}}_\ell \langle \dim \mathbb{W} \rangle)|_{T^*\mathbb{V}^\circ} = \mathrm{CC}(\sigma^\circ_* \bar{\mathbb{Q}}_\ell \langle \dim \mathbb{W} \rangle) = 2[\mathrm{Im}(b \circ r)]$. The multiplicity 2 arises because σ° has degree 2 over its image, and corresponds to the decomposition (6). Taking f as in (6), we therefore have

$$\mathrm{CC}(f)|_{T^*\mathbb{V}^\circ} = [\mathrm{Im}(b \circ r)]. \tag{22}$$

9.1.3. Before going further, we describe $\mathbb{Y}$ explicitly. First, $\mathbb{W}^\circ \times_{\mathbb{V}^\circ} T^*\mathbb{V}^\circ$ parameterises diagrams

$$(\mathscr{L}, \gamma, w, v') : \quad \begin{tikzcd} \mathscr{L}^\vee \arrow[r, "\gamma_1"] \arrow[d, "w^\vee"'] & \mathscr{V} \arrow[r, "\gamma_2"] \arrow[ddl, "v'"] & \mathscr{L} \\ \Omega \arrow[d, "w"'] & & \\ \mathscr{L} \otimes \Omega & & \end{tikzcd}$$

over $(\mathscr{L}, \gamma, v') \in T^*\mathbb{V}^\circ$ and $(\mathscr{L}, w) \in \mathbb{W}^\circ$. The projection to $T^*\mathbb{W}^\circ$ is given by

$$b(\mathscr{L}, \gamma, w, v') = (\mathscr{L}, w, \beta, w')$$

with $\beta := (w^\vee)_*(\gamma)$ and $w' := w \oplus v' \colon \Omega \oplus_{\mathscr{L}^\vee} \mathscr{V} \to \mathscr{L} \otimes \Omega$. Therefore $\mathbb{Y}$ parameterises diagrams

$$(\mathscr{L}, \gamma, w, w_0) : \quad \begin{tikzcd} \mathscr{L}^\vee \arrow[r, "\gamma_1"] \arrow[d, "w^\vee"'] & \mathscr{V} \arrow[r, "\gamma_2"] \arrow[dl, "w_0"] & \mathscr{L} \\ \Omega & & \end{tikzcd}$$

over $(\mathscr{L}, w) \in \mathbb{W}^\circ$ and $(\mathscr{L}, \gamma, w, v' := w \circ w_0) \in \mathbb{W}^\circ \times_{\mathbb{V}^\circ} T^*\mathbb{V}^\circ$.

9.1.4. Next we apply Theorem 7.2 to describe $\mathrm{CC}(\mathfrak{F}f)|_{T^*\mathbb{V}^{*\circ}}$. According to that corollary, we should check first that $(\bar{\pi}^\circ_{\mathbb{V}})_! \,\mathrm{cf}(f)$ is locally constant on $\mathbb{B}$; but it equals $(\bar{\pi}^\circ_{\mathbb{W}})_! 1$ (pushforward of the constant function), and therefore has locally constant value $h^0(\mathscr{L})$ at $\mathscr{L}$.

Second, we must show that $\mathrm{CC}(f)|_{T^*\mathbb{V}^\circ} \cap (\mathbb{V}^\circ \times_{\mathbb{B}} T^*\mathbb{B})$ is contained in the zero section $\mathbb{V}^\circ$. Now, $\mathbb{V}^\circ \times_{\mathbb{B}} T^*\mathbb{B}$ parameterises diagrams

$$(\mathscr{L}, v, \gamma, v') : \quad \begin{tikzcd} \mathscr{L}^\vee \arrow[r, "\mathrm{id}"] \arrow[d, "v"'] & \mathscr{L}^\vee \oplus \mathscr{L} \arrow[r, "\mathrm{id}"] \arrow[dl, "v'=(v,\phi)"] & \mathscr{L} \\ \mathscr{L} \otimes \Omega & & \end{tikzcd}$$

over $(\mathscr{L},\phi)\in T^*\mathbb{B}$ and $(\mathscr{L},v)\in\mathbb{V}^\circ$; we are required to show that if such $(\mathscr{L},v,\gamma,v')$ lies in $\operatorname{Im}(b\circ r)$, then $\phi=0$. But $(\mathbb{V}^\circ\times_{\mathbb{B}}T^*\mathbb{B})\times_{T^*\mathbb{V}}\mathbb{Y}$ parameterises diagrams

$$(\mathscr{L},\gamma,w,w'):\qquad \begin{tikzcd} \mathscr{L}^\vee \arrow[r, "\mathrm{id}"] \arrow[d, "w^\vee"'] & \mathscr{L}^\vee\oplus\mathscr{L} \arrow[r, "\mathrm{id}"] \arrow[dl, "w'=(w^\vee,\phi_0)"] & \mathscr{L} \\ \Omega & & \end{tikzcd}$$

over $(\mathscr{L},v:=w\circ w^\vee,\gamma,v':=w\circ w')\in\mathbb{Y}$. Since $H^0(\mathscr{L}^\vee)=0$ (see Remark 5.1), we have $\phi_0=0$, hence $\phi=w\circ\phi_0=0$, as desired.

9.1.5. Let $\mathbb{Y}'\subset\mathbb{Y}$ be the open substack consisting of those $(\mathscr{L},\gamma,w,w_0)$ with $(\mathscr{L},\gamma)\in\mathbb{V}^{*\circ}$. From Theorem 7.2 we conclude that $\mathrm{CC}(\mathfrak{F}_{\mathbb{V}}f)|_{T^*\mathbb{V}^{*\circ}}\cap(\mathbb{V}^{*\circ}\times_{\mathbb{B}}T^*\mathbb{B})$ is contained in the zero section $\mathbb{V}^{*\circ}$, and moreover from (22) that

$$\mathrm{CC}(\mathfrak{F}_{\mathbb{V}}f)|_{T^*\mathbb{V}^{*\circ}-\mathbb{V}^{*\circ}}=[\operatorname{Im}(\Upsilon\circ b\circ r|_{\mathbb{Y}'})].$$

Here

$$F:=\Upsilon\circ b\circ r\colon(\mathscr{L},\gamma,w,w_0)\mapsto(\mathscr{L},\gamma,v':=w\circ w_0)\in T^*\mathbb{V}^{*\circ}.$$

9.1.6. Finally, let $\mathbb{M}$, as in the statement of the theorem, be the stack over $\mathrm{Bun}_{\mathrm{Mp}}$ whose fiber over $(\mathscr{V},d)$ is $\operatorname{Hom}(\mathscr{V},\Omega)-\{0\}$. We have $\mathbb{Y}'=\mathbb{V}^{*\circ}\times_{\mathrm{Bun}_{\mathrm{Mp}}}\mathbb{M}=(\mathbb{V}^{*\circ}\times_{\mathrm{Bun}_{\mathrm{Mp}}}T^*\mathrm{Bun}_{\mathrm{Mp}})\times_{T^*\mathrm{Bun}_{\mathrm{Mp}}}\mathbb{M}$, in other words a Cartesian square

$$\begin{tikzcd} \mathbb{Y}' \arrow[r, "p_1"] \arrow[d, "p_2"] & \mathbb{V}^{*\circ}\times_{\mathrm{Bun}_{\mathrm{Mp}}}T^*\mathrm{Bun}_{\mathrm{Mp}} \arrow[r, "a"] \arrow[d, "b"] & T^*\mathbb{V}^{*\circ} \\ \mathbb{M} \arrow[r, "s"] & T^*\mathrm{Bun}_{\mathrm{Mp}} & \end{tikzcd}$$

with s as in Theorem 1.1 and a,b as in 7.1.1(b). By definition of Θ_ϵ,

$$\begin{aligned}\mathrm{CC}((p^{\mathbb{V}^{*\circ}}_{\mathrm{Bun}_{\mathrm{Mp}}})^*\Theta_\epsilon\langle\dim\mathbb{V}^*-\dim\mathrm{Bun}_{\mathrm{Mp}}\rangle)|_{T\mathbb{V}^{*\circ}-\mathbb{V}^{*\circ}}&=\mathrm{CC}(\mathfrak{F}_{\mathbb{V}}f)|_{T\mathbb{V}^{*\circ}-\mathbb{V}^{*\circ}}\\&=[\operatorname{Im}(F|_{\mathbb{Y}'})]=[\operatorname{Im}(a\circ p_1)]\end{aligned}$$

On the other hand, it also equals $a_*b^*\,\mathrm{CC}(\Theta_\epsilon)$. The map $\Upsilon\circ b\circ r\colon\mathbb{Y}'\to T^*\mathbb{V}^*$ is $a\circ p_1$. The only way this can happen is if $\mathrm{CC}(\Theta_\epsilon)|_{T^*\mathrm{Bun}_{\mathrm{Mp}}-\mathrm{Bun}_{\mathrm{Mp}}}=[\operatorname{Im}b\circ p_1]=[\operatorname{Im}s]$. This determines $\mathrm{CC}(\Theta_\epsilon)$ as in the desired statement, up to the multiplicity of the zero section; but that multiplicity is clear from Proposition 6.1. $\square$

9.2. Remark. Instead of appealing to Proposition 6.1, one can calculate the multiplicity of the zero section (and indeed the whole characteristic function of Θ_ϵ) directly, using Lemma 7.3. In any case one has the following result.

Proposition 9.1. *As a constructible function of* $(\mathscr{V},\delta)\in\mathrm{Bun}_{\mathrm{Mp}}$,

$$\mathrm{CF}(\Theta_\epsilon[\dim\mathrm{Bun}_{\mathrm{Mp}}])(\mathscr{V},\delta)=\tfrac{1}{2}((-1)^{h^0(\mathscr{V})}+\epsilon).\tag{23}$$

In particular, the zero section $[\mathrm{Bun}_{\mathrm{Mp}}]$ *occurs with multiplicity* $(1+\epsilon)/2$ *in* $\mathrm{CC}(\Theta_\epsilon)$.

The calculation comes down to finding the Euler characteristic of a quadratic cone, for which one has the following elementary result.

Lemma 9.2. *Let* $s\colon V\to V^*$ *be a symmetric map of vector spaces, and* Λ *the quadratic cone* $\Lambda:=\{v\mid\langle s(v),v\rangle=0\}$. *Then the Euler characteristic*

$$\chi(\mathbb{P}\Lambda)=\dim\ker s+2\lfloor\tfrac{\dim V-\dim\ker s}{2}\rfloor.$$

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