# Structure Constants of a Global Hecke Algebra
## Part I: Project Summary

**0.1.** This project concerns the theory of automorphic forms over function fields. The basic objects are a smooth algebraic curve $X$ over a finite field $\mathbb{F}_q$, and the set $\mathrm{Bun}_n$ of isomorphism classes of rank-$n$ vector bundles on $X$. Our 'automorphic forms' are just finitely supported functions $\mathrm{Bun}_n \to \mathbb{Q}$. The space $C_0$ of all such functions carries an action of the commutative 'global Hecke algebra' $\mathfrak{H} \subset \mathrm{End}\,(C_0)$, which is generated by certain correspondences, called 'Hecke operators.' The structure of $\mathfrak{H}$ is of primary importance. For example, according to the Langlands conjectures (in this situation a theorem of Lafforgue), the spectrum of $\mathfrak{H} \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_\ell$ is the set of $n$-dimensional unramified $\ell$-adic representations of the Weil group (essentially the Galois group) of $X$.

I propose to study the structure constants of $\mathfrak{H}$. In §1 I explain how one can make sense of these structure constants as a function $c$ on $\mathrm{Bun}_n \times \mathrm{Bun}_n \times \mathrm{Bun}_n$. This idea was suggested to me by Vladimir Drinfeld, and a version of it was independently discovered by Maxim Kontsevich, who describes a closely related project in [Ko].

**0.2.** One of the basic ideas in the general theory is to replace functions on the set $\mathrm{Bun}_n$ by complexes of $\ell$-adic sheaves on the underlying moduli stack $\underline{\mathrm{Bun}}_n$. Grothendieck's 'sheaf-function correspondence' (see [Ga]) allows one to recover a function from such a complex. One can thus apply the techniques of algebraic geometry to the study of automorphic forms.

**Question 1.** *Is $c$ geometric, that is, how can one represent it by an $\ell$-adic complex $\underline{c}$ on $(\underline{\mathrm{Bun}}_n)^3$?*

One motivation is that the complex $\underline{c}$ might enable us to define a symmetric monoidal structure on the derived category of $\ell$-adic sheaves on $\underline{\mathrm{Bun}}_n$. The result would be a 'categorified' version of $\mathfrak{H}$.

Even if it is hard to represent $c$ by a complex, it should be relatively easy to find its class in the Grothendieck group of sheaves. If $X$ were a curve in characteristic zero, to such a class would be associated a 'characteristic cycle' in $(T^*\underline{\mathrm{Bun}}_n)^3$, and understanding this cycle might help to construct and understand the underlying complex. In §2.1 I will sketch a way to make sense of the characteristic cycle of $c$ for a curve in characteristic zero, without using sheaves. So let us pose

**Question 2.** *What is the characteristic cycle corresponding to $c$?*

In §2.2 I present a conjectural answer in terms of the Hitchin fibration.

**0.3.** Now suppose $n = 2$. As I explain in §3, $c$ is conjecturally related to a version of Jacobi's theta function. In the classical picture, the theta function may be seen as a function on the metaplectic group $\mathrm{Mp}_{2n}(\mathbb{R})$, the 2-fold cover of $\mathrm{Sp}_{2n}(\mathbb{R})$. In the present situation, $\theta$ is a function on the set $\mathrm{Bun}_{\mathrm{Mp}_8}$ of 'metaplectic bundles' on $X$. One can pull back $\theta$ to a function on $(\mathrm{Bun}_2)^3$, and I denote the result by $\theta_+$.

**Conjecture 1** (Drinfeld)**.** *Up to normalisation, $c = \theta_+$.*

Here are two reasons why the conjecture is important. First, it answers Question 1 in the case $n = 2$, and may help with Question 2: indeed, as Sergey Lysenko [Ly] has explained, $\theta$ corresponds to a perverse sheaf $\underline{\theta}$ on the moduli stack $\underline{\mathrm{Bun}}_{\mathrm{Mp}_8}$, so $\underline{c}$ should just be the pullback of $\underline{\theta}$ to $(\underline{\mathrm{Bun}}_2)^3$. Second, as I explain in §1, there is some ambiguity in the definition of the function $c$; when $n = 2$, the theta function provides a canonical normalisation, and I hope that a proper understanding of Conjecture 1 will allow me to define $c$ on the nose for all $n$.

**0.4.** So far we have considered unramified automorphic forms. A simple generalisation is to allow 'tame ramification' at several points $x_1, x_2, \ldots, x_m \in X(\mathbb{F}_q)$. In §4 I describe some previous calculations giving an explicit formula for $c$ in case $X = \mathbb{P}^1$, $n = 2$, $m = 4$. A similar calculation is presented by Kontsevich in [Ko]. It may be possible to find the characteristic cycle directly in this case, using my calculations and the ideas of [FK].

# Structure Constants of a Global Hecke Algebra
## Part II: Project Description

### 1. Definition of the Structure Constants Function.

**1.1.** The structure constants of an algebra depend on the choice of a basis. As I will explain in §1.2, $C_0$ contains a canonical element $\delta$ that freely generates a large submodule $\mathfrak{H}(\delta) \subset C_0$. If we had $\mathfrak{H}(\delta) = C_0$, we could identify $\mathfrak{H}$ with $C_0$, and since $C_0$ has a basis of delta functions, we would obtain the structure constants of $\mathfrak{H}$ as a function $c$ on $\mathrm{Bun}_n \times \mathrm{Bun}_n \times \mathrm{Bun}_n$. As $\mathfrak{H}(\delta)$ is slightly smaller than $C_0$, $c$ is only defined up to some 'degenerate' functions. For example, in case $n = 2$, a function $f$ on $\mathrm{Bun}_2$ is 'degenerate' if $f(\mathcal{L})$ depends only on the line bundle $\wedge^2\mathcal{L}$.

**1.2.** To be more precise, let $C$ be the space of functions $\mathrm{Bun}_n \to \mathbb{Q}$, so again $C_0 \subset C$ is the space of finitely supported ones. Operators in the Hecke algebra $\mathfrak{H}$ are self-adjoint with respect to the inner product on $C_0$ defined by

$$\langle f, g\rangle := \sum_{\mathcal{L}\in\mathrm{Bun}_n} \frac{1}{|\mathrm{Aut}\,\mathcal{L}|} f(\mathcal{L})g(\mathcal{L}^*). \tag{1}$$

Moreover, this formula identifies $C$ with the dual $C_0^*$.

Here is the definition of $\delta$ when $n = 2$. The definition for $n > 2$ is similar, but requires more notation, so I omit it.

**Definition.** Define the *first Fourier coefficient*[1] function $\delta \in C_0$ by

$$\langle f, \delta\rangle = f(\mathcal{O}_X \oplus \Omega^1_X) - f(\mathcal{N}_0) \qquad \text{for any } f \in C.$$

Here $\mathcal{N}_0$ is the unique non-trivial extension of $\mathcal{O}_X$ by $\Omega^1_X$.

Here is the desired property of $\delta$.

**Lemma.** *The map $\mathfrak{H} \to C_0$ given by $A \mapsto A(\delta)$ is injective. As for the image $\mathfrak{H}(\delta)$, its annihilator $C_d := (\mathfrak{H}(\delta))^\perp \subset C$ coincides with the space of degenerate functions.*

Thus $\mathfrak{H}(\delta)$ becomes an algebra isomorphic to $\mathfrak{H}$, and it acts on $C_0$; let us write $\star$ for the action. The functional

$$\gamma\colon u \otimes v \otimes w \mapsto \langle u \star v, w\rangle$$

makes sense for $u \otimes v \otimes w \in \mathfrak{H}(\delta) \otimes C_0 \otimes C_0$. In fact, because operators in $\mathfrak{H}$ are commutative and self-adjoint, one can naturally extend $\gamma$'s domain to

$$\gamma\colon \mathfrak{H}(\delta) \otimes C_0 \otimes C_0 + C_0 \otimes \mathfrak{H}(\delta) \otimes C_0 + C_0 \otimes C_0 \otimes \mathfrak{H}(\delta) \to \mathbb{Q}.$$

**Definition.** The pairing (1) identifies $(C_0 \otimes C_0 \otimes C_0)^*$ with the completed tensor product $C\hat{\otimes}C\hat{\otimes}C$, i.e. the space of all functions $(\mathrm{Bun}_n)^3 \to \mathbb{Q}$. The **structure constants function** $c$ is the element of $(C\hat{\otimes}C\hat{\otimes}C)/(C_d\hat{\otimes}C_d\hat{\otimes}C_d)$ representing $\gamma$.

### 2. Sheaves and Characteristic Cycles

**2.1. Construction of the Characteristic Cycle.** Given a complex $S$ of $\ell$-adic sheaves on $(\underline{\mathrm{Bun}}_n)^3$, or even just an element of the Grothendieck group of $\ell$-adic sheaves, one obtains two objects: the 'stalk-wise Euler characteristic' $\chi_S$, a constructible function on $(\underline{\mathrm{Bun}}_n)^3$; and the 'characteristic cycle' $C_S$, a Lagrangian cycle in $(T^*\underline{\mathrm{Bun}}_n)^3$. Moreover, there is a map from constructible functions to Lagrangian cycles so that $\chi_S \mapsto C_S$ (see [Go]).

To define the characteristic cycle for our function $c$, it suffices to define the corresponding 'Euler characteristic' function $\chi_c$ on $(\underline{\mathrm{Bun}}_n)^3$. We first define $\chi_c$ over finite fields, and then extend it to fields of characteristic zero (or indeed to any field at all).

To avoid talking about degenerate functions, let us fix from the beginning some *ad hoc* normalisation of $c$.[2]

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[1] The name comes from the equivalent formula $\langle f, \delta\rangle = \sum_{\alpha\in\mathrm{Ext}^1(\mathcal{O}_X,\Omega^1_X)} f([\alpha])\psi(\alpha)$, where $\psi$ is any non-trivial additive character of $\mathrm{Ext}^1(\mathcal{O}_X, \Omega^1_X) = \mathbb{F}_q$, and for any extension $\alpha$, $[\alpha]$ denotes the underlying vector bundle.

[2] E.g. we can require $c(\mathcal{O}_X \oplus \mathcal{A}, \mathcal{O}_X \oplus \mathcal{B}, \mathcal{O}_X \oplus \mathcal{C}) = 0$ for all line bundles $\mathcal{A}, \mathcal{B}, \mathcal{C}$.

**2.1.1.** *Definition of $\chi_c$ over finite fields.* Let $X$ again be defined over $\mathbb{F}_q$, define $X^{[n]} = X \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$, and for $\mathcal{L} \in \mathrm{Bun}_n$ let $\mathcal{L}^{[m]}$ be its pullback to $X^{[m]}$. We then form the zeta function

$$\zeta(\mathcal{L}, \mathcal{M}, \mathcal{N}; T) := \exp\left(\sum c(\mathcal{L}^{[m]}, \mathcal{M}^{[m]}, \mathcal{N}^{[m]}) \cdot T^m/m\right)$$

where $\mathcal{L}, \mathcal{M}, \mathcal{N} \in \mathrm{Bun}_n$ and $T$ is a formal parameter. We expect that $\zeta$ is a rational function of $T$, in which case the following definition makes sense.

**Definition.** Let $\chi_c(\mathcal{L}, \mathcal{M}, \mathcal{N})$ be the order of the zero of $\zeta(\mathcal{L}, \mathcal{M}, \mathcal{N}; T)$ at $T = \infty$.

**2.1.2.** *Definition of $\chi_c$ for any field.* Let $X$ now be a curve over any field $k$ (e.g. of characteristic zero), and fix $\mathcal{L}, \mathcal{M}, \mathcal{N} \in \mathrm{Bun}_n$. Then in fact all of these objects are defined over a subring $R \subset k$ finitely generated over $\mathbb{Z}$. For each closed point $v$ of $\mathrm{Spec}\, R$, one has a curve $X_v$ over a finite field, and hence a number $\chi_c(\mathcal{L}|_{X_v}, \mathcal{M}|_{X_v}, \mathcal{N}|_{X_v})$.

**Definition.** We expect that the number $\chi_c(\mathcal{L}|_{X_v}, \mathcal{M}|_{X_v}, \mathcal{N}|_{X_v})$ will be independent of $v$ after a finite localisation of $R$. Call the generic value $\chi_c(\mathcal{L}, \mathcal{M}, \mathcal{N})$.

**2.2. Conjectural Form of the Characteristic Cycle.** The conjecture describes the characteristic cycle of $c$ in terms of the Hitchin fibration $\pi$. This $\pi$ is a Lagrangian fibration of $T^*\underline{\mathrm{Bun}}_n$ over an affine space $\bar{B}$. Let $B \subset \bar{B}$ be the locus over which $\pi$ is smooth; then $A := \pi^{-1}(B)$ is a commutative group stack over $B$ (the neutral component is a gerbe over an abelian scheme). Let $S$ be the inverse image of the zero section of $A$ by the multiplication $A \times_B A \times_B A \to A$.

**Conjecture 2** (Drinfeld)**.** *The characteristic cycle of $c$, restricted to $A \times A \times A$, is $S$.*

**Remark 1.** Even though the characteristic cycle of $c$ is *a priori* only defined up to degenerate cycles, the statement of the conjecture is precise because such cycles do not intersect $A$.

**Remark 2.** In §3.2 I list some conditions characterising $c$, and these also determine properties of the characteristic cycle. One may hope that these properties essentially determine the cycle (they motivate the conjectural form).

## 3. The Theta Function

**3.1. Definition.** For us, a 'symplectic bundle' is a vector bundle $\mathcal{L}$ on $X$ with a non-degenerate skew pairing $\wedge^2\mathcal{L} \to \Omega^1_X$. Let $\mathrm{Bun}_{\mathrm{Sp}_8}$ be the set of isomorphism classes of rank-8 symplectic bundles over $X$. Lysenko [Ly] has defined the notion of a 'metaplectic bundle' on $X$. Such an object consists of a pair $(\mathcal{L}, \ell)$ where $\mathcal{L} \in \mathrm{Bun}_{\mathrm{Sp}_8}$ and $\ell$ is a square-root of the one-dimensional vector space $\det R\Gamma(X, \mathcal{L})$. Let $\mathrm{Bun}_{\mathrm{Mp}_8}$ denote the set of isomorphism classes of metaplectic bundles on $X$; it has a natural projection to $\mathrm{Bun}_{\mathrm{Sp}_8}$.

There is a canonical function $\theta$ on $\mathrm{Bun}_{\mathrm{Mp}_8}$, known classically as the theta function on the two-fold cover $\mathrm{Mp}_8(\mathbb{A})$ of the adelic group $\mathrm{Sp}_8(\mathbb{A})$. A precise definition can be found in [Ly].

Let $Z \subset (\mathrm{Bun}_2)^3$ be the set of triples $(\mathcal{L}, \mathcal{M}, \mathcal{N})$ of bundles with $(\wedge^2\mathcal{L}) \otimes (\wedge^2\mathcal{M}) \otimes (\wedge^2\mathcal{N}) = \Omega^1_X$. For such a triple, $\mathcal{L}\otimes\mathcal{M}\otimes\mathcal{N}$ is rank-8 bundle with a symplectic structure defined by the natural map

$$\wedge^2(\mathcal{L} \otimes \mathcal{M} \otimes \mathcal{N}) \to (\wedge^2\mathcal{L}) \otimes (\wedge^2\mathcal{M}) \otimes (\wedge^2\mathcal{N}) = \Omega^1_X.$$

**Lemma.** *The tensor product map $t\colon Z \to \mathrm{Bun}_{\mathrm{Sp}_8}$ lifts canonically to $\tilde{t}\colon Z \to \mathrm{Bun}_{\mathrm{Mp}_8}$.*

**Definition.** Define $\theta_+$ to be $\theta \circ \tilde{t}$, extended by zero from $Z$ to all of $(\mathrm{Bun}_2)^3$.

**3.2. Approach to Conjecture 1.** Here is a way one should be able to prove (or disprove) Conjecture 1. The function $c$ should be characterised (up to degenerate functions) by the following properties:

1. Our $c$ is fixed by the action of the symmetric group $S_3$ on $C^{\hat{\otimes}3}$.
2. The three actions of $\mathfrak{H}$ on $C^{\hat{\otimes}3}$ coincide on $c$.

(3) For any $\mathcal{L}, \mathcal{M} \in \mathrm{Bun}_n$, the first Fourier coefficient $\langle c(\mathcal{L}, \mathcal{M}, -), \delta \rangle$ is the number of isomorphisms $\mathcal{L} \to \mathcal{M}$.

It then remains to check conditions (1,2,3) for $\theta_+$, which should be possible because $\theta$ is defined via Fourier decomposition.

**Question 3.** *How can one add further conditions that characterise $c = \theta_+$ on the nose, and not merely up to degenerate functions?*

**Remark.** Even though $\theta$ does not seem to make sense for $n > 2$, one may hope that a good answer to Question 3 would allow one to define structure constants on the nose for any rank.

More generally, the existence of $\theta$ should help me to study general questions about $c$ in the rank-2 case.

**3.3. The Theta Sheaf.** Lysenko described not only the function $\theta$ but also a corresponding perverse sheaf $\underline{\theta}$ on $\mathrm{Bun}_{\mathrm{Mp}_8}$. His definition makes sense in characteristic zero, so Conjecture 1 suggests that in the rank-2 case we ask:

**Question 4.** *What is the characteristic cycle of the complex corresponding to $\theta_+$?*

Question 4 should reduce to the following problem, interesting in its own right:

**Question 4′.** *What is the characteristic cycle of Lysenko's complex $\underline{\theta}$ on $\underline{\mathrm{Bun}}_{\mathrm{Mp}_{2n}}$?*

I hope that Question 4′ will be straightforward using the method of Laumon [La].

## 4. Previous Calculations: Tamely Ramified Case

The calculations in this section are described in more detail in [Th 2]. Kontsevich sketches a closely related calculation in [Ko].

Let $X = \mathbb{P}^1$ over $\mathbb{F}_q$, and fix a set of four points $S = \{x_1, x_2, x_3, x_4\} \subset X(\mathbb{F}_q)$. I will work with automorphic forms on PGL(2), so all the vector bundles are considered up to the action of the Picard group of $X$.

**4.1.** Let P be the set of isomorphism classes of indecomposable 'parabolic bundles.' These are data $(\mathcal{L}, \ell_1, \ell_2, \ell_3, \ell_4)$ where $\mathcal{L}$ is a rank-2 vector bundle and each $\ell_i$ is a line in the fiber $\mathcal{L}|_{x_i}$; 'indecomposable' means that there are only scalar automorphisms. Let $\tilde{\mathrm{P}}$ be the set of isomorphism classes of 'enriched' indecomposable parabolic bundles, meaning that for each $x_i$ we also fix an isomorphism $\phi_i \colon \mathcal{L}|_{x_i}/\ell_i \to \ell_i$. Thus there is a projection $\pi \colon \tilde{\mathrm{P}} \to \mathrm{P}$ on whose fiber acts $(\mathbb{F}_q^\times)^4$. One can show (cf. [AL 2]) that P is the set of $\mathbb{F}_q$ points of a non-separated scheme $\underline{\mathrm{P}}$ having two components, corresponding to even and odd degrees: $\underline{\mathrm{P}} = \underline{\mathrm{P}}_{\mathrm{ev}} \sqcup \underline{\mathrm{P}}_{\mathrm{od}}$, and as it turns out,

$$\underline{\mathrm{P}}_{\mathrm{ev}} \cong \underline{\mathrm{P}}_{\mathrm{od}} \cong (X \text{ with } S \text{ doubled}).$$

Similarly, $\tilde{\mathrm{P}}$ comes from a $(\mathbb{G}_m)^4$-torsor $\pi \colon \underline{\tilde{\mathrm{P}}} \to \underline{\mathrm{P}}$, $\underline{\tilde{\mathrm{P}}} = \underline{\tilde{\mathrm{P}}}_{\mathrm{ev}} \sqcup \underline{\tilde{\mathrm{P}}}_{\mathrm{od}}$.

**4.2.** Now fix a generic[3] character $\mu \colon (\mathbb{F}_q^\times)^4 \to \bar{\mathbb{Q}}^\times$. Our space $C_0$ of automorphic forms consists of the functions $\tilde{\mathrm{P}} \to \bar{\mathbb{Q}}$ that are $\mu$-equivariant on the fibres of $\pi$.

The Hecke algebra $\mathfrak{H}$ acting on $C_0$ is generated by one operator $T_v$ for each $v \in X(\mathbb{F}_q) - S$ and two operators $T_{x_i}^{\pm}$ for each $x_i \in S$. Thus these generators are again parameterised by $X$ with $S$ doubled.

[3]Here is the precise condition on $\mu$. If $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$ where $\mu_i \colon \mathbb{F}_q^\times \to \bar{\mathbb{Q}}^\times$, then we require first that $\mu_i^2 \neq 1$ for each $i$; and second that $\prod \mu_i^{\epsilon_i} \neq 1$ for each $(\epsilon_1, \ldots, \epsilon_4) \in \{\pm 1\}^4$.

There are two purposes for this condition. First, *a priori*, we should consider all parabolic bundles, not just indecomposable ones; but because $\mu$ is generic, any $\mu$-equivariant function is supported on the indecomposable locus. Second, this condition ensures that all our automorphic forms are cuspidal.

**4.3.** As in the unramified case (§1), there is a distinguished function $\delta \in C_0$ representing the 'first Fourier coefficient' functional.[4] The 'structure constants' function $c$ on $(\tilde{\mathrm{P}})^3$ is then defined in exactly the same way as in §1. However, our $c$ is defined on the nose because (in contrast to Lemma 1.2) in this case $\mathfrak{H}(\delta) = C_0$.

**4.4. Formula for the Structure Constants.** I was able to derive a formula for $c$ in terms of the matrix coefficients of the Hecke operators [Th 2]. The formula takes the following geometric format. First of all, $c$ is supported over triples $(\mathcal{L}, \mathcal{M}, \mathcal{N}) \in \underline{\tilde{\mathrm{P}}}^3(\mathbb{F}_q)$ for which the sum of the degrees is even. Let us just describe $c$ as a function on $(\underline{\tilde{\mathrm{P}}}_{\mathrm{od}} \times \underline{\tilde{\mathrm{P}}}_{\mathrm{od}} \times \underline{\tilde{\mathrm{P}}}_{\mathrm{ev}})(\mathbb{F}_q)$. Let $p \colon T \to (\underline{\tilde{\mathrm{P}}}_{\mathrm{od}} \times \underline{\tilde{\mathrm{P}}}_{\mathrm{od}} \times \underline{\tilde{\mathrm{P}}}_{\mathrm{ev}})$ be the trivial bundle with fiber $(\mathbb{G}_m)^4$. There is a certain closed subscheme $M \subset T$ whose generic fiber under $p$ is an affine rational variety of dimension 3. The character $\mu$ defines a function on $M(\mathbb{F}_q)$ and the function $c$ is obtained by summing over the fibers of $p$.

**Remarks.** I was able to interpret the associativity and commutatitivity of $\mathfrak{H}$ in the geometric terms of 4.4. Everything can be described by explicit equations.

The case considered here, with $X = \mathbb{P}^1$ ramified at four points, is closely related to the theory of the sixth Painlevé equation. As explained in [AL, Bo], the situation has many special symmetries, and because of this I do not expect my calculations to generalise directly.

## 5. Other Previous Work

My published work has focused on the theory of the metaplectic group Mp and its 'Weil representation,' which play an important role in the proposed project, as explained in §3. In [Th 1] I described a new theory of the Maslov index (the combinatorial structure underlying the metaplectic group) and in [Th 3] I calculated the character of the Weil representation. A summary of this work is available as [Th 4]. In my master's thesis [Th 5] I gave an account of classical Hecke theory in the language of vector bundles used in this proposal.

## References


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[4] The functional can be defined by evaluation at a canonical element $\mathcal{N}_0' \in \tilde{\mathrm{P}}_{\mathrm{ev}}$ constructed as follows. Let $\mathcal{N}_0$ be the non-trivial extension of $\mathcal{O}_X$ by $\Omega^1_X$. If one makes an upper modification along each of the lines $\Omega^1_X|_{x_i} \subset \mathcal{N}_0|_{x_i}$, one obtains a bundle $\mathcal{N}_0'$ with a canonical enriched parabolic structure.