Structure Constants Formulary (Summary)

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Warning. These notes have not been carefully proof-read.

Contents

1. Basic Setup

Let $X = \mathbb{P}^1$ over $\mathbb{F}_q$, and fix a set of four points $S = \{x_1, x_2, x_3, x_4\} \subset X(\mathbb{F}_q)$. For simplicity of exposition, we work with automorphic forms on $PGL(2)$, so all the vector bundles are considered up to the action of the Picard group of X.

2. Geometry of the space of parabolic ("qp") bundles

Let P be the set of isomorphism classes of indecomposable 'qp' bundles. These are data $(\mathcal{L}, \ell_1, \ell_2, \ell_3, \ell_4)$ where $\mathcal{L}$ is a rank-2 vector bundle and each ℓ_i is a line in the fiber $\mathcal{L}|_{x_i}$. 'Indecomposable' means that we cannot write $\mathcal{L} = \mathcal{A} \oplus \mathcal{B}$ with each l_i lying in one of the summands.

Let $\tilde{\mathrm{P}}$ be the set of isomorphism classes of indecomposable 'eqp' bundles, meaning that for each x_i we also fix an isomorphism $\phi_i : \mathcal{L}|_{x_i}/\ell_i \to \ell_i$. Thus there is a projection $\pi : \tilde{\mathrm{P}} \to \mathrm{P}$ whose fiber is a $(\mathbb{F}_q^\times)^4$-torsor.

Lemma 2.1. *An indecomposable qp bundle has height 0,1,or 2. (By definition, the height of $\mathcal{O}(i) \oplus \mathcal{O}(j)$ is $i - j$ if $i \geq j$.)*

Proof. It suffices to consider $\mathcal{L} = \mathcal{O} \oplus \mathcal{O}(3)$, and to assume that none of the lines l_i lies in the summand $\mathcal{O}(3)$. The space of sections of $\mathcal{L}$ has dimension $\dim H^0(\mathcal{L}) = 5$. Thus there is a trivial summand of $\mathcal{L}$ containing any four lines l_i. $\square$

Lemma 2.2. *There is a unique indecomposable qp bundle $\mathcal{N}$ of height 2. In fact, it has a canonical eqp structure.*

Proof. The same argument as previous lemma shows there is only one. It and the canonical eqp structure can be obtained as follows. Let $\mathcal{N}_0$ be the non-trivial extension of $\mathcal{O}$ by Ω. Take upper modifications along the four lines $\Omega|_{x_i}$. We get a degree 2 bundle $\mathcal{N}$, canonically an extension of $\mathcal{O}$ by $\Omega(S)$. The lines giving the qp structure are the images of $\mathcal{N}_0$ in the fibers $\mathcal{N}|_{x_i}$. The canonical eqp structure comes from the fact that the fibers $\mathcal{N}|_{x_i}$ are essentially trivialized. $\square$

PROPOSITION 1. P *is the set of* $\mathbb{F}_q$ *points of a non-separated scheme* $\mathbf{P}$ *having two components, corresponding to even and odd degrees:* $\mathbf{P} = \mathbf{P}_{\text{ev}} \sqcup \mathbf{P}_{\text{od}}$*, and*

$$\mathbf{P}_{\text{ev}} \cong \mathbf{P}_{\text{od}} = \hat{X} := (X \text{ with } S \text{ doubled}).$$

Similarly, $\tilde{\mathrm{P}}$ *comes from a* $(\mathbb{G}_m)^4$*-torsor* $\pi : \tilde{\mathbf{P}} \to \mathbf{P}$, $\tilde{\mathbf{P}} = \tilde{\mathbf{P}}_{\text{ev}} \sqcup \tilde{\mathbf{P}}_{\text{od}}$.

PROOF. In fact the identification of $\mathbf{P}_{\text{od}}$ with $\hat{X}$ is canonical. Consider $\mathcal{L} \in \mathbf{P}_{\text{od}}$. We can assume $\deg \mathcal{L} = 1$. There is a unique hom $s : O(-1) \to \mathcal{L}$ so that the image of s in $\mathcal{L}|_{x_i}$ is contained in l_i. On the other hand $\mathcal{L}$ is an extension of $\mathcal{O}$ by $\mathcal{O}(1)$; let $p : \mathcal{L} \to \mathcal{O}$. Then $p \circ s : \mathcal{O}(-1) \to \mathcal{O}$ vanishes somewhere. The claim is that the map "location of the zero of $p \circ s$" gives the identification $\mathbf{P}_{\text{od}} = \hat{X}$.

For $\mathbf{P}_{\text{ev}}$: an isomorphism $\mathbf{P}_{\text{ev}} \cong \mathbf{P}_{\text{od}}$ is given by the taking a modification along the line l_1. Or, roughly speaking, we can take the cross ratio of the lines l_i. $\square$

3. The Function Space, Hecke Operators, and Cyclic Vector

Now fix a generic character $\mu : (\mathbb{F}_q^\times)^4 \to \bar{\mathbb{Q}}^\times$. Our space C_0 of automorphic forms consists of the functions on $\tilde{\mathrm{P}}$ that are μ-equivariant on the fibres of π. Write $C_0 = C_{\text{od}} \oplus C_{\text{ev}}$ where functions in C_{od} are supported over $\tilde{\mathrm{P}}_{\text{od}}$, and similarly for C_{ev}.

3.0.1. Here is the precise condition on μ. Write $\mu = \mu_1 \otimes \cdots \otimes \mu_4$. First, we require that $\mu_i^2 \neq 1$ for all i. We also require that $\prod \mu_i^{\epsilon_i} \neq 1$ for any $(\epsilon_1, \ldots, \epsilon_4) \in \{\pm 1\}^4$.

There are two purposes for this latter condition (the first condition is probably not serious). First, *a priori*, we should consider all parabolic bundles, not just indecomposable ones; but, because μ is generic, any μ-equivariant function is supported on the indecomposable locus.

Second, this condition ensures that all our automorphic forms are cuspidal.

3.1. Description of the Hecke operators. The Hecke algebra $\mathfrak{H}$ acting on C_0 is generated by one operator T_v for each $v \in X(\mathbb{F}_q) - S$ and two operators $T_{x_i}^{\pm}$ for each $x_i \in S$. Thus these generators are again parameterized by $\hat{X}(\mathbb{F}_q)$. Each generator interchanges C_{od} and C_{ev}. Moreover, $T_{x_i}^+ T_{x_i}^- = q$.

Consider the Hermitian form on C_0, defined by

$$\langle f, g \rangle := \sum_{\mathcal{L} \in \mathrm{P}} f(\mathcal{L}) \overline{g(\mathcal{L})}$$

(each summand is well-defined, i.e. we can choose an arbitary eqp structure on each $\mathcal{L}$). For $v \notin S$ the Hecke operator T_v is self-adjoint, while $T_{x_i}^+$ is adjoint to $T_{x_i}^-$.

3.2. The Cyclic Vector. The delta function[1] $\delta_{\mathcal{N}}$ (where $\mathcal{N}$ is the canonical eqp bundle, see Lemma 2.2) represents the 'first Fourier coefficient' functional. It is a cyclic vector for $\mathfrak{H}$, by the proof of the multiplicity one theorem and the fact that all our automorphic forms are cuspidal.

4. Key Calculation: Orthogonality of Hecke Basis

PROPOSITION 2. *The functions* $\{T_v \delta_{\mathcal{N}}\}_{v \in \hat{X}}$ *form an orthogonal basis for* C_{od}*, and*

$$\langle T_v \delta_{\mathcal{N}}, T_v \delta_{\mathcal{N}} \rangle = q.$$

Similarly, fix one of the special Hecke operators, say $T_{x_1}^+$*. Then the functions* $\{T_v T_{x_1}^+ \delta_{\mathcal{N}}\}_{v \in \hat{X}}$ *form an orthogonal basis for* C_{ev}*, and*

$$\left\langle T_v T_{x_1}^+ \delta_{\mathcal{N}}, T_v T_{x_1}^+ \delta_{\mathcal{N}} \right\rangle = q^2.$$

[1] To be precise, $\delta_{\mathcal{N}} \in C_0$ is supported over $\pi^{-1}\pi(\mathcal{N})$ and $\delta_{\mathcal{N}}(\mathcal{N}) = 1$.

REMARKS. The second statement follows from the first, using the fact that $T^+_{x_1}$ is adjoint to $T^-_{x_1}$ and $T^+_{x_1}T^-_{x_1} = q$.

The first statement requires a (rather short) calculation. One can say that the Hecke and delta bases are related by "Radon transform."

5. Algebraic Formulae for the Structure Constants.

According to Proposition 2, we have two orthogonal bases for C_0: a basis of Hecke operators as well as the obvious basis of delta functions. Moreover, **proposition 2 allows us to write the structure constants, in either basis, in terms of the matrix coefficients of the Hecke operators.**

5.1. Delta basis. For $\mathcal{L} \in \tilde{\mathrm{P}}_{\mathrm{od}}$ we can express the delta function $\delta_{\mathcal{L}}$ as

$$\delta_{\mathcal{L}} = \tfrac{1}{q} \sum_{v \in \hat{X}(\mathbb{F}_q)} \langle \delta_{\mathcal{L}}, T_v \delta_{\mathcal{N}} \rangle \, T_v \delta_{\mathcal{N}} = \tfrac{1}{q} \sum_{v \in \hat{X}(\mathbb{F}_q)} \overline{T_v \delta_{\mathcal{N}}(\mathcal{L})} \cdot T_v \delta_{\mathcal{N}}$$

so for $\mathcal{L}, \mathcal{M}$ odd and $\mathcal{K}$ even eqp bundles, the structure constants c are given by

$$\boxed{c(\mathcal{L}, \mathcal{M}, \mathcal{K}) = \frac{1}{q^2} \sum_{v,w \in \hat{X}(\mathbb{F}_q)} \overline{T_v \delta_{\mathcal{N}}(\mathcal{L}) \cdot T_w \delta_{\mathcal{N}}(\mathcal{M})} \cdot T_v T_w \delta_{\mathcal{N}}(\mathcal{K}).} \tag{1}$$

For $\mathcal{L}, \mathcal{M}, \mathcal{K}$ all even degree, a similar formula.

5.1.1. *Delta basis, version 2.* We also have the 'smaller' but less symmetrical formula

$$c(\mathcal{L}, \mathcal{M}, \mathcal{K}) = \tfrac{1}{q} \sum_{v \in \hat{X}(\mathbb{F}_q)} \overline{T_v \delta_{\mathcal{N}}(\mathcal{L})} \cdot T_v \delta_{\mathcal{M}}(\mathcal{K}).$$

5.2. Hecke basis. By orthogonality,

$$T_v T_w = \tfrac{1}{q^2} \sum_{T_z} \left\langle T_v T_w \delta_{\mathcal{N}}, T_z T^+_{x_1} \delta_{\mathcal{N}} \right\rangle T_z T^+_{x_1}.$$

So we can define a structure constants function s on $\hat{X}(\mathbb{F}_q)^3$,

$$\boxed{s(T_v, T_w, T_z) := \left\langle T_v T_w \delta_{\mathcal{N}}, T_z T^+_{x_1} \delta_{\mathcal{N}} \right\rangle.} \tag{2}$$

5.3. Eigenfunction basis. If $\{g\}$ is a complete set of Hecke eigenfunctions with $T_v g = t^g_v g$ then it is easy to prove formulas like

$$s(T_v, T_w, T_z) = \sum_g \frac{t^g_v t^g_w \overline{t^g_z t^{+g}_{x_1}}}{\langle g, g \rangle}$$

where, moreover, $\langle g, g \rangle = \frac{2}{q} \sum_v |t_v|^2$.

6. 'Geometric Reformulation' of the Algebraic Formulae

Let us give a geometric interpretation of the formula (2). Of course one can (and probably should) work with the delta basis instead. But the bases are closely related (by "Radon transform") and the Hecke basis is more convenient because: (a) The structure constants function is defined on $\hat{X}(\mathbb{F}_q)^3$ rather than on a torsor over this space; (b) it is slightly annoying to formulate "associativity" for the delta basis (because of odd vs even degree).

6.1. Format for the geometric description. Let $p : T \to \hat{X}^3$ be the trivial bundle with fiber $(\mathbb{G}_m)^4$. There is a certain closed subscheme $M \subset T$ whose generic fiber under p is an affine rational variety of dimension 3. The character μ defines a function on $M(\mathbb{F}_q)$ and the function s is obtained by summing over the fibers of p.

6.2. Definition of the space M. M is the moduli space of diagrams

$$M(\mathbb{F}_q) := \left\{ \left(\mathcal{N} \overset{T_v}{\hookrightarrow} \mathcal{K}_1 \overset{T_w}{\hookrightarrow} \mathcal{K}_2 \overset{T_z}{\hookleftarrow} \mathcal{K}_3 \overset{T^+_{x_1}}{\hookleftarrow} \mathcal{N} \right) \right\}.$$

Let me explain the notation. $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$ are qp bundles that are allowed to vary, and T_v, T_w, T_z are allowed to vary in $\hat{X}(\mathbb{F}_q)$. An arrow like $\mathcal{N} \overset{T_v}{\hookrightarrow} \mathcal{K}_1$ means that we represent $\mathcal{N}$ as a lower modification of $\mathcal{K}_1$ in the sense used to define the Hecke operator T_v.

6.3. The map $p : M \to \hat{X}^3$. It is just the projection to (T_v, T_w, T_z). The fiber is clearly rational since it is parameterized by quadruples of lines (the lines along which we take the modifications at v, w, z, x_1). For the same reason it is easy to see that the fiber is affine. The fiber is actually three dimensional, not four, because if we fix $\mathcal{K}_3$, there is a unique way to modify $\mathcal{K}_3$ at x_1 to get $\mathcal{N}$.

6.4. The embedding $\mathrm{pdet} : M \to T$ **over** $\hat{X}^3$**.** Given an element of $M(\mathbb{F}_q)$ represented by a diagram as shown above, $\mathcal{K}_2$ inherits two different eqp structures (because it is represented as a modification of $\mathcal{N}$ in two different ways). The ratio of these eqp structures is an element of $(\mathbb{F}_q^\times)^4$. Thus we obtain a map $\mathrm{pdet} : M \to T$. One can show that pdet is a closed embedding.

Remark 1. Everything can be described by explicit equations.

Remark 2. A very similar picture holds for the delta basis (except now "T" is a bundle over $\tilde{\mathbf{P}}^3$, not just over $\hat{X}^3$.)

7. Associativity Property.

Associativity for structure constants s means, for every u, v, w and y

$$\sum_{T_z} s(T_u, T_v, T_z) s(T_z, T_w, T_y) = \sum_{T_z} s(T_v, T_w, T_z) s(T_z, T_u, T_y). \tag{3}$$

7.1. Algebraic argument. In terms of the Hermitian product, (3) becomes

$$\sum_{T_z} \left\langle T_u T_v, T_z T^+_{x_1} \right\rangle \left\langle T_z T_w, T_y T^+_{x_1} \right\rangle = \sum_{T_z} \left\langle T_v T_w, T_z T^+_{x_1} \right\rangle \left\langle T_z T_u, T_y T^+_{x_1} \right\rangle .$$

But the left-hand-side, for example, is

$$\sum_{T_z} \left\langle T_u T_v T^{+\,*}_{x_1}, T_z \right\rangle \left\langle T_z, T^*_w T_y T^+_{x_1} \right\rangle$$

which, by the orthogonality of the Hecke basis, is just

$$= \left\langle T_u T_v T^{+\,*}_{x_1}, T^*_w T_y T^+_{x_1} \right\rangle = \left\langle T_u T_v T_w, T_y {T^+_{x_1}}^2 \right\rangle .$$

Similar manipulations on the right-hand-side give the same result.

7.2. 'Geometric' reformulation. For every $u, v, w, y \in \hat{X}(\mathbb{F}_q)$, we want to consider the spaces

$$A^1_{u,v,w,y}(\mathbb{F}_q) = \bigcup_{T_z} M_{u,v,z} \times M_{z,w,y}$$
$$A^2_{u,v,w,y}(\mathbb{F}_q) = \bigcup_{T_z} M_{v,w,z} \times M_{z,u,y}$$

where $M_{u,v,z}$ is the fiber of M over (u, v, z) (see 6.3). In other words, A^1, A^2 are collections of diagrams:

$$A^1(\mathbb{F}_q) = \left\{ \mathcal{N} \overset{T_u}{\hookrightarrow} \mathcal{K}_1 \overset{T_v}{\hookrightarrow} \mathcal{K}_2 \overset{T^+_{x_1}}{\hookleftarrow} \mathcal{K}_3 \overset{T_z}{\hookleftarrow} \mathcal{N} \overset{T_z}{\hookrightarrow} \mathcal{L}_3 \overset{T_w}{\hookrightarrow} \mathcal{L}_2 \overset{T^+_{x_1}}{\hookleftarrow} \mathcal{L}_1 \overset{T_y}{\hookleftarrow} \mathcal{N} \right\}.$$

$$A^2(\mathbb{F}_q) = \left\{ \mathcal{N} \overset{T_v}{\hookrightarrow} \mathcal{K}_1 \overset{T_w}{\hookrightarrow} \mathcal{K}_2 \overset{T^+_{x_1}}{\hookleftarrow} \mathcal{K}_3 \overset{T_z}{\hookleftarrow} \mathcal{N} \overset{T_z}{\hookrightarrow} \mathcal{L}_3 \overset{T_u}{\hookrightarrow} \mathcal{L}_2 \overset{T^+_{x_1}}{\hookleftarrow} \mathcal{L}_1 \overset{T_y}{\hookleftarrow} \mathcal{N} \right\}.$$

As in 6.4, $A^1_{u,v,w,y}$ and $A^2_{u,v,w,y}$ come with maps pdet to $\mathbb{G}_m^4$, and **we want to show that**

$$\sum_{E \in A^1_{u,v,w,y}(\mathbb{F}_q)} \mu \circ \mathrm{pdet}(E) = \sum_{E \in A^2_{u,v,w,y}(\mathbb{F}_q)} \mu \circ \mathrm{pdet}(E)$$

Definition. Inside A^i consider the divisor D^i defined by $\mathcal{K}_3 = \mathcal{L}_3$.

Proposition 3.

(1) *There is a natural isomorphism* $D^1_{u,v,w,y} \to D^2_{u,v,w,y}$ *over* $\mathbb{G}_m^4$.

(2) *The sum of* $\mu \circ \mathrm{pdet}$ *over the complement of* $D^i_{u,v,w,y}$ *in* $A^i_{u,v,w,y}$ *vanishes, i.e.*

$$\sum_{E \in A^i_{u,v,w,y}(\mathbb{F}_q)} \mu \circ \mathrm{pdet}(E) = \sum_{E \in D^i_{u,v,w,y}(\mathbb{F}_q)} \mu \circ \mathrm{pdet}(E).$$

Sketch of Proof. **(1)** Fix $\mathcal{K}_3 = \mathcal{L}_3$. For any fixed T_z, $\mathcal{K}_3$ occurs as a modification of $\mathcal{N}$ in at most one way (at least generically). So essentially

$$D^1 \approx \left\{ \mathcal{N} \overset{T_u}{\hookrightarrow} \mathcal{K}_1 \overset{T_v}{\hookrightarrow} \mathcal{K}_2 \overset{T^+_{x_1}}{\hookleftarrow} \mathcal{K}_3 = \mathcal{L}_3 \overset{T_w}{\hookrightarrow} \mathcal{L}_2 \overset{T^+_{x_1}}{\hookleftarrow} \mathcal{L}_1 \overset{T_y}{\hookleftarrow} \mathcal{N}) \right\} \times \{T_z\}$$

and

$$D^2 \approx \left\{ \mathcal{N} \overset{T_v}{\hookrightarrow} \mathcal{K}_1 \overset{T_w}{\hookrightarrow} \mathcal{K}_2 \overset{T^+_{x_1}}{\hookleftarrow} \mathcal{K}_3 = \mathcal{L}_3 \overset{T_u}{\hookrightarrow} \mathcal{L}_2 \overset{T^+_{x_1}}{\hookleftarrow} \mathcal{L}_1 \overset{T_y}{\hookleftarrow} \mathcal{N} \right\} \times \{T_z\}.$$

But now these diagrams are in bijection because the Hecke operators commute.

(2) The idea is that $A^1 - D^1$ (sim. for $i = 2$) is fibred over a base

$$\left\{ \mathcal{N} \overset{T_u}{\hookrightarrow} \mathcal{K}_1 \overset{T_v}{\hookrightarrow} \mathcal{K}_2 \overset{T^+_{x_1}}{\hookleftarrow} \mathcal{K}_3 \neq \mathcal{L}_3 \overset{T_w}{\hookrightarrow} \mathcal{L}_2 \overset{T^+_{x_1}}{\hookleftarrow} \mathcal{L}_1 \overset{T_y}{\hookleftarrow} \mathcal{N} \right\}$$

with fibres isomorphic to $\mathbb{G}_m$ (i.e. the fiber parameterizes the choice of $T_z \in \hat{X}$, but only $\mathbb{G}_m$ worth of choices are admissible). The sum of $\mu \circ \mathrm{pdet}$ over each fibre is the sum of a non-trivial character. $\square$

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