COBORDISMS, CATEGORIES, QUADRATIC FORMS

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1. Introduction

These are some notes of a talk I gave at the Oxford topology seminar on 11 February 2008. The first object, in §3, is to explain how the classical theory of the signature of manifolds can be formulated as a TFT, that is, as a functor from the cobordism category to a certain symmetric monoidal category of quadratic forms. I then sketch two closely related generalisations. The first, in §5, describes the Maslov index as the signature on a category of "decorated cobordisms." The second, in §7 explains how to define a signature 2-functor on a 2-category of cobordisms.

So far, of course, this is a story of old wine in new bottles. One point of categories is to clarify structural relationships, and for me, that is already a powerful motive. A more concrete point is the role of categories in K-theory since Quillen. In §4 I describe very briefly a connection between the present work and Hermitian K-theory. In some sense these notes are prospective to studying that relationship.

Finally, in §8, I explain in categorical terms the relationship between the Maslov index and the universal covering space of the Lagrangian Grassmannian (or of the symplectic group). This was one reason the Maslov index was originally studied (by Leray, Hormander, Kashiwara, etc.; another route was through Wall's work in topology, as in §6). On the other hand, this clean categorical statement is hard to find; in the TFT literature, one sees vague and possibly incorrect statements.

Serge Lang once told me that what is called 2π should have been called π. In a similar spirit, my "Maslov index" τ in these notes differs from the traditional one by a sign.

2. Signatures

Let F be any field; when F has characteristic 2, one probably has to be more careful than I have been.

2.1. Quadratic spaces. A *quadratic space* is a vector space V over F equipped with a non-degenerate quadratic form q. *Non-degenerate* means that $V^\perp = 0$.

Now suppose our ground field $F = \mathbb{R}$. The signature of V can be defined (as apparently by Sylvester) in the following way. Choose an orthonormal basis $\{e_i\}$ for V, so that $q(e_i) = \pm 1$. Then

$$\mathrm{Sig}(V) := \sum q(e_i).$$

It is an integer, and is independent of the choice of basis.

When $F \neq \mathbb{R}$, the signature of V can be defined not as an integer but as a class in the Witt group $W(F)$, as I explain in §3.

2.2. Manifolds. Let M be a closed oriented $4n$-manifold. Then $V_M := H^{2n}(M, F)$ is a quadratic space under Poincaré duality. That is, one has the cup product $V_M \otimes V_M \to H^{4n}(M, F)$, and the orientation gives an isomorphism $H^{4n}(M, F) \to F$.

One now defines (following Thom) $\mathrm{Sig}(M) := \mathrm{Sig}(V_M)$, and one has the following well-known properties.

(a) $\mathrm{Sig}(M \sqcup N) = \mathrm{Sig}(M) + \mathrm{Sig}(N)$

(b) $\mathrm{Sig}(\partial M) = 0$ whenever M is a $(4n+1)$-manifold.

In other words, Sig is a cobordism invariant.

3. The Signature Functor

Let $(4n+1)$-cob be the category of cobordisms between closed, oriented $4n$-manifolds. It is a symmetric monoidal (under $\sqcup$) category with duality (orientation-reversal). The idea of this section is to define a corresponding symmetric monoidal category Q^+ of quadratic spaces and a monoidal 'signature' functor $(4n+1)$-cob $\to Q^+$. One recovers the classical signature theory from the fact that $\pi_0(Q^+) = \mathbb{Z}$ when $F = \mathbb{R}$; in general, $\pi_0(Q^+)$ is the Witt group $W(F)$.

3.1. **The monoidal category Q^+.** Here is the definition.

Objects:	Quadratic spaces.
Monoidal structure:	Direct sum.
Duals:	If $V = (V, q)$ then the dual is $\overline{V} := (V, -q)$.
Morphisms:	$\mathrm{Hom}_{Q^+}(V, W) := \mathrm{Lagr}(\overline{V} \oplus W)$.

Here $\mathrm{Lagr}(V) := \{L \subset V \mid L = L^\perp\}$ is the Lagrangian Grassmannian of V.

Thus a morphism is a certain kind of correspondence between V and W. Composition is given by the composition of correspondences. It is easy to see, but a bit surprising, that this composition is well-defined. The monoidal and dual structures affect morphisms in an obvious way.

Theorem 3.1. *There is a functor* $\underline{\mathrm{Sig}}\colon (4n+1)$*-cob* $\to Q^+$ *defined as follows.*

(a) *If M is a $4n$-manifold, define* $\underline{\mathrm{Sig}}(M) = V_M$.

(b) *For N a $(4n+1)$-manifold, define* $\underline{\mathrm{Sig}}(N) = \mathrm{image}[H^{2n}(N, F) \to \underline{\mathrm{Sig}}(\partial N)]$.

3.2. **Symplectic version.** We also define a category Q^- in the same way, but using symplectic spaces rather than quadratic spaces as the objects. Then one gets a functor $\underline{\mathrm{Sig}}\colon (4n+3)$-cob $\to Q^-$.

Since $\pi_0(Q^-) = 0$, the corresponding classical theory is trivial (or, anyway, one needs to be cleverer). However, Q^- has non-trivial higher homotopy groups.

4. Digression on K-theory.

In the literature on on Hermitian K-theory, it is not Q^+ that appears, but its subcategory Q^+_{mon}, having as its morphisms all monomorphisms in Q^+. This is a version of "Giffen's category". It was proved by Marco Schlichting [Sch] that there is a homotopy fibration sequence

$$K \to K^+ \to Q^+_{\mathrm{mon}}$$

where the first map, from usual to Hermitian K-theory, is the hyperbolic map: it is induced by the functor that takes a vector space V to the quadratic space $(V \oplus V^*, (a, \lambda) \mapsto \lambda(a))$.

5. Decorated Manifolds and the Maslov Index.

One can give a simple-minded generalisation of cobordism categories by replacing the constant sheaf F by a more general sheaf satisfying a 'self-duality' property.

Let us say that a *decorated manifold* is a triple (M, P, ϕ) where M is an oriented manifold with boundary, P a constructible sheaf of F-vector spaces, and $\phi\colon P \otimes P \to F$ a (skew, say) pairing satisfying the following property:

Let j be the inclusion of the interior of M into M, and let $\mathbb{D}$ be Verdier duality. Then we require that the natural map

$$j_! j^* P \to \mathbb{D}P[-\dim M],$$

induced by ϕ, is an isomorphism.

The idea is to mimic the duality between $H^i(M, F)$ and $H^{\dim M - i}(M, \partial M, F)$.

The boundary of a decorated manifold, equipped with the restrictions of P and ϕ, is again decorated. As before, one obtains a symmetric monoidal category $(4n+1)\text{-}\widetilde{\mathrm{cob}}$ of decorated cobordisms, and a signature functor

$$\underline{\mathrm{Sig}}\colon (4n+1)\text{-}\widetilde{\mathrm{cob}} \to Q^-$$

that takes a decorated $4n$-manifold (M, P, ϕ) to $H^{2n}(M, j_! j^* P)$. This last is naturally a symplectic space, because I have (arbitrarily) assumed that each ϕ is a skew pairing. One similarly gets a functor $(4n+3)\text{-}\widetilde{\mathrm{cob}} \to Q^+$.

5.1. **The Maslov Index.** As explained in [Th], the Maslov index appears naturally in the context of decorated cobordisms. To a symplectic space V and Lagrangians $L_1, \ldots, L_n \in \mathrm{Lagr}(V)$ one associates a quadratic space and therefore a class $\tau(L_1, L_2, \ldots, L_n) \in \pi_0(Q^+)$ in the following way.

On a 2-sphere S draw an n-gon G and define a sheaf P on S that is constructible with respect to the stratification defined by the vertices, the edges, and the interior and exterior of G. Outside G, the stalk of P is 0; inside, the stalk is V; the stalk on the ith edge is L_i, and the stalk at the vertex between edges i and j is $L_i \cap L_j$. The transition maps between stalks are the natural inclusions.

The symplectic form on V defines a pairing $\phi\colon P \otimes P \to F$, and, as explained in [Th], the resulting triple (M, P, ϕ) is a decorated manifold whose signature is $\tau(L_1, \ldots, L_n)$.

Moreover, one can prove identities involving the Maslov index by considering decorated cobordisms between decorated manifolds. For example, one has

$$\tau(L_1, L_2, L_3) + \tau(L_1, L_3, L_4) = \tau(L_1, L_2, L_3, L_4). \tag{1}$$

This identity and its corollary (2) play a key role in the classical theory of the Maslov index as developed by Kashiwara [KS] and others.

6. Wall's Non-Additivity

Now I push a little further to consider 2-categories of cobordisms. Suppose that M, N are $4n$-manifolds, and that we have boundary decompositions

$$\partial M = X_1 \cup_A X_2 \qquad \partial N = X_2 \cup_A X_3$$

with the X_i being $(4n-1)$-manifolds with boundary $A = X_1 \cap X_2 = X_2 \cap X_3$. Then one can glude M, N along X_2 to get a new manifold M' with boundary $X_1 \cup_A X_3$.

In this situation, when M has a boundary, it is $W := H^{2n}(M, \partial M, F)$ that carries a quadratic form; this quadratic form, however, is usually degenerate, and one defines $\mathrm{Sig}(M)$ to be the signature of the quadratic space $W/W^\perp$.

Wall's theorem [Wa] states:

Theorem 6.1. $\mathrm{Sig}(M') = \mathrm{Sig}(M) + \mathrm{Sig}(N) + \tau(L_1, L_2, L_3)$, *where* $L_i := \underline{\mathrm{Sig}}(X_i)$ *is a Lagrangian in the symplectic space* $\underline{\mathrm{Sig}}(A)$*.*

Wall considered the case $F = \mathbb{R}$, but the result makes sense and is true in general.

7. The Signature 2-Functor

Let $(4n)\text{-cob}_2$ be the 2-category of cobordisms (so an object is a closed, oriented $(4n-2)$-manifold). The object here is to define a 2-categorical version of Q^- that will receive a 2-functor from $(4n)\text{-cob}_2$.

7.1. **Step 1.** Given a symplectic space V, define the Maslov category M_V to have $\mathrm{Lagr}(V)$ as its set of objects, $\mathrm{Hom}_{M_V}(L_1, L_2) = W(F)$, and composition given by

$$\begin{tikzcd} & L_2 \arrow[dr, "q_2"] & \\ L_1 \arrow[ur, "q_1"] \arrow[rr, "q_1+q_2+\tau(L_1,L_2,L_3)"'] & & L_3 \end{tikzcd}$$

The only problem is to show that this composition is associative. This boils down to checking the cocycle condition

$$\tau(L_1, L_2, L_3) + \tau(L_1, L_3, L_4) = \tau(L_1, L_2, L_4) + \tau(L_2, L_3, L_4). \tag{2}$$

According to (1), both sides equal $\tau(L_1, L_2, L_3, L_4)$.

7.2. **Step 2.** Define a 2-category Q_2^-.

Objects:	Symplectic spaces.
Morphisms:	$\mathrm{Hom}_{Q_2^-}(V, W)$ is the category $M_{\overline{V}\oplus W}$.
Composition	
on objects:	composition of Lagrangian correspondences.
on morphisms:	addition in W_F twisted by the Maslov index.

To be precise, suppose given

$$q_1 \in W_F = \mathrm{Hom}_{M_{\overline{V}\oplus W}}(L_1, L_2) \qquad q_2 \in W_F = \mathrm{Hom}_{M_{\overline{W}\oplus X}}(M_1, M_2).$$

Then the composition $q_3 \in W_F = \mathrm{Hom}_{M_{\overline{V}\oplus X}}(M_1 \circ L_1, M_2 \circ L_2)$ equals

$$q_1 + q_2 + \tau(M_1 \oplus L_1, M_1 \circ L_1 \oplus \Delta_W, M_2 \oplus L_2).$$

Here Δ_W is the diagonal in $\overline{W} \oplus W$ and the Maslov index is of Lagrangians in $\overline{V} \oplus W \oplus \overline{W} \oplus X$.

That this composition is associative and functorial can be proved in the same spirit as (2).

7.3. **Signature 2-functor.**

Theorem 7.1. *There is a functor $\underline{\mathrm{Sig}}_2\colon (4n)\text{-}cob_2 \to Q_2^-$ defined as follows.*

(a) *If M is a $(4n-2)$-manifold then $\underline{\mathrm{Sig}}_2(M)$ is the symplectic space $\underline{\mathrm{Sig}}(M)$.*
(b) *If L is a $(4n-1)$-manifold then $\underline{\mathrm{Sig}}_2(L)$ equals $\underline{\mathrm{Sig}}(L) \in \mathrm{Lagr}(\underline{\mathrm{Sig}}(\partial L))$.*
(c) *If N is a $4n$-manifold then $\underline{\mathrm{Sig}}_2(N)$ equals $\mathrm{Sig}(N) \in W(F)$.*

7.4. **Higher Categories.** When defining the Maslov categories M_V, it is tempting to let the morphisms $\mathrm{Hom}_{M_V}(L_1, L_2)$ to be the category Q^+ instead of the set $W(F) = \pi_0(Q^+)$. The result would be a *lax* 2-category.[1]

One can try to "keep going" to define an ω-categorical version of M_V and thence of Q^+. However, taking cobordism theory into account, it seems more natural to start dealing with quadratic complexes rather than vector spaces, in the style of Balmer [Ba] and Ranicki [Ra].

[1] In a usual weak 2-category one has isomorphisms of the form $f_1 \circ (f_2 \circ f_3) \cong (f_1 \circ f_2) \circ f_3$. *Lax* means, rather, that one has n-ary composites for each n, and morphisms (not necessarily isomorphisms) of the form

$$f_1 \circ (f_2 \circ f_3) \to f_1 \circ f_2 \circ f_3 \leftarrow (f_1 \circ f_2) \circ f_3.$$

See [Le] for more details.

8. Relation to the fundamental groupoid

The Maslov category M_V associated to a symplectic space V is a $W(F)$-gerbe, that is, a connected groupoid such that the automorphism group of any object is $W(F)$. The action of these automorphism groups on the hom-sets makes each hom-set into a $W(F)$-torsor.

When $F = \mathbb{R}$, $W(F) = \mathbb{Z}$; on the other hand, $\mathrm{Lagr}(V)$ is a manifold with $\pi_1(\mathrm{Lagr}(V)) = \mathbb{Z}$, making the fundamental groupoid $\Pi_V := \Pi_1(\mathrm{Lagr}(V)$ into a $\mathbb{Z}$-gerbe as well.

Since M_V and Π_V have "the same" objects and morphisms, it is natural to conjecture (and common to assume) that the identity map $M_V \to \Pi_V$ extends to an isomorphism of categories; but this is not true.

What is true is that the identity map on objects extends to a functorial isomorphism

$$\Pi_V^{\otimes 2} \to M_V.$$

Here $\Pi_V^{\otimes 2}$ is the following $\mathbb{Z}$-groupoid. It has the same objects as Π_V, but the set of morphisms $\mathrm{Hom}_{\Pi_V^{\otimes 2}}(L, L')$ is the tensor product $\mathrm{Hom}_{\Pi_V}(L, L') \otimes_{\mathbb{Z}} \mathrm{Hom}_{\Pi_V}(L, L')$ of $\mathbb{Z}$-torsors; composition is induced by that in Π_V.

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