Unramified Hecke Theory for $\mathbf{GL_2}$ over a Function Field

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My aim is to give a self-contained account of the Hecke theory of unramified automorphic forms and L-functions for GL_2 over a function field. In this case an automorphic form is a function on the set of isomorphism classes of rank-two vector bundles, and the question "when does an irreducible admissible representation of $GL_2(\mathbb{A})$ appear in the space of cusp forms?" amounts to the more classical question, "when does a given collection of numbers appear as eigenvalues of Hecke operators on the space of cusp forms?"

The answer has two parts: first, I prove the Multiplicity One Theorem, that for any character η on the Picard group Pic there exists a cusp function Φ on the space of flags, unique up to scale, that transforms via η under the action of Pic and that is a Hecke eigenvector with the given eigenvalues. Next, I define the L-function associated to Φ, and show that Φ descends to a function on bundles if and only if the L-function is a polynomial satisfying a certain functional equation.

This work essentially reinterprets the adelic proofs of Weil [1] in the language of vector bundles (although he works in a more general setting). I thank Vladimir Drinfeld for proposing this topic and making many helpful suggestions.

Contents

I. Preliminaries.

Let X be a smooth projective curve over the field of constants $\mathbb{F}_q$. By 'a point of X' we always mean a closed one. $\mathcal{O}$ is the structure sheaf of X and Ω the sheaf of regular differentials. Fix also a field F of characteristic zero in which all our functions and characters take values. For any function or section f, denote by $\operatorname{div} f$ its associated divisor. Div^+ is the semigroup of effective divisors, including zero. If D is an effective divisor, then there is a subscheme of X, also denoted D, corresponding to the ideal sheaf $\mathcal{O}(-D)$.

1. Vector Bundles, Flags, and Extensions

In our situation, an automorphic form is a function on the set of isomorphism classes of rank-two vector bundles on X. In this section we lay out some basic definitions and conventions concerning vector bundles and related objects.

1.1. By a *vector bundle* we mean a locally free coherent sheaf. If $\mathcal{L}$ is a vector bundle on X and D is an effective divisor, then write $\mathcal{L}|_D$ for the quotient sheaf $\mathcal{L}/\mathcal{L}(-D)$. This sheaf is supported on D, and we can view it as a vector bundle on the sub-scheme D—it is a free module over $\mathcal{O}|_D$, which is itself naturally the structure sheaf of D. If $D = v$ is a single point, then we can identify $\mathcal{L}|_v$ with the fibre of $\mathcal{L}$ at v.

Let us draw a distinction between *vector sub-bundles* and *locally free sub-sheaves* of a given vector bundle $\mathcal{L}$: a locally free sub-sheaf $\mathcal{A} \subset \mathcal{L}$ of rank n is a vector sub-bundle of $\mathcal{L}$ if the image of $\mathcal{A}$ in every fibre $\mathcal{L}|_v$ is a vector sub-space of dimension n. Equivalently, a vector sub-bundle is maximal among locally free sub-sheaves of the same rank. Equivalently, the quotient of a vector bundle by a vector sub-bundle is again a vector bundle.

1.2. By a *line bundle* we mean a vector bundle of rank 1. As usual, the Picard group Pic is the group of isomorphism classes of line bundles equipped with tensor product $\otimes$, which symbol we frequently omit. We will write $\mathcal{A}^\vee$ for the inverse of $\mathcal{A}$.

If $D > 0$ is an effective divisor, *a line bundle with level-D structure* is a pair $(\mathcal{A}, \sigma)$ consisting of a line bundle $\mathcal{A}$ and a trivialisation $\sigma : \mathcal{O}|_D \xrightarrow{\sim} \mathcal{A}|_D$ of $\mathcal{A}$ over D. The trivialisation can also be seen as a section $s = \sigma(1) \in H^0(\mathcal{A}|_D)^\times := \{s \in H^0(\mathcal{A}|_D) \text{ with } \operatorname{div} s = 0\}$.

Two line bundles with level-D-structure $(\mathcal{A}, s)$ and $(\mathcal{A}', s')$ are called isomorphic if there is an isomorphism $f : \mathcal{A} \to \mathcal{A}'$ with $s' = f(s)$. So $(\mathcal{A}, s) \cong (\mathcal{A}, s')$ if and only if $s = \lambda s$ for some $\lambda \in \mathbb{F}_q^\times$.

The isomorphism classes of line bundles with level-D-structure again form a group Pic_D under tensor product. If C is an effective divisor disjoint from D, then $\mathcal{O}(C)$, considered as a subsheaf of the constant sheaf of rational functions, has a natural level-D structure 1_C that is the restriction of the rational function 1.

For $D = 0$ let us agree that formally $H^0(\mathcal{A}|_0)^\times = H^0(\mathcal{A}|_0) = \{1\}$, so that $\mathrm{Pic}_0 = \mathrm{Pic} \times \{1\} \cong \mathrm{Pic}$.

1.3. Let Bun be the set of isomorphism classes of rank-two vector bundles on X.

1.4. A *flag* is a pair $(\mathcal{L}, \mathcal{A})$ with $\mathcal{L}$ a rank-two vector bundle and $\mathcal{A}$ a line sub-bundle of $\mathcal{L}$. Two flags $(\mathcal{L}, \mathcal{A})$ and $(\mathcal{L}', \mathcal{A}')$ are said to be isomorphic if there is a commutative diagram of the form

$$\begin{array}{ccc} \mathcal{A} & \subset & \mathcal{L} \\ \cong\downarrow & & \downarrow\cong \\ \mathcal{A}' & \subset & \mathcal{L}'. \end{array}$$

Let Flag denote the set of isomorphism classes of flags.

1.5. For any two line bundles $\mathcal{A}$ and $\mathcal{B}$, write $\mathrm{Ext}(\mathcal{B},\mathcal{A})$ for the set of (classes of) extensions of $\mathcal{B}$ by $\mathcal{A}$. Explicitly, $\mathrm{Ext}(\mathcal{B},\mathcal{A})$ is the set of all short exact sequences $0 \to \mathcal{A} \to \mathcal{L} \to \mathcal{B} \to 0$ with $\mathcal{L}$ a rank-two vector bundle, modulo an equivalence given by commutative diagrams

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{A} \arrow[r] \arrow[d, equal] & \mathcal{L} \arrow[r] \arrow[d, "\cong"'] & \mathcal{B} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{A} \arrow[r] & \mathcal{L}' \arrow[r] & \mathcal{B} \arrow[r] & 0. \end{tikzcd}$$

Given an extension $\alpha = [0 \to \mathcal{A} \xrightarrow{i} \mathcal{L} \xrightarrow{j} \mathcal{B} \to 0]$, the short exact sequence

$$0 \longrightarrow \mathcal{B}^\vee\mathcal{A} \longrightarrow \mathcal{B}^\vee\mathcal{L} \longrightarrow \mathcal{B}^\vee\mathcal{B} = \mathcal{O} \longrightarrow 0$$

induces a boundary map $\delta_\alpha : H^0(\mathcal{O}) \to H^1(\mathcal{B}^\vee\mathcal{A})$. The association $\alpha \mapsto \delta_\alpha(1)$ establishes a bijection $\mathrm{Ext}(\mathcal{B},\mathcal{A}) \to H^1(\mathcal{B}^\vee\mathcal{A})$. In particular, $\mathrm{Ext}(\mathcal{B},\mathcal{A})$ becomes an $\mathbb{F}_q$-vector space, and its dual $\mathrm{Ext}(\mathcal{B},\mathcal{A})^\vee$ can be identified with $H^0(\Omega\mathcal{A}^\vee\mathcal{B})$ under Serre duality. For α as above and $n \in \mathbb{F}_q^\times$, the extension $n\alpha$ is represented explicitly by

$$0 \longrightarrow \mathcal{A} \xrightarrow{i\circ[n]} \mathcal{L} \xrightarrow{\ j\ } \mathcal{B} \longrightarrow 0$$

where $[n] : \mathcal{A} \to \mathcal{A}$ is just multiplication-by-n.

1.6. Finally, define a groupoid **Ext** with objects $\mathrm{Ob}(\mathbf{Ext}) = \coprod_{\mathcal{A},\mathcal{B}} \mathrm{Ext}(\mathcal{B},\mathcal{A})$, and with morphisms induced by isomorphisms of short exact sequences, that is, commutative diagrams of the form

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{A} \arrow[r] \arrow[d, "\cong"'] & \mathcal{L} \arrow[r] \arrow[d, "\cong"'] & \mathcal{B} \arrow[r] \arrow[d, "\cong"'] & 0 \\ 0 \arrow[r] & \mathcal{A}' \arrow[r] & \mathcal{L}' \arrow[r] & \mathcal{B}' \arrow[r] & 0. \end{tikzcd}$$

There is a forgetful map $\mathrm{Ob}(\mathbf{Ext}) \to \mathrm{Flag}$, sending α as above to $(\mathcal{L}, i(\mathcal{A}))$. This map is clearly surjective. From the various commutative diagrams one deduces:

(a) The forgetful map establishes a bijection between Flag and the isomorphism classes in **Ext**.
(b) Two extensions α, β in the same group $\mathrm{Ext}(\mathcal{B},\mathcal{A})$ are isomorphic in **Ext** if and only if they are in the same $\mathbb{F}_q^\times$ orbit.

2. Lower Modifications

2.1. Here we introduce our main tool for manipulating bundles. Given rank-two vector bundles $\mathcal{L}$ and $\mathcal{M}$, $\mathcal{M}$ is called a *lower modification of $\mathcal{L}$ at an effective divisor* Δ if $\mathcal{M}$ is a locally free sub-sheaf of $\mathcal{L}$ and $\mathcal{L}/\mathcal{M}$ is isomorphic to $\mathcal{O}|_\Delta$. Equivalently, there exists a line sub-bundle l of $\mathcal{L}|_\Delta$ on Δ such that $\mathcal{M}$ is the pre-image of l under the restriction $\mathcal{L} \to \mathcal{L}|_\Delta$. In that case, we write $\mathcal{M} = \mathcal{L}_{\Delta,l}$. We also say that $\mathcal{L}$ is an *upper modification* of $\mathcal{M}$. Notice that

$$\mathcal{L}^{\Delta,l} := \mathcal{L}_{\Delta,l} \otimes \mathcal{O}(\Delta) = (\mathcal{L} \otimes \mathcal{O}(\Delta))_{\Delta,l(\Delta)}$$

is an upper modification of $\mathcal{L}$, and this establishes a bijection between upper and lower modifications of $\mathcal{L}$ at Δ.

2.2. Given an extension $\alpha = [0 \to \mathcal{A} \xrightarrow{i} \mathcal{L} \xrightarrow{j} \mathcal{B} \to 0]$ and a lower modification $\mathcal{L}_{\Delta,l}$ of $\mathcal{L}$, we can define correspondingly $\alpha_{\Delta,l}$ as the extension class

$$\alpha_{\Delta,l} := [0 \longrightarrow i^{-1}(\mathcal{L}_{\Delta,l}) \xrightarrow{i} \mathcal{L}_{\Delta,l} \xrightarrow{j} j(\mathcal{L}_{\Delta,l}) \longrightarrow 0].$$

Thus in the case of a flag $(\mathcal{L}, \mathcal{A})$, i is just the inclusion and $(\mathcal{L}, \mathcal{A})_{\Delta,l} = (\mathcal{L}_{\Delta,l}, \mathcal{A} \cap \mathcal{L}_{\Delta,l})$. As before, $\alpha^{\Delta,l} := \alpha_{\Delta,l} \otimes \mathcal{O}(\Delta)$, and similarly for flags.

Let us say that l is *transverse* (to $\mathcal{A}$) if $i(\mathcal{A})|_\Delta \cap l = 0$, that is, if $i^{-1}(\mathcal{L}_{\Delta,l}) = \mathcal{A}(-D)$, or equivalently, if $j(\mathcal{L}_{\Delta,l}) = \mathcal{B}$. Then we can also say that $\alpha_{\Delta,l}$ (or $\alpha^{\Delta,l}$) is a transverse lower (or upper) modification of α. If l is transverse then $(\mathcal{L}, \mathcal{A})$ is a *non*-transverse lower modification of $(\mathcal{L}, \mathcal{A})^{\Delta,l}$.

2.3. Here is another useful way of picturing things. For any $\mathcal{A}$ consider the inclusion $\mathcal{A} \to \mathcal{A}(\Delta)$. For any $\mathcal{B}$, it induces natural projections

$$\cdots \xrightarrow{j_\Delta} H^1(\mathcal{B}^\vee \mathcal{A}(-\Delta)) \xrightarrow{j_\Delta} H^1(\mathcal{B}^\vee \mathcal{A}) \xrightarrow{j_\Delta} H^1(\mathcal{B}^\vee \mathcal{A}(\Delta)) \xrightarrow{j_\Delta} \cdots$$

which we can interpret, for example, as

$$\cdots \xrightarrow{j_\Delta} \mathrm{Ext}(\mathcal{B}, \mathcal{A}(-\Delta)) \xrightarrow{j_\Delta} \mathrm{Ext}(\mathcal{B}, \mathcal{A}) \xrightarrow{j_\Delta} \mathrm{Ext}(\mathcal{B}, \mathcal{A}(\Delta)) \xrightarrow{j_\Delta} \cdots.$$

For $\alpha \in \mathrm{Ext}(\mathcal{B}, \mathcal{A})$ it is easy to see that $j_\Delta(\alpha)$ is precisely the non-transverse upper modification $\alpha^{\Delta,\mathcal{A}|_\Delta}$. Similarly, the transverse lower modifications of α parameterise the elements of $j_\Delta^{-1}(\alpha)$ inside $H^1(\mathcal{B}^\vee \mathcal{A}(-\Delta))$.

Of course, it is in general possible that lower modifications of the same α along distinct l will be isomorphic as extensions: since

$$|j_\Delta^{-1}(\alpha)| = \frac{|\mathrm{Ext}(\mathcal{B}, \mathcal{A}(-\Delta)|}{|\mathrm{Ext}(\mathcal{B}, \mathcal{A})|},$$

each element of $\mathrm{Ext}(\mathcal{B}, \mathcal{A}(-\Delta))$ is realised as a transverse lower modification at Δ in

$$\frac{q^{\deg \Delta}}{|j_\Delta^{-1}(\alpha)|} = q^{\deg \Delta} \frac{|\mathrm{Ext}(\mathcal{B}, \mathcal{A})|}{|\mathrm{Ext}(\mathcal{B}, \mathcal{A}(-\Delta)|}$$

ways.

II. Main Results

3. Function Spaces and Hecke Operators.

3.1. Denote the spaces of F-valued functions on Bun, Flag, and Ob(**Ext**) by FunBun, FunFlag, and Fun**Ext**. The forgetful surjections Ob(**Ext**) $\twoheadrightarrow$ Flag $\twoheadrightarrow$ Bun induce injections FunBun $\hookrightarrow$ FunFlag $\hookrightarrow$ Fun**Ext**.

An element of FunFlag is equivalent to an element of Fun**Ext** that commutes with all morphisms in **Ext** (see §1.6); in particular, such a function is $\mathbb{F}_q^\times$-invariant on each group $\mathrm{Ext}(\mathcal{B}, \mathcal{A})$.

Obviously, an element of FunBun is equivalent to an element of FunFlag that takes the same value on any two flags $(\mathcal{L}, \mathcal{A})$ and $(\mathcal{L}, \mathcal{A}')$. One of our main goals is to give a more useful criterion in the case where Φ is an eigenfunction for the *Hecke operators*, which we now define.

3.2. Suppose as usual given $\Phi \in \mathrm{Fun}\mathbf{Ext}$ and an extension $\alpha = [0 \to \mathcal{A} \xrightarrow{i} \mathcal{L} \xrightarrow{j} \mathcal{B} \to 0]$. Then define

$$\mathbb{V}_v\Phi(\alpha) := \Phi(\alpha \otimes \mathcal{O}(-v))$$

and, in the notation of §2,

$$\mathbb{T}_v\Phi(\alpha) := \sum_{l \in \mathbb{P}(\mathcal{L}|_v)} \Phi\left(\alpha_{v,l}\right).$$

The algebra generated by $\mathbb{T}_v$ and $\mathbb{V}_v$ is clearly commutative. We would like to understand the $\Phi \in \mathrm{FunBun}$ that are *Hecke eigenfunctions*, i.e. eigenvectors for every $\mathbb{T}_v$ and every $\mathbb{V}_v$ with eigenvalues t_v, u_v.

To begin with, being an eigenfunction for every $\mathbb{V}_v$ is precisely the condition that there exists some character η on Pic such that Φ satisfies

$$\Phi(\mathcal{C} \otimes \alpha) = \eta^{-1}(\mathcal{C})\Phi(\alpha) \tag{1}$$

for any line bundle $\mathcal{C}$. Explicitly, $\eta(\mathcal{O}(v)) = u_v$. We call η the *central character* due to its role in representation theory, and denote by $\mathrm{Fun}\mathbf{Ext}(\eta)$, etc., the spaces of functions obeying (1).

4. Multiplicity One Theorem.

4.1. We will limit our attention to those functions Φ that are *cuspidal.* A function $\Phi \in \mathrm{Fun}\mathbf{Ext}$ is said to be cuspidal if, for any line bundles $\mathcal{A}$ and $\mathcal{B}$, it satisfies the condition

$$\sum_{\alpha \in \mathrm{Ext}(\mathcal{B},\mathcal{A})} \Phi(\alpha) = 0. \tag{2}$$

Let $\mathrm{Fun}_0\mathbf{Ext}$, etc., denote the spaces of cuspidal functions.

THEOREM 4.1 (Multiplicity One). *For any character η of* Pic *and any collection of numbers $\{t_v \in F\}_{v \in X}$, there is up to scale a unique cuspidal Hecke eigenfunction $\Phi \in \mathrm{Fun}_0\mathrm{Flag}(\eta)$ with eigenvalues t_v.*

The key to the proof is understanding how the Fourier coefficients on the Ext-groups transform under the Hecke operators. The result is Lemma 4.3 below; but first we must fix some notation for the Fourier coefficients.

4.2. Fix a non-trivial additive character $\psi : \mathbb{F}_q \to F$. Then the characters of $\mathrm{Ext}(\mathcal{B},\mathcal{A})$ correspond to differentials $\omega \in \mathrm{Ext}(\mathcal{B},\mathcal{A})^\vee = H^0(\Omega\mathcal{A}^\vee\mathcal{B})$ via $\omega \leftrightarrow \psi \circ \omega$. For simplicity we write $\langle\omega|\alpha\rangle$ for $\psi(\omega(\alpha))$. Thus, for any $\Phi \in \mathrm{Fun}\mathbf{Ext}$, we can speak of the ω-Fourier coefficient, with ψ implicit,

$$c_\Phi(\omega; \mathcal{B}, \mathcal{A}) := \frac{1}{|\mathrm{Ext}(\mathcal{B},\mathcal{A})|} \sum_{\alpha \in \mathrm{Ext}(\mathcal{B},\mathcal{A})} \Phi(\alpha)\left\langle \omega | -\alpha \right\rangle. \tag{3}$$

Here the leading factor is normalisation so that, for $\alpha \in \mathrm{Ext}(\mathcal{B},\mathcal{A})$,

$$\Phi(\alpha) = \sum_{\omega \in H^0(\Omega\mathcal{A}^\vee\mathcal{B})} c_\Phi(\omega; \mathcal{B}, \mathcal{A})\left\langle \omega | \alpha \right\rangle.$$

What more can be said when Φ comes from $\mathrm{Fun}_0\mathrm{Flag}(\eta)$?

First, cuspidality is just the condition that $c_\Phi(0; \mathcal{B}, \mathcal{A}) = 0$, for any $\mathcal{A}$ and $\mathcal{B}$. So let us assume $\omega \neq 0$.

Next, since Φ is defined on flags, $\Phi(\alpha)$ depends only on the orbit $\mathbb{F}_q^\times \alpha$ (see §1.6). Hence c_Φ is actually independent of the additive character ψ, and $c_\Phi(\omega; \mathcal{B}, \mathcal{A})$ depends merely on the orbit $\mathbb{F}_q^\times \omega$, that is, on the divisor $D := \operatorname{div} \omega$. Moreover, $\mathcal{A}$ can be recovered from $\mathcal{B}$ and D since $\mathcal{A} \cong \mathcal{B}\Omega(-D)$. So $\Phi \in \mathrm{Fun}_0\mathrm{Flag}$ is determined by the numbers

$$c_\Phi(D; \mathcal{B}) := c_\Phi(\omega; \mathcal{B}, \mathcal{B}\Omega(-D))$$

for every $\mathcal{B} \in \mathrm{Pic}$ and every effective divisor D.

Finally, introduce the central character η. Then Φ is determined by its values on flags $(\mathcal{L}, \mathcal{A})$ where the quotient $\mathcal{L}/\mathcal{A}$ is $\mathcal{O}$. On this assumption, Φ is determined by the numbers

$$c_\Phi(D) := c_\Phi(D; \mathcal{O})$$

for each effective divisor D.

Conversely, suppose given a map $c_\Phi : \mathrm{Div}^+ \to F$, and define a function Φ on $\mathrm{Ext}(\mathcal{O}, \mathcal{A})$ by

$$\Phi(\alpha) = \sum_{\substack{\omega \in H^0(\Omega\mathcal{A}^\vee) \\ \omega \neq 0}} c_\Phi(\operatorname{div} \omega) \, \langle \omega | \alpha \rangle \, . \tag{4}$$

The function Φ can be extended to all of $\mathrm{Ob}(\mathbf{Ext})$ by demanding that it be constant on each isomorphism class and transform by (1) under the Picard group. In summary:

Proposition 4.2. *For a fixed character η, giving a function $\Phi \in \mathrm{Fun}_0\mathrm{Flag}(\eta)$ is equivalent to giving a "Fourier coefficients" map $c_\Phi : \mathrm{Div}^+ \to F$.*

4.3. Here is how the Fourier coefficients transform under the Hecke operators. The proof is written out below (§4.5).

Lemma 4.3. *Hecke operators preserve cuspidality, and for $\Phi \in \mathrm{FunFlag}(\eta)$, we have*

$$c_{\mathbb{T}_v\Phi}(D) = q^{\deg v} c_\Phi(D + v) + \eta(\mathcal{O}(v)) c_\Phi(D - v) \tag{5}$$

(where the second term vanishes unless $D - v \geq 0$).

Note that $c_\Phi(D)$ is linear in Φ. Therefore, if Φ is a Hecke eigenfunction, the equation of Lemma 4.3 gives a recursion relation on the Fourier coefficients, which can be solved uniquely up to scale. *This implies the Multiplicity One Theorem 4.1.*

4.4. We can give an explicit solution to the recursion relation. To fix the scale of our solution, let us assume that Φ is normalised to have $c_\Phi(0) = 1$ (equivalently, $\Phi(\Omega \oplus \mathcal{O}, \Omega) = q - 1$). It is convenient to encode the Fourier coefficients in a generating function. Introduce a variable $\mathbf{T}_v$ for each point $v \in X$ and for any divisor D put $\mathbf{T}_D := \prod_{v \in X} \mathbf{T}_v^{\mathrm{ord}_v D}$. Then define a formal sum

$$\zeta(\Phi, \{\mathbf{T}_v\}_{v \in X}) := \sum_{D \in \mathrm{Div}^+} c_\Phi(D) \mathbf{T}_D.$$

Proposition 4.4. *The generating funtion is a product of local factors*

$$\zeta(\Phi, \{\mathbf{T}_v\}_{v \in X}) = \prod_{v \in X} \sum_{n \geq 0} c_\Phi(nv) \mathbf{T}_{nv} \tag{6}$$

and the local factors are given by

$$\sum_{n\geq 0} c_\Phi(nv)\mathbf{T}_{nv} = \left(1 - \frac{t_v}{q^{\deg v}}\mathbf{T}_v + \frac{u_v}{q^{\deg v}}\mathbf{T}_{2v}\right)^{-1}. \tag{7}$$

To prove (7), rearrange the recurrence relation of Lemma 4.3 as

$$c_\Phi(D+v) - \frac{t_v}{q^{\deg v}}c_\Phi(D) + \frac{u_v}{q^{\deg v}}c_\Phi(D-v) = 0. \tag{8}$$

If we take $D = nv$ then the left hand side of (8) is precisely the coefficient of $\mathbf{T}_{(n+1)v}$, $n \geq 0$, in the expansion of

$$\left(\sum_{n\geq 0} c_\Phi(nv)\mathbf{T}_{nv}\right)\left(1 - \frac{t_v}{q^{\deg v}}\mathbf{T}_v + \frac{u_v}{q^{\deg v}}\mathbf{T}_{2v}\right) \stackrel{?}{=} 1.$$

The constant term is obviously 1, since we demand $c_\Phi(0) = 1$.

To prove (6), it is necessary and sufficient that $c_\Phi(D+nv) = c_\Phi(D)c_\Phi(nv)$ whenever D is disjoint from v. But applying the relation (8) and induction on n we can write

$$\begin{aligned} c_\Phi(D+nv) &= \frac{t_v}{q^{\deg v}}c_\Phi(D)c_\Phi((n-1)v) - \frac{u_v}{q^{\deg v}}c_\Phi(D)c_\Phi((n-2)v) \\ &= c_\Phi(D)\left(\frac{t_v}{q^{\deg v}}c_\Phi((n-1)v) - \frac{u_v}{q^{\deg v}}c_\Phi((n-2)v)\right) \\ &= c_\Phi(D)c_\Phi(nv), \text{ q.e.d.} \end{aligned}$$

4.5. To prove Lemma 4.3, consider the action of the Hecke operators on the Fourier coefficients $c_{\mathbb{T}_v\Phi}(\omega;\mathcal{B},\mathcal{A})$ defined by (3). By definition,

$$|\mathrm{Ext}(\mathcal{B},\mathcal{A})| \cdot c_{\mathbb{T}_v\Phi}(\omega;\mathcal{B},\mathcal{A}) = \sum_{\alpha\in\mathrm{Ext}(\mathcal{B},\mathcal{A})}\sum_l \Phi(\alpha_{v,l})\left\langle -\omega|\alpha\right\rangle.$$

According to §2.3, every element of $\mathrm{Ext}(\mathcal{B},\mathcal{A}(-v))$ is obtained in

$$a := q^{\deg v}\frac{|\mathrm{Ext}(\mathcal{B},\mathcal{A})|}{|\mathrm{Ext}(\mathcal{B},\mathcal{A}(-v))|}$$

ways as the transverse modification of some α, while every element of $\mathrm{Ext}(\mathcal{B}(-v),\mathcal{A})$ is obtained in

$$b := \frac{|\mathrm{Ext}(\mathcal{B},\mathcal{A})|}{|\mathrm{Ext}(\mathcal{B}(-v),\mathcal{A})|}$$

ways as a non-transverse lower modification. Separating these two cases, we obtain

$$\begin{aligned} |\mathrm{Ext}(\mathcal{B},\mathcal{A})| \cdot c_{\mathbb{T}_v\Phi}(\omega;\mathcal{B},\mathcal{A}) = &\sum_{\alpha\in\mathrm{Ext}(\mathcal{B},\mathcal{A}(-v))} a\cdot\Phi(\alpha)\left\langle -\omega|\alpha^{v,\mathcal{A}(-v)|_v}\right\rangle \\ &+ \sum_{\alpha\in\mathrm{Ext}(\mathcal{B}(-v),\mathcal{A})}\frac{b\cdot\Phi(\alpha)}{q^{\deg v}}\sum_{l\text{ transverse}}\left\langle -\omega|\alpha^{v,l}\right\rangle. \end{aligned} \tag{9}$$

Now the cuspidality of $\mathbb{T}_v\Phi$ follows from the cuspidality of Φ if we take $\omega = 0$. To deal with $\omega \neq 0$, consider the chain of natural inclusions:

$$\cdots \xrightarrow{i_v} H^0(\Omega(-v)\mathcal{A}^\vee\mathcal{B}) \xrightarrow{i_v} H^0(\Omega\mathcal{A}^\vee\mathcal{B}) \xrightarrow{i_v} H^0(\Omega(v)\mathcal{A}^\vee\mathcal{B}) \xrightarrow{i_v} \cdots.$$

One needs the the following facts:

(a) For $\omega \in H^0(\Omega \mathcal{A}^\vee \mathcal{B})$ and $\alpha \in \mathrm{Ext}(\mathcal{B}, \mathcal{A}(-v))$ one has

$$\left\langle \omega | \alpha^{v, \mathcal{A}|_v} \right\rangle = \langle i_v(\omega) | \alpha \rangle .$$

(b) For $\omega \in H^0(\Omega \mathcal{A}^\vee \mathcal{B})$ and $\alpha \in \mathrm{Ext}(\mathcal{B}, \mathcal{A}(v))$ one has

$$\sum_{l \text{ transverse}} \langle \omega | \alpha_{v,l} \rangle = \begin{cases} 0 & \text{if } \mathrm{ord}_v(\omega) = 0 \\ q^{\deg v} \left\langle i_v^{-1}(\omega) | \alpha \right\rangle & \text{if } \mathrm{ord}_v(\omega) > 0. \end{cases}$$

Here (a) follows from the fact (§2.3) that taking the non-transverse upper modification coincides with the natural projection $j_v : H^1(\mathcal{B}^\vee \mathcal{A}(-v)) \to H^1(\mathcal{B}^\vee \mathcal{A})$, which is indeed adjoint to i_v.

To deduce (b), consider $j_v : H^1(\mathcal{B}^\vee \mathcal{A}) \to H^1(\mathcal{B}^\vee \mathcal{A}(v))$. The set of all $\alpha_{v,l}$ inside $H^1(\mathcal{B}^\vee \mathcal{A})$ is just the affine subspace $j_v^{-1}(\alpha)$, so the sum is the sum of a non-trivial character unless $\ker j_v \subseteq \ker \omega$; but then ω descends to a functional on $H^1(\mathcal{B}^\vee \mathcal{A}(v))$, i.e. it is in the image of i_v, and the value given derives from (a).

Returning to our proof, fact (a) applies to the first sum of equation (9). Hence that first sum is just

$$a \cdot |\mathrm{Ext}(\mathcal{B}, \mathcal{A}(-v))| \cdot c_\Phi(i_v(\omega); \mathcal{B}, \mathcal{A}(-v)).$$

After rewriting $\alpha^{v,l}$ as $(\alpha \otimes \mathcal{O}(v))_{v,l(v)}$, the sum over l in (9) can be evaluated by fact (b). Let us just consider the case $\mathrm{ord}_v \omega > 0$, so that

$$\begin{aligned} \sum_{\alpha \in \mathrm{Ext}(\mathcal{B}(-v), \mathcal{A})} \frac{b \cdot \Phi(\alpha)}{q^{\deg v}} \sum_{l \text{ trans.}} \left\langle -\omega | \alpha^{v,l} \right\rangle &= \sum_{\alpha \in \mathrm{Ext}(\mathcal{B}(-v), \mathcal{A})} b \cdot \Phi(\alpha) \left\langle -i_v^{-1} \omega | \alpha \otimes \mathcal{O}(v) \right\rangle \\ &= \sum_{\alpha \in \mathrm{Ext}(\mathcal{B}, \mathcal{A}(v))} b \cdot \Phi(\alpha \otimes \mathcal{O}(-v)) \left\langle -i_v^{-1} \omega | \alpha \right\rangle \end{aligned}$$

which is just

$$b \cdot \eta(\mathcal{O}(v)) |\mathrm{Ext}(\mathcal{B}, \mathcal{A}(v))| \cdot c_\Phi(i_v^{-1}(\omega); \mathcal{B}, \mathcal{A}(v)).$$

Plugging in the expressions for a and b, we have all in all

$$\begin{aligned} |\mathrm{Ext}(\mathcal{B}, \mathcal{A})| \cdot c_{\mathbb{T}_v \Phi}(\omega; \mathcal{B}, \mathcal{A}) = {} & q^{\deg v} |\mathrm{Ext}(\mathcal{B}, \mathcal{A})| \cdot c_\Phi(i_v(\omega); \mathcal{B}, \mathcal{A}(-v)) \\ & + \eta(\mathcal{O}(v)) |\mathrm{Ext}(\mathcal{B}, \mathcal{A})| \cdot c_\Phi(i_v^{-1}(\omega); \mathcal{B}, \mathcal{A}(v)). \end{aligned}$$

At last we specialise to the case $\mathcal{B} = \mathcal{O}$ and note that $\operatorname{div} i_v(\omega) = \operatorname{div} \omega + v$ to obtain the desired expression.

5. When does $\Phi \in \mathrm{FunFlag}$ come from FunBun?

5.1. We have shown that for any fixed character η of Pic, any choice of eigenvalues $\{t_v\}$ occurs in $\mathrm{Fun}_0\mathrm{Flag}(\eta)$. We now investigate the question of when such a choice of eigenvalues occurs in $\mathrm{Fun}_0\mathrm{Bun}(\eta)$, or equivalently, when the unique eigenfunction $\Phi \in \mathrm{Fun}_0\mathrm{Flag}(\eta)$ with the given eigenvalues descends to a function in $\mathrm{Fun}_0\mathrm{Bun}(\eta)$.

5.2. The main result will be stated in §6.2; but let us first try to understand from the beginning what it means for $\Phi \in \mathrm{FunFlag}$ to descend to FunBun. By definition, we must have $\Phi(\mathcal{L}, \mathcal{A}) = \Phi(\mathcal{L}, \mathcal{A}')$ whenever $\mathcal{A}$ and $\mathcal{A}'$ are distinct subbundles of a rank-two bundle $\mathcal{L}$.

Let Δ be the largest divisor such that $\mathcal{A}|_\Delta = \mathcal{A}'|_\Delta$. Then $\mathcal{L}_{\Delta,\mathcal{A}|_\Delta} = \mathcal{A} \oplus \mathcal{A}'$, and more precisely

$$(\mathcal{L},\mathcal{A})_{\Delta,\mathcal{A}|_\Delta} = (\mathcal{A}\oplus\mathcal{A}',\mathcal{A})$$
$$(\mathcal{L},\mathcal{A}')_{\Delta,\mathcal{A}|_\Delta} = (\mathcal{A}\oplus\mathcal{A}',\mathcal{A}').$$

For some $l \subset (\mathcal{A}\oplus\mathcal{A}')|_\Delta$ we have, conversely, $(\mathcal{L},\mathcal{A}) = (\mathcal{A}\oplus\mathcal{A}',\mathcal{A})^{\Delta,l}$, etc. This l must be transverse to both $\mathcal{A}$ and $\mathcal{A}'$. It is convenient to think of such an l as the graph $\Gamma(\sigma)$ of some isomorphism $\sigma : \mathcal{A}'|_\Delta \to \mathcal{A}|_\Delta$.

Let us introduce some notation for this situation. Given line bundles $\mathcal{A},\mathcal{A}'$, an effective divisor Δ, and a homomorphism $\sigma : \mathcal{A}'|_\Delta \to \mathcal{A}|_\Delta$, we will write $E_\Delta(\mathcal{A}',\mathcal{A},\sigma)$ for the class

$$E_\Delta(\mathcal{A}',\mathcal{A},\sigma) := [0 \to \mathcal{A} \to \mathcal{A}\oplus\mathcal{A}' \to \mathcal{A}' \to 0]_{\Delta,\Gamma(\sigma)} \tag{10}$$

in $\mathrm{Ext}(\mathcal{A}',\mathcal{A}(-\Delta))$, or for the corresponding flag. By convention, when $\Delta = 0$, $\mathrm{Hom}(\mathcal{A}'|_0,\mathcal{A}|_0) = \{1\}$ and $E_0(\mathcal{A}',\mathcal{A},1) = \mathcal{A}\oplus\mathcal{A}'$. In summary

- (**•**) $\Phi \in \mathrm{FunFlag}$ descends to Bun if and only if, for every pair of line bundles $\mathcal{A},\mathcal{A}'$, every effective divisor Δ, and every isomorphism $\sigma : \mathcal{A}'|_\Delta \to \mathcal{A}|_\Delta$,

$$\Phi(E_\Delta(\mathcal{A}',\mathcal{A},\sigma)) = \Phi(E_\Delta(\mathcal{A},\mathcal{A}',\sigma^{-1})).$$

So far we have ignored the central character η; using it, we can assume that $\mathcal{A}' = \mathcal{O}$. Let us write

$$E_\Delta(\mathcal{A},\sigma) := E_\Delta(\mathcal{O},\mathcal{A},\sigma) \in \mathrm{Ext}(\mathcal{O},\mathcal{A}(-\Delta)). \tag{11}$$

As a flag (rather than an extension), $E_\Delta(\mathcal{A},\sigma)$ only depends on the orbit $\mathbb{F}_q\sigma$—in other words, on the class of $(\mathcal{A},\sigma)$ in Pic_Δ. So (noting that $E_\Delta(\mathcal{A},\mathcal{O},\sigma^{-1}) = \mathcal{A}\otimes E_\Delta(\mathcal{O},\mathcal{A}^\vee,\sigma^{-1})$), our condition becomes

- (**•′**) $\Phi \in \mathrm{FunFlag}(\eta)$ descends to Bun if and only if, for every effective divisor Δ and every $(\mathcal{A},\sigma) \in \mathrm{Pic}_\Delta$, we have

$$\Phi(E_\Delta(\mathcal{A},\sigma)) = \eta^{-1}(\mathcal{A})\Phi(E_\Delta(\mathcal{A}^\vee,\sigma^{-1})).$$

Now that (**•′**) is a statement about the equality of functions on a group Pic_Δ, we may investigate its Fourier transform; let us fix notation.

5.3. Pic_Δ is infinite, but we can make use of the fact that $\mathrm{Pic}_\Delta \overset{\deg}{\to} \mathbb{Z}$ has finite kernel Pic^0_Δ. Define for each character χ of Pic_Δ a formal sum

$$Z_\Delta(\Phi,\chi,\mathbf{T}) := \sum_{(\mathcal{A},s)\in\mathrm{Pic}_\Delta} \Phi\left(E_\Delta(\mathcal{A},s)\right)\chi^{-1}(\mathcal{A},s)\mathbf{T}^{-\deg\mathcal{A}}. \tag{12}$$

This ‘Fourier transform’ can be inverted as follows. For fixed $(\mathcal{C},s) \in \mathrm{Pic}_\Delta$, let $f(\chi)$ be the constant term of $Z_\Delta(\Phi,\chi,\mathbf{T})\chi(\mathcal{C},s)\mathbf{T}^{\deg\mathcal{C}}$. Then $f(\chi)$ depends merely on the restriction χ_0 of χ to Pic^0_Δ, and

$$\Phi(E_\Delta(\mathcal{C},s)) = \frac{1}{|\mathrm{Pic}^0_\Delta|}\sum_{\chi_0\in\mathrm{Hom}(\mathrm{Pic}^0_\Delta,F^\times)} f(\chi_0).$$

5.4. With this notation, our condition (•′) becomes

Proposition 5.4. *$\Phi \in \mathrm{FunFlag}(\eta)$ comes from* FunBun *if and only if, for every effective divisor Δ and every character χ of Pic^0_Δ, we have*

$$Z_\Delta(\Phi, \chi, \mathbf{T}) = Z_\Delta(\Phi, \eta^{-1}\chi^{-1}, \mathbf{T}^{-1}). \tag{13}$$

We have not so far stipulated that Φ be a Hecke eigenfunction. In that case, we would like to re-express (13) purely in terms of the eigenvalues t_v. There are apparently two ways one might proceed.

5.5. First, in analogy to the proof of the Multiplicity One Theorem 4.1, we could try to understand $Z_\Delta(\mathbb{T}_v\Phi, \chi, \mathbf{T})$ in terms of $Z_\Delta(\Phi, \chi, \mathbf{T})$, thereby extracting information about the eigenvalues.

Recall that a character χ on Pic_Δ is called *primitive* on D if $D \le \Delta$ is the smallest effective divisor such that χ factors through Pic_D. Then the analogous result to Lemma 4.3 is:

Lemma 5.5. *Suppose given $\Phi \in \mathrm{FunFlag}(\eta)$ and χ primitive on $D \le \Delta$. Then, if $v \in \Delta$, we have*

$$Z_\Delta(\mathbb{T}_v\Phi, \chi, \mathbf{T}) = Z_{\Delta+v}(\Phi, \chi, \mathbf{T}) + q^{\deg v}\eta(\mathcal{O}(v))Z_{\Delta-v}(\Phi, \chi, \mathbf{T})$$

(the second term is zero unless $D \le \Delta - v$). Otherwise,

$$\begin{aligned} Z_\Delta(\mathbb{T}_v\Phi, \chi, \mathbf{T}) =& Z_{\Delta+v}(\Phi, \chi, \mathbf{T}) + \chi^{-1}(\mathcal{O}(v), 1_v)Z_\Delta(\Phi, \chi, \mathbf{T})\mathbf{T}^{-\deg v} \\ &+ \eta(\mathcal{O}(v))\chi(\mathcal{O}(v), 1_v)Z_\Delta(\Phi, \chi, \mathbf{T})\mathbf{T}^{\deg v}. \end{aligned}$$

Unfortunately, this calculation indicates that the Hecke operators relate values of Z for different Δ, whereas the functional equation (13) relates values of Z for different χ. So it does not seem possible to turn the functional equation into a condition on the eigenvalues by this means. However, using the Lemma inductively to write $Z_\Delta(\Phi, \chi, \mathbf{T})$ in terms of $Z_D(\Phi, \chi, \mathbf{T})$ we do find as consolation

Corollary 5.5. *When $\Phi \in \mathrm{FunFlag}(\eta)$ is a Hecke eigenfunction, $Z_\Delta(\Phi, \chi, \mathbf{T})$ for arbitary Δ and χ is determined by the cases when χ is primitive on Δ. In particular, in Proposition 5.4 it suffices to consider only the case when χ is primitive.*

The simple proof of Lemma 5.5 is left for the reader; the main observation is that $E_\Delta(\mathcal{B}, \mathcal{A}, \sigma)_{v,l}$ takes on the following values:

- $v \notin \Delta$ $\begin{cases} l = \mathcal{A}(-\Delta)|_v: & E_\Delta(\mathcal{B}(-v), \mathcal{A}, \sigma). \\ l = \mathcal{B}(-\Delta)|_v: & E_\Delta(\mathcal{B}, \mathcal{A}(-v), \sigma). \\ \text{otherwise:} & \text{Ranges over } E_{\Delta+v}(\mathcal{B}, \mathcal{A}, \tilde\sigma) \text{ for all } \tilde\sigma|_\Delta = \sigma, \\ & \tilde\sigma|_v \text{ an isomorphism } (q^{\deg v} - 1 \text{ possibilities}). \end{cases}$
- $v \in \Delta$ $\begin{cases} l = \mathcal{A}(-\Delta)|_v: & E_{\Delta-v}(\mathcal{B}, \mathcal{A}, \sigma|_{\Delta-v}) \otimes \mathcal{O}(-v). \\ \text{otherwise:} & \text{Ranges over } E_{\Delta+v}(\mathcal{B}, \mathcal{A}, \tilde\sigma) \text{ for all } \tilde\sigma|_\Delta = \sigma \\ & (q^{\deg v} \text{ possibilities}). \end{cases}$

5.6. The second (and more successful) way to try to relate the functional equation to the eigenvalues is to plug the Fourier expansion of Φ on the Ext-groups into the definition of $Z_\Delta(\phi, \chi, \mathbf{T})$. Then the functional equation becomes a condition on the Fourier coefficients c_Φ, which we know from Proposition 4.4 are closely related to the eigenvalues. In §6.2 we formulate the ultimate result, and in §7 carry out the calculation; but let us first try to understand what Proposition 5.4 means for $\Delta = 0$.

5.7. The Case $\Delta = 0$. In this case χ is just a character on Pic and $E_0(\mathcal{A}, \sigma)$ is just $\mathcal{A} \oplus \mathcal{O}$. Suppose, then, we are given $\Phi \in \mathrm{Fun}_0\mathrm{Bun}(\eta)$. By definition

$$Z_0(\Phi, \chi, \mathbf{T}) = \sum_{\mathcal{A} \in \mathrm{Pic}} \Phi(\mathcal{A} \oplus \mathcal{O}) \chi^{-1}(\mathcal{A}) \mathbf{T}^{-\deg \mathcal{A}}.$$

Inserting the Fourier expansion (4),

$$= \sum_{\mathcal{A} \in \mathrm{Pic}} \sum_{0 \neq \omega \in H^0(\Omega\mathcal{A}^\vee)} c_\Phi(\operatorname{div} \omega) \chi^{-1}(\mathcal{A}) \mathbf{T}^{-\deg \mathcal{A}}.$$

($\mathcal{A} \oplus \mathcal{O}$ is a trivial extension, so the pairing with ω vanishes.) Now change variables $\mathcal{A} \mapsto \Omega\mathcal{A}^\vee$:

$$\begin{aligned} &= \sum_{\mathcal{A} \in \mathrm{Pic}} \sum_{0 \neq \omega \in H^0(\mathcal{A})} c_\Phi(\operatorname{div} \omega) \chi(\mathcal{A}) \mathbf{T}^{\deg \mathcal{A}} \cdot \chi^{-1}(\Omega) \mathbf{T}^{\deg \Omega} \\ &= (q-1) \sum_{D \geq 0} c_\Phi(D) \chi(\mathcal{O}(D)) \mathbf{T}^{\deg D} \cdot \chi^{-1}(\Omega) \mathbf{T}^{\deg \Omega}. \end{aligned}$$

Thus if we define

$$L(\Phi, \chi, \mathbf{T}) = \sum_{D \geq 0} c_\Phi(D) \chi(\mathcal{O}(D)) \mathbf{T}^{\deg D}$$

we obtain

$$L(\Phi, \chi, \mathbf{T}) = \frac{1}{q-1} \chi(\Omega) \mathbf{T}^{\deg \Omega} Z_0(\Phi, \chi, \mathbf{T}).$$

The functional equation for $Z_\Delta(\Phi, \chi, \mathbf{T})$ then immediately establishes

$$L(\Phi, \chi, \mathbf{T}) = \eta(\Omega) \chi(\Omega)^2 \mathbf{T}^{2\deg \Omega} L(\Phi, \eta^{-1}\chi^{-1}, \mathbf{T}^{-1}). \tag{14}$$

To write this in terms of eigenvalues t_v, we can obtain $L(\Phi, \chi, \mathbf{T})$ from the generating function $\zeta(\Phi, \{\mathbf{T}_v\}_{v \in X})$ by putting $\mathbf{T}_v = \chi(\mathcal{O}(v)) \mathbf{T}^{\deg v}$. It follows from Proposition 4.4 that when Φ is a normalised eigenfunction,

$$L(\Phi, \chi, \mathbf{T}) = \prod_{v \in X} \left(1 - \frac{t_v}{q^{\deg v}} \chi(\mathcal{O}(v)) \mathbf{T}^{\deg v} + \frac{u_v}{q^{\deg v}} \chi(\mathcal{O}(v))^2 \mathbf{T}^{2\deg v} \right)^{-1}.$$

Now we turn to the general situation.

6. The L-function and Its Functional Equation.

6.1. Fix an effective divisor $\Delta = \sum n_v v$. To $\Phi \in \mathrm{Fun}_0\mathrm{Flag}(\eta)$ associate a formal power series in a variable $\mathbf{T}$ and depending on a primitive character χ of Pic_Δ:

$$L(\Phi, \chi, \mathbf{T}) := \sum_{D \in \mathrm{Div}^+, D \cap \Delta = \emptyset} c_\Phi(D) \chi\left(\mathcal{O}(D), 1_D\right) \mathbf{T}^{\deg D}.$$

Remarks. (i) We can obtain $L(\Phi, \chi, \mathbf{T})$ from the generating function $\zeta(\Phi, \{\mathbf{T}_v\}_{v \in X})$ by putting $\mathbf{T}_v = 0$ if $v \in \Delta$ and $\mathbf{T}_v = \chi(\mathcal{O}(v), 1_v) \mathbf{T}^{\deg v}$ otherwise. It follows from Proposition 4.4 that when Φ is a normalised eigenfunction,

$$L(\Phi, \chi, \mathbf{T}) = \prod_{v \notin \Delta} \left(1 - \frac{t_v}{q^{\deg v}} \chi(\mathcal{O}(v), 1_v) \mathbf{T}^{\deg v} + \frac{u_v}{q^{\deg v}} \chi(\mathcal{O}(v), 1_v)^2 \mathbf{T}^{2\deg v} \right)^{-1}. \tag{15}$$

(ii) If one puts $\mathbf{T} = q^{-s}$, one can consider the L-function as a function in a complex variable s (subject to convergence). Under this substitution, one obtains the 'Hecke normalisation' of the L-function (see Deligne [2], §3.2.6). Besides agreeing with Weil [1], this normalisation allows us to avoid factors of $q^{\frac{1}{2}}$ and otherwise makes certain expressions simpler.

On the other hand, it is sometimes preferable to use the 'unitary normalisation,' which one can derive in our situation simply by putting $\mathbf{T} = q^{\frac{1}{2}-s}$.

6.2. Here is the main result:

THEOREM 6.2. *A cuspidal Hecke eigenfunction* Φ *on* Flag *with central character* η *descends to* Bun *if and only if, for every choice of* Δ *and every* χ *primitive on* Δ*, the associated* L*-function satisfies the functional equation*

$$L(\Phi, \chi, \mathbf{T}) = \eta(\Omega(\Delta))\epsilon(\chi, \mathbf{T})L(\Phi, \chi^{-1}\eta^{-1}, \mathbf{T}^{-1}). \tag{16}$$

Moreover, in that case $L(\Phi, \chi, \mathbf{T})$ *is a polynomial in* $\mathbf{T}$ *of degree* $2\deg\Omega(\Delta)$*.*

Remarks. (i) The factor $\epsilon(\chi, \mathbf{T})$ is defined as follows.

$$\epsilon(\chi, \mathbf{T}) := q^{-\deg\Delta}\mathbf{T}^{2\deg\Omega(\Delta)}\left(\sum_{\sigma\in H^0(\Omega(\Delta)|_\Delta)^\times}\psi\left(\mathrm{Res}_\Delta\sigma\right)\chi\left(\Omega(\Delta), \sigma\right)\right)^2 \tag{17}$$

(see §7.2 for the meaning of Res_Δ). This is clearly independent of the choice of the additive character ψ. The quantity in parentheses is a Gauss sum (see Lemma 7.3). In case $\Delta = 0$, we should interpret the definition to mean

$$\epsilon(\chi, \mathbf{T}) = \chi(\Omega)^2\mathbf{T}^{2\deg\Omega}$$

according to the calculation in §5.7.

(ii) If one substitutes $\mathbf{T} = q^{\frac{1}{2}-s}$ (cf. remark (ii) in §6.1), $\epsilon(\chi, \mathbf{T})$ becomes a perfect square in $\mathbb{F}_q[q^{-s}]$.

(iii) In light of the Multiplicity One Theorem 4.1, the net result is that a given collection of numbers $\{t_v\}$ appears as Hecke eigenvalues in $\mathrm{Fun}_0\mathrm{Bun}(\eta)$ if and only if all the products (15) obey the functional equation (16); the Fourier expansion of the unique normalised eigenfunction can be computed explicitly from the product expansion (6,7) of the generating function.

7. Proof of the Functional Equation.

7.1. As promised in §5, the functional equation of Theorem 6.2 is a translation of Proposition 5.4 by inserting the Fourier expansion (4) of Φ into the definition (12) of $Z_\Delta(\Phi, \chi, \mathbf{T})$. Here we need only consider the case when χ is primitive on Δ, by Corollary 5.5.[1] The calculation for $\Delta = 0$ was already done in §5.7. In what follows the formulas make more sense if we assume $\Delta \neq 0$.

[1]This reduction simplifies the application of Lemma 7.3 below; the reader is invited to carry out the calculation without it. One gets an understandable, but more complicated answer than the conventional one given is §6.2.

7.2. In carrying out the calculation, we will inevitably have to apply a functional $\omega \in H^0(\Omega(\Delta)\mathcal{A}^\vee)$ to extensions of the form $E_\Delta(\mathcal{A},\sigma)$. Let us pre-emptively observe that the pairing is given by

$$(\omega, E_\Delta(\mathcal{A},\sigma)) \mapsto \sum_{v\in\Delta} \mathrm{Tr}_{\mathbb{F}_q}^{\mathbb{F}_q(v)} \mathrm{res}_v \left(\omega|_\Delta \otimes \sigma\right) =: \mathrm{Res}_\Delta(\omega\sigma). \tag{18}$$

Here $\omega|_\Delta \otimes \sigma$ is a section of $\Omega(\Delta)|_\Delta$, in other words, a 'principal part' for the sheaf of meromorphic differentials with poles supported on Δ; $\mathrm{res}_v \left(\omega|_\Delta \otimes \sigma\right)$ is the residue of this principal part at v; and Tr is the trace in the sense of Galois theory.

7.3. One immediately obtains

$$\begin{aligned} Z_\Delta(\Phi,\chi,\mathbf{T}) &= \sum_{(\mathcal{A},s)\in\mathrm{Pic}_\Delta} \sum_{\substack{\omega\in H^0(\Omega(\Delta)\mathcal{A}^\vee)\\ \omega\neq 0}} c_\Phi(\operatorname{div}\omega)\psi\left(\mathrm{Res}_\Delta(\omega s)\right)\chi^{-1}(\mathcal{A},s)\mathbf{T}^{-\deg\mathcal{A}} \\ &= \frac{1}{q-1}\sum_{\mathcal{A}\in\mathrm{Pic}}\sum_{\omega\neq 0} c_\Phi(\operatorname{div}\omega)\mathbf{T}^{-\deg\mathcal{A}} \sum_{s\in H^0(\mathcal{A}|_\Delta)^\times} \psi\left(\mathrm{Res}_\Delta(\omega s)\right)\chi^{-1}(\mathcal{A},s). \end{aligned}$$

(The factor of $\frac{1}{q-1}$ enters because s now ranges over all level-Δ structures on $\mathcal{A}$, rather than over isomorphism classes of them; if we allow $\Delta = 0$, this factor does not appear). The inner sum is a Gauss sum; let us recall some well-known facts about Gauss sums—the proofs are precisely analogous to the classical situation of Dirichlet characters on $\mathbb{Z}/n\mathbb{Z}$ (see, e.g., Bump [3] pp. 4-6).

LEMMA 7.3. *Suppose given a line bundle $\mathcal{A}$, an effective divisor D, an additive character Ψ on $H^0(\mathcal{A}|_D)$ and a character Λ on Pic_D. Say that Ψ is primitive if it doesn't factor through $H^0(\mathcal{A}|_{D'})$ for any $D' < D$. Consider the Gauss sum $\Gamma(\mathcal{A},\Psi,\Lambda) := \sum_{s\in H^0(\mathcal{A}|_D)^\times}\Psi(s)\Lambda(\mathcal{A},s)$.*

(a) *If η is a character on* Pic*, then $\Gamma(\mathcal{A},\Psi,\eta\Lambda) = \eta(\mathcal{A})\Gamma(\mathcal{A},\Psi,\Lambda)$.*
(b) *If Λ is primitive but Ψ is not, then $\Gamma(\mathcal{A},\Psi,\Lambda) = 0$.*
(c) *If Ψ is primitive, then $\Gamma(\mathcal{A},\Psi,\Lambda) \neq 0$ and in fact*

$$\Gamma(\mathcal{A},\Psi,\Lambda)\Gamma(\mathcal{A},\Psi,\Lambda^{-1}) = q^{\deg D}.$$

In our situation, $\Lambda := \chi^{-1}$ is assumed to be primitive, and $\Psi := [s \mapsto \psi(\mathrm{Res}_\Delta(\omega s))]$ is only primitive if $\operatorname{div}\omega$ is disjoint from Δ: for if ω vanishes at $v \in \Delta$, then $\mathrm{Res}_\Delta(\omega s)$ depends only on $s|_{\Delta - v}$. So, by Lemma 7.3(b), we can restrict the sum over ω to that case, and then it also makes sense to change variables $s \mapsto s/\omega$ to remove the ω-dependency from the inner sum:

$$\begin{aligned} Z_\Delta(\Phi,\chi,\mathbf{T}) = \frac{1}{q-1}\sum_{\mathcal{A}} \sum_{\substack{\omega\in H^0(\Omega(\Delta)\mathcal{A}^\vee)\\ \Delta\cap\operatorname{div}\omega=\emptyset}} & c_\Phi(\operatorname{div}\omega)\mathbf{T}^{-\deg\mathcal{A}}\chi(\Omega(\Delta)\mathcal{A}^\vee,\omega|_\Delta) \\ \cdot \sum_{s\in H^0(\Omega(\Delta)|_\Delta)^\times} & \psi(\mathrm{Res}_\Delta(s))\chi^{-1}(\Omega(\Delta),s). \end{aligned}$$

We can clean up this expression by noting that $\chi(\Omega(\Delta)\mathcal{A}^\vee,\omega|_\Delta)$ is the same as $\chi(\Omega(\Delta)\mathcal{A}^\vee,1_{\operatorname{div}\omega})$, the arguments coinciding as classes in Pic_Δ. Finally, we can change variables $\mathcal{A} \mapsto \Omega(\Delta)\mathcal{A}^\vee$ to obtain

$$Z_\Delta(\Phi, \chi, \mathbf{T}) = \frac{1}{q-1} \sum_{\mathcal{A} \in \mathrm{Pic}} \sum_{\substack{\omega \in H^0(\mathcal{A}), \\ \Delta \cap \mathrm{div}\,\omega = \emptyset}} c_\Phi(\mathrm{div}\,\omega)\chi(\mathcal{A}, 1_{\mathrm{div}\,\omega})\mathbf{T}^{\deg \mathcal{A}} \cdot \sum_{s \in H^0(\Omega(\Delta)|_\Delta)^\times} \psi(\mathrm{Res}_\Delta(s))\chi^{-1}(\Omega(\Delta), s)\mathbf{T}^{-\deg \Omega(\Delta)}.$$

That is,

$$Z_\Delta(\Phi, \chi, \mathbf{T}) = L(\Phi, \chi, \mathbf{T}) \cdot \Gamma(\Omega(\Delta), \psi \circ \mathrm{Res}_\Delta, \chi^{-1}) \cdot \mathbf{T}^{-\deg \Omega(\Delta)}. \tag{19}$$

Therefore the functional equation (13) for $Z_\Delta(\Phi, \chi, \mathbf{T})$ can be re-written

$$\begin{aligned} L(\Phi, \chi, \mathbf{T}) &= \frac{\Gamma(\Omega(\Delta), \psi \circ \mathrm{Res}_\Delta, \eta\chi)}{\Gamma(\Omega(\Delta), \psi \circ \mathrm{Res}_\Delta, \chi^{-1})} \cdot \mathbf{T}^{2\deg \Omega(\Delta)} \cdot L(\Phi, \eta^{-1}\chi^{-1}, \mathbf{T}^{-1}) \\ &= \frac{\eta(\Omega(\Delta))}{q^{\deg \Delta}} \cdot \Gamma(\Omega(\Delta), \psi \circ \mathrm{Res}_\Delta, \chi)^2 \cdot \mathbf{T}^{2\deg \Omega(\Delta)} \cdot L(\Phi, \eta^{-1}\chi^{-1}, \mathbf{T}^{-1}) \end{aligned}$$

using the standard properties of the Gauss sum. Defining

$$\epsilon(\chi, \mathbf{T}) := q^{-\deg \Delta} \cdot \Gamma(\Omega(\Delta), \psi \circ \mathrm{Res}_\Delta, \chi)^2 \cdot \mathbf{T}^{2\deg \Omega(\Delta)}$$

we have obtained precisely the desired expression (16).

7.4. *It remains to prove that the L-function is a polynomial of the stated degree* $2\deg \Omega(\Delta)$. For it to be a polynomial, according to equation (19) it suffices to show that in $Z_\Delta(\Phi, \chi, \mathbf{T})$ only powers of $\mathbf{T}$ between $-\deg \Omega(\Delta)$ and $\deg \Omega(\Delta)$ occur. Actually, we need only show that the coefficient of $\mathbf{T}^{-n}$ in $Z_\Delta(\Phi, \chi, \mathbf{T})$ vanishes for $n > \deg \Omega(\Delta)$—then the functional equation (13) implies that the coefficient of $\mathbf{T}^n$ vanishes as well.

So it suffices to show that $\Phi(E_\Delta(\mathcal{A}, \sigma)) = 0$ whenever $\deg \mathcal{A} > \deg \Omega(\Delta)$. But in that case $E_\Delta(\mathcal{A}, \sigma)$ is a class in $\mathrm{Ext}(\mathcal{O}, \mathcal{A}(-\Delta)) = H^0(\Omega(\Delta)\mathcal{A}^\vee)^\vee = 0$. Hence $\Phi(E_\Delta(\mathcal{A}, \sigma)) = \sum_{\alpha \in \mathrm{Ext}(\mathcal{O}, \mathcal{A}(-\Delta))} \Phi(\alpha) = 0$, by definition of cuspidality.

To show that the L-function has precisely the degree given, we recall that Φ can be normalised so that the constant term of $L(\Phi, \chi, \mathbf{T})$ is 1. Then, according to the functional equation, the top-degree term is precisely $\eta(\Omega(\Delta))\epsilon(\chi, \mathbf{T})$. Lemma 7.3 guarantees that this quantity does not vanish!

References

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[2] Deligne, P. "Formes Modulaires et représentations de $GL(2)$" in *Modular Functions of One Variable II.* Springer LNM 349, 1973.

[3] Bump, D. *Automorphic Forms and Representations.* Cambridge Studies in Advanced Mathematics 55, 1997.