# The Maslov Index

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# Chapter 1

## Teruji Thomas:

# The Maslov Index

## 1.1 Introduction: The landscape of the Maslov index

There are several different things called "the Maslov index". See [CLM] for the connections between them. In this chapter we will consider only the "algebraic" version developed in [Th] and references.

Let $(V, \omega)$ be a vector space over any field (with characteristic $\neq 2$) with symplectic form $\omega$. There is a set of Lagrangian subspaces

$$\mathrm{Lagr}(V, \omega) := \left\{ L < V : L = L^{\perp} \right\} ,$$

where $W^{\perp} = \{v \in V : \omega(v, w) = 0 \text{ for all } w \in W\}$.

**Idea.** The Maslov index is an invariant associated to any $n$-tuple in $\mathrm{Lagr}(V)$, more precisely a quadratic form. So it is a morphism

$$\tau \colon \big(\mathrm{Lagr}(V, \omega)\big)^n \longrightarrow \{ \text{ quadratic forms } \} .$$

(This will be made precise later.) Commonly we have $n = 3$, and $\tau$ is invariant (at least) under the action of the symplectic group.

Here are some contexts in which the Maslov index plays an important role.

**Example 1.1.1.** If $M$ is any smooth manifold, $T^*M$ is canonically a symplectic manifold with symplectic form $\omega = -d\lambda$, where $\lambda$ is the canonical Liouville 1-form. So $T(T^*M)$ is a family of symplectic vector spaces. For example, the vertical tangent spaces are Lagrangians. This is the original setting for the Maslov index in the theory of Maslov-Arnold [MA], Leray [Le], Kashiwara-Schapira [KS], under the name of "micro-local analysis".

**Example 1.1.2.** Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}^*$ is a disjoint union of coadjoint orbits (i.e. orbits under the coadjoint action of $G$ on $\mathfrak{g}^*$). Each of these is again a symplectic manifold. Kirillov studied these structures in the context of representation theory, under the name "orbit method". The role of the Maslov index was explained by Lion, Perrin [LP], and others. Something similar can also be done for finite nilpotent groups [KT].

**Example 1.1.3.** Let $M$ be a real manifold of dimension $4n + 2$. Then $H^{2n+1}(M; \Bbbk)$ is a symplectic vector space. The role of the Maslov index here was independently developed by Wall [Wa], and is related to surgery theory and higher cobordisms.

**Example 1.1.4.** Hermitian $K$-theory (Karoubi). The role of the Maslov index has not been satsifactorily developed yet.

### 1.1.1 Symplectic vs. projective geometry

**Motto.** The Maslov index is to symplectic geometry as the cross ratio is to projective geometry.

The projective linear group $PGL(2;\ \Bbbk)$ acts transitively on $\mathbb{P}^1_{\Bbbk}$ and simply transitively on $P_3$, the space of triples of distinct points. So consider the space $P_4$ of quadruples $(L_1, L_2, L_3, L_4)$. By the action of $PGL(2)$ we can map it to $(0, 1, \infty, L)$, and $L \in \Bbbk \setminus \{0, 1\}$ is the *cross ratio* of the quadruple. Thus two quadruples of distinct points are in the same $PGL(2;\ \Bbbk)$-orbit if and only if they have the same cross ratio.

Let let us look at the symplectic group. Note that for any choice of symplectic form on $\Bbbk^2$, we have $Sp(2;\ \Bbbk) = SL(2;\ \Bbbk)$, and $\mathrm{Lagr}(\Bbbk^2) = \mathbb{P}^1_{\Bbbk}$. In general, $PSL(2)$ is a proper subgroup of $PGL(2)$, so it does not act transitively on $P_3$. Rather, the set of $PSL(2)$-orbits of $P_3$ corresponds to the quotient $PGL(2)/PSL(2)$. Taking determinants gives an isomorphism

$$\det\colon PGL(2;\ \Bbbk)\big/PSL(2;\ \Bbbk) \xrightarrow{\sim} \Bbbk^*\big/\left(\Bbbk^*\right)^2 .$$

The right-hand side is the set of 1-dimensional quadratic forms up to isomorphism. This map $\tau\colon P_3 \to \Bbbk^*/(\Bbbk^*)^2$ is a version of the Maslov index (and it is independent of any choices).

**Example 1.1.5.** If $\Bbbk = \mathbb{R}$, then $\mathbb{R}^*/(\mathbb{R}^*)^2 = \{-1, +1\}$, and $\mathbb{RP}^1 \cong S^1$. The Maslov index of $L_1, L_2, L_3$ is positive if the three lines are in cyclic order, and negative if they are not.

**Theorem 1.1.6** ([RR, Theorem 2.11])**.** *Consider the action of $Sp(V,\omega)$ on $\mathrm{Lagr}(V,\omega)^3$. Two triples $L, L' \in \mathrm{Lagr}(V,\omega)^3$ are in the same $Sp(V,\omega)$-orbit if and only if they are in the same $GL(V)$-orbit and $\tau(L) = \tau(L')$.*

## 1.2 The Maslov index of three transverse Lagrangians

Here is a more somewhat general construction of the Maslov index. For any symplectic vector space $(V,\omega)$, consider three transverse Lagrangians $L_1, L_2, L_3$, so $V = L_1 \oplus L_3$. Think of $L_2$ as the graph of a linear map $f\colon L_1 \to L_3$. So we have maps

$$L_1 \xrightarrow{f} L_3 \xrightarrow{\sim} V\big/L_1 = L_1^* ,$$

and we can think of $L_2$ as the graph of a map $\gamma\colon L_1 \to L_1^*$. Moreover, $L_2$ is Lagrangian if and only if $\gamma$ is symmetric considered as a map $L_1 \otimes L_1 \to \Bbbk$. So we define $\tau(L_1, L_2, L_3)$ to be $\gamma$ considered as a quadratic form. Explicitly, $\tau(L_1, L_2, L_3)$ is the symmetric form on $L_1$ given by

$$a \otimes b \mapsto \omega(a, f(b)).$$

This was Leray's definition [Le]. The full definition will consider arbitrary $n$-tuples of Lagrangians.

## 1.3 Categories of symplectic/symmetric spaces

**Terminology.** We will say "symplectic space" for a symplectic vector space, and "quadratic space" for a vector space with a non-degenerate, symmetric bilinear form. Since $2 \neq 0$, symmetric bilinear forms and quadratic forms determine each other uniquely.

We construct a category $Q_+$, whose objects are quadratic spaces. If $V_1, V_2$ are quadratic spaces, then morphisms are given by

$$\mathrm{Hom}_{Q_+}(V_1, V_2) := \mathrm{Lagr}(V_1^\circ \oplus V_2) ,$$

where $(V,q)^\circ := (V,-q)$, and (just as in the symplectic case) $L < V$ is "Lagrangian" if $L = L^\perp$ with respect to the quadratic form. $\mathrm{Lagr}(V_1^\circ \oplus V_2)$ is sometimes called the space of "Lagrangian correspondences". Composition of morphisms is like composition of general correspondences (think of composing functions $f\colon A \to B$ and $g\colon B \to C$ via the subsets of $A \times B$ and $B \times C$).

**Example 1.3.1.** If $f\colon V_1 \to V_2$ is an isometry, then the graph $\Gamma_f \subset V_1^\circ \oplus V_2$ is Lagrangian.

We construct another category $Q_-$ in the same way, only that we take the objects to be symplectic spaces, and $(V,\omega)^\circ := (V,-\omega)$.

**Definition 1.3.2.** The *Witt group* of a field $\Bbbk$ is $W(\Bbbk) := \pi_0(Q_+)$. To spell this out, we say that two quadratic spaces are *Witt equivalent* if there exists a Lagrangian corrrespondence between them. For example, $(V,q)$ is equivalent to 0 if and only if it contains a Lagrangian. THen $W(\Bbbk)$ is the group whose elements are Witt equivalence-classes of quadratic spaces, with addition induced by direct sum. Note that inverses are given by $-(V,q) = (V,q)^\circ$.

For example, $\big(\Bbbk^2, \left(\begin{smallmatrix} +1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\big) \equiv 0$. For $\Bbbk = \mathbb{R}$,

$$\mathrm{index}\colon W(\mathbb{R}) \xrightarrow{\sim} \mathbb{Z} ,$$

where the index$(q)$ is the number of positive eigenvalues minus the number of negative eigenvalues. For $\Bbbk = \mathbb{C}$ (or any algebraically closed field) we have

$$\dim\colon W(\mathbb{C}) \xrightarrow{\sim} \mathbb{Z}/2\mathbb{Z} .$$

What we are going to do is to define a function $L \mapsto (T_L, q_L)$ from the set $\mathrm{Lagr}(V)^n$ of $n$-tuples of Lagrangians in a symplectic space, to the category $Q_+$ of quadratic spaces. The class of $(T_L, q_L)$ in $W(\Bbbk)$ will be, by definition, the Maslov index of $L$.

## 1.4 Signatures of manifolds

First we consider a parallel construction coming from topology.

There is a relation between Lagrangian correspondences and the cobordism category. Recall that the category of cobordisms $\mathfrak{Cob}_{4n+1}$ has as objects closed, oriented $4n$-manifolds. For any such manifold $M$, write $M^\circ$ for $M$ with the opposite orientation. A morphism from $M_1$ to $M_2$ is a cobordism $(W, M_1, M_2)$, where $W$ is a $(4n+1)$-dimensional manifold-with-boundary such that $\partial W = M_1^\circ \sqcup M_2$.

There is a functor $\mathrm{sig}\colon \mathfrak{Cob}_{4n+1} \to Q_+$. On objects,

$$\mathrm{sig}(M) = (H^{2n}(M, \Bbbk), \cup).$$

Note that $\mathrm{sig}(M^\circ) = \mathrm{sig}(M)^\circ$. On morphisms,

$$\mathrm{sig}(W, M_1, M_2) = \mathrm{im}\big(H^{2n}(W) \to H^{2n}(M_1^\circ) \oplus H^{2n}(M_2)\big).$$

The point is that this image is Lagrangian. The traditional "signature of $M$" is the class of $\mathrm{sig}(M)$ in $W(\Bbbk)$. The existence of this functor shows that class to be "cobordism invariant".

## 1.5 Definition and properties of the Maslov index

Following [Th], we will define the Maslov Index as an $n$-ary map

$$\tau\colon \operatorname{Lagr}(V,\omega)^{\mathbb{Z}/n\mathbb{Z}} \to \mathbf{ob}Q_+ \to W(\Bbbk)\ .$$

Given an $n$-tuple $L = (L_1, \dots, L_n)$ of Lagrangians, we have a cochain complex

$$C_L := \bigoplus (L_i \cap L_{i+1}) \xrightarrow{\partial} \bigoplus L_i \xrightarrow{\Sigma} V\ ,$$

concentrated in degrees 0, 1 and 2. Here $\Sigma$ just sums the components, while $\partial$ takes $a \in L_i \cap L_{i+1}$ to $(a, -a) \in L_i \oplus L_{i+1}$. As Beilinson pointed out, one can think of this cochain complex topologically as a "decorated cell complex," namely a $n$-gon with the face labelled by $V$, edges labelled by the $L_i$, and vertices labelled by $L_i \cap L_{i+1}$ (see Figure 1.1).

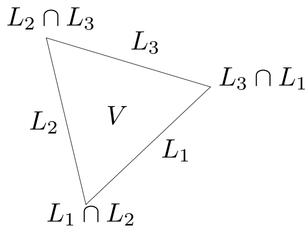


Figure 1.1: The cell complex when $n = 3$.

In analogy to the situation with signatures of manifolds, the "middle cohomology" $T_L := H^1(C_L) = \ker \Sigma / \operatorname{im} \partial$ of the complex is equipped with a "cup product" symmetric form, given explicitly by

$$q_L(a, b) = \sum_{i>j} \omega(a_i, b_j).$$

Here we have lifted $a, b \in T_L$ to representatives $(a_i), (b_i) \in \bigoplus L_i$.

**Proposition 1.5.1.** *$(T_L, q_L)$ is a quadratic space, i.e. $q_L$ is well defined, symmetric, and non-degenerate. It is called the "Maslov form".*

**Definition 1.5.2.** The *Maslov index* $\tau(L_1, \dots, L_n)$ is the quadratic space $(T_L, q_L) \in W(\Bbbk)$.

**Proposition 1.5.3.** *The obvious identities*

$$\begin{aligned} T(L_1, \dots, L_n) &= T(L_n, L_1, \dots, L_{n-1}) \\ &= T(L_n, \dots, L_1)^\circ \end{aligned}$$

*are isometries.*

In our topological interpretation, Figure 1, these isometries come from the dihedral symmetry of the polygon.

**Proposition 1.5.4.** *There exist canonical Lagrangian correspondences*

$$T(L_1, \dots, L_k) \oplus T(L_1, L_k, \dots, L_n) \to T(L_1, \dots, L_n)$$

*for each $k < n$. In particular, these give rise to equalities for $\tau$ in the Witt group.*

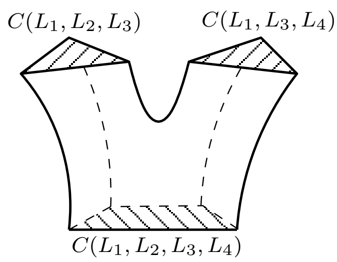


Figure 1.2: A "cobordism" between cell complexes defining the Lagrangian correspondence of Proposition 1.5.4 ($n = 4, k = 3$).

Using this proposition, one can always reduce to the case of three Lagrangians. In the topological interpretation, these Lagrangian correspondences come from a "cobordism" (Figure 1.5). These pictures are given a precise meaning in [Th].

Propositions 1.5.3 and 1.5.4 show that "$\tau$ is a cocycle":

$$\tau(L_1, L_2, L_3) - \tau(L_1, L_2, L_4) + \tau(L_1, L_3, L_4) - \tau(L_2, L_3, L_4) = 0. \tag{1.5.1}$$

This is the most important property of $\tau$.

## 1.6 The topology of the Lagrangian Graßmannian

### 1.6.1 Over the real numbers

Recall the identities

$$W(\mathbb{R}) \cong \mathbb{Z} \cong \pi_1(\mathrm{Lagr}(V, \omega)) \ .$$

Let $\pi\colon \widetilde{\mathrm{Lagr}}(V, \omega) \to \mathrm{Lagr}(V, \omega)$ be the universal cover of the Lagrangian Graßmannian.

**Theorem 1.6.1** ([Le])**.** *There exists a function $m\colon (\widetilde{\mathrm{Lagr}}(V))^2 \to \mathbb{Z}$ such that*

$$\tau\big(\pi(\widetilde{L}_1), \dots, \pi(\widetilde{L}_n)\big) = \sum_{i \in \mathbb{Z}/n\mathbb{Z}} m(\widetilde{L}_i, \widetilde{L}_{i+1}) \ .$$

**Remark 1.6.2.** We can think of $m$ as a function of homotopy classes of paths in the Lagrangian Graßmannian.

We now describe the construction of Leray's function $m$. For the standard Graßmannian $Gr_k(V)$ of $k$-dimensional planes in $V$, the tangent spaces are

$$T_L Gr_k(V) = \mathrm{Hom}\big(L, V/L\big) \ .$$

In the Lagangian case, $V/L = L^*$ and the tangent space consists of *symmetric* maps,

$$T_L \,\mathrm{Lagr}(V) = \mathrm{Sym}\big(L, L^*\big) = \big\{\text{quadratic forms on } L\,\big\} \ .$$

So we can speak of tangents being positive, negative, etc. A *path* is positive, etc, if all its tangents are. The identification $\pi_1(\mathrm{Lagr}(V)) = (Z)$ is chosen so that $1 \in \mathbb{Z}$ is represented by a positive path.

**Lemma 1.6.3.** *Between any $L_1$ and $L_2$ there exists a unique homotopy class $\gamma_\pm$ of paths $L_1 \to L_2$ such that*

- $\gamma_+$ *is positive,* $\gamma_-$ *is negative.*
- $\gamma_\pm(t) \cap L_1 = \gamma_\pm(t) \cap L_2 = L_1 \cap L_2$ *for* $t \in (0,1)$.

**Definition 1.6.4.** Suppose that $\gamma\colon L_1 \to L_2$ is a path. Then

$$m(\gamma) = \gamma_+(L_2, L_1) \circ \gamma + \gamma_-(L_2, L_1) \circ \gamma\ .$$

Here each summand is an element of $\pi_1(\mathrm{Lagr}(V), L_1) = \mathbb{Z}$.

**Remark 1.6.5.** $\mathrm{Lagr}(V)$ has a unique double cover $L^{(2)}$. For any pair $\widetilde{L}_1, \widetilde{L}_2 \in L^{(2)}$, the number $m(\widetilde{L}_1, \widetilde{L}_1)$ is well-defined modulo 4.

### 1.6.2 General ground fields

In the previous section, we studied real vector spaces. To generalise the notions to general ground fields, we observe the following correspondences.

- The double cover $L^{(2)}$ corresponds to the set $\Lambda$ of oriented Lagrangians $(L, o)$, where $o \in \Lambda^{\mathrm{top}} L\big/(\mathbb{k}^*)^2$ is the 'orientation.'
- $\mathbb{Z}$ corresponds to the general Witt group $W(\mathbb{k})$.
- The ideals $2\mathbb{Z}$, $4\mathbb{Z}$, etc. correspond to ideals $I := \ker\big(\dim\colon W(\mathbb{k}) \to \mathbb{Z}/2\big)$, $I^2$, etc. (Note that the Witt group is in fact a ring with multiplication given by $\otimes$.)

**Theorem 1.6.6** ([PPS])**.** *There exists a function* $m\colon \Lambda \times \Lambda \to W(\mathbb{k})$ *such that*

$$\tau(L_1, \dots, L_n) = \sum_i m(L_i, L_{i+1}) \mod I^2\ .$$

*Moreover,* $m$ *is invariant under the natural action of* $Sp(V)$ *on* $\Lambda \times \Lambda$.

## 1.7 The Metaplectic Group (Postscript)

The cocycle property (1.5.1) has the following consequence. Choose $L \in \mathrm{Lagr}(V)$. Let $Mp_1(V)$ be the set $W(\mathbb{k}) \times Sp(V)$ equipped with the multiplication

$$(q, g)(q', g') = (q + q' + \tau(L, gL, gg'L), gg').$$

Then $Mp_1(V)$ is a group and gives a central extension $0 \to W(\mathbb{k}) \to Mp_1(V) \to Sp(V) \to 1$.

Moreover, Theorem 1.6.6 implies that, choosing $\tilde{L} \in \Lambda$ over $L \in \mathrm{Lagr}(V)$, the subset

$$Mp_2(V) = \Big\{(m(g\tilde{L}, \tilde{L}) + q, g) : q \in I^2, g \in Sp(V)\Big\} \subset Mp_1(V)$$

is a subgroup, giving a central extension $0 \to I^2 \to Mp_2(V) \to Sp(V) \to 1$. Finally, it is traditional to quotient $I^2$ by $I^3$ to define a central extension

$$0 \to I^2/I^3 \to Mp(V) \to Sp(V) \to 1$$

in which $Mp(V)$ is called *the metaplectic group.*

Over $\mathbb{R}$, $I^2/I^3 = \mathbb{Z}/2\mathbb{Z}$, so $Mp(V)$ is then the (unique) double cover of $Sp(V)$. In this case $Mp_1(V)$ has four connected components, among which $Mp_2(V)$ is the identity. $Mp_2(V)$ is the universal covering group of $Sp(V)$.

**Remark 1.7.1.** Instead of choosing a Lagrangian $L \in \mathrm{Lagr}(V)$, one can construct the metaplectic group more canonically by observing that $Sp(V)$ embeds into $\mathrm{Lagr}(V^\circ \oplus V)$, by $g \mapsto \Gamma_g$, the graph of $g$. Then define multiplication on $Mp_2(V)$ by

$$(q, g)(q', g') = (q + q' + \tau(\Gamma_1, \Gamma_g, \Gamma_{gg'}), gg').$$

Moreover, $\Gamma_g$ has a canonical orientation. But the first construction using $L$ is convenient in many contexts.

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