Some Aspects of Greek Geometry on Papyrus

While it is generally agreed that the origins of western mathematics can be traced back to the mathematics of the ancient Greeks — one can note the mathematical vocabulary in use in European languages today which, although often Latinized, is clearly derived from Greek,[1] and further that the name of at least one ancient figure is still attached to a theorem, viz. Pythagoras[2] —, the extant sources for this history have been somewhat neglected since for the most part they are not contemporary. The text of Euclid, for example, comes via a manuscript tradition from the edition of Theon of Alexandria (fl. c. AD 380), some six centuries after Euclid. A further difficulty arises from the indirect nature of many of the sources, in that they are often incidental references or citations in texts concerned with other topics;[3] they may, therefore, consist solely of an oblique reference to or quotation from another author. As a result, despite the apparent high status of mathematics (and geometry in particular) in ancient Greece, the details of the developments which must have taken place during this period remain often tantalisingly obscure.

Nevertheless, as work has progressed during the last century on new sources of evidence for the ancient world in general, namely the transcription and study of papyri, wax tablets and ostraka, material has come to light which affords new insights into this area. It cannot be claimed that all the questions are or indeed could be resolved through study of these new texts, but in providing direct evidence that can be put together with the traditional sources, the increasing number of sources is invaluable.

Of course, the field is in itself something of a grey area: there are no well-defined boundaries separating mathematics from other sciences, still less separating different branches within the

[1] This is particularly true of geometry: e.g., the *names* of shapes (trapezium (τραπέζιον from τράπεζα), rhombus (ῥόμβος), pentagon, hexagon etc.), solids (octahedron, dodecahedron etc.) and other terms (hypotenuse etc.).

[2] In fact, a good deal of debate rightly concerns whether this theorem or its proof can properly be attributed to Pythagoras himself. The evidence is far from clear and one may have at best to conclude that this question will remain unresolved. For further discussion see Heath (1921) and Gow (1889).

[3] e.g., in Simplicius' and Philoponus' commentaries on Aristotle.

subject. Certain divisions can, however, be traced back as far as the fourth century BC and Plato, who separated arithmetic, logistic, geometry, stereometry, astronomy and harmonics.[4] It seems that by Plato's day mathematics was a field which had specialists in particular areas. It is not clear whether someone in an earlier period who carried out research in the science would have regarded himself as investigating ‘geometry’ or ‘arithmetic’ specifically, rather than mathematics in general. However, our focus is on what constitutes the contents of Euclid's *Elements*, i.e. mainly geometry and stereometry (with some number theory), although it will be necessary to bring in selectively such other areas as are necessary to place this and the other evidence in context.[5]

Before turning to our period (the earliest new source is dated the third century BC) it is worth pointing out that there remains considerable uncertainty about the details of the earlier history of Greek mathematics and its possible connections with and origins in Egyptian mathematics. Many points have been debated, and the significance of various sources of evidence has been contested, but it is not possible to dwell on these here.[6]

The most important figure for our purposes is Euclid.[7] This is because of the impact which his work had. While most of what he wrote was <u>known</u> about much earlier, the development of a complete synthetic system was revolutionary since it created a clear progression from the most basic (and most easily known) to those things less easily known, by valid logical steps. As a process for philosophy, and indeed geometry, this was not *per se* new (cf. the Aristotelian methodological principle set out in *Physics* I.1), but the application of it to reduce the whole

[4] Plato, *Republic* VII, 525a-530d

[5] A good deal of papyrus evidence exists for the closely-related science of astronomy, but this falls outside the scope of the present study and will not be considered.

[6] The bibliography on the history of most areas of Greek mathematics is well developed; this is not to deny, however, the importance of recent theories about exactly who discovered what and when (e.g. Szabó (1978) on the importance of the Eleatics, and Fowler (1999) on ratios, proportions and *anthyphairesis*), but to note that these historical investigations shed little light on the later geometric papyri with which we are presently concerned.

[7] His dates are not certain, but most of his work is thought to date from the reign of Ptolemy I, i.e. 306-283 BC. For a good account of what may be said about Euclid, see Fraser (1972: i.386-8).

of geometry to a set of (descriptive) definitions,[8] postulates[9] and (provable) propositions (each proposition building on the previous ones) was on a scale not previously seen. Euclid became a standard reference and teaching handbook, superseding all previous works so that few now even survive, and rapidly gained an exalted status which left its influence on education undiminished even into this century; in Greek he was known as ὁ στοιχειωτής.[10] As we shall see, his standard type of proof[11] and his logical ordering of material[12] provide an excellent means of classifying the new sources' content and approach, while his style and vocabulary also left a clear impression.[13]

We can now turn to those new sources[14] themselves and try to see what can be learnt from them. Because they do not, despite (and, in fact, due to) their chronological and geographical spread, allow us to chronicle the development of the field on the basis of them alone, a re-examination of these sources can be more effectively used, not to write a history of who discovered or was doing what, where and when, but to establish key facts about the development of this ‘genre’ on papyrus. While literary texts are numerous, and the

[8] These stipulate the meanings of terms.

[9] ‘Postulates’: these cannot be proved from the definitions but (a) seem to be true in all cases and (b) are required for the proof of propositions. They, together with the ‘common notions’ (κοιναὶ ἔννοιαι), are the minimum description of the kind of system in which the definitions are to operate.

The problems of the infinite and of parallel lines, which still occasion much spilt geometrical ink, are brilliantly dealt with in the descriptive nature of the Euclidean idealised system. He does not claim that this reflects the real world, only implicitly that the results obtained seem not to be obviously divorced from it. By thus removing the philosophical uncertainties, he created a firm place to stand from which he could move the rest of geometry, rather than endlessly digging to find a secure foundation but creating a hole from which he could not escape.

[10] The title of his handbook, the *Elements*, is significant, since it may reflect the structure which he deliberately adopts; on this, see Proclus (70-72), Heath (1926: i.114-7) and Fraser (1972: ii.390).

[11] See below, p. 27, in connection with the structure of geometrical texts in the Euclidean style.

[12] See below, pp. 26 & 28, and also Fraser (1972: i.390-1).

[13] Cf. POxy inv. 75/13(a).

[14] A full list of all the geometric texts on papyrus, ostraka and wax tablets which have been published is to be found at appendix A, together with a brief description of each, although not all are referred to in the main text here; the intention here is to bring out specific points to be made from these. Nevertheless, there is, at present, no up-to-date list of these sources for this subject area, so all published texts (even those which are unremarkable or too fragmentary to be of use here) have been included in it for the sake of completeness.

conventions and features exhibited in their presentation (e.g. punctuation, ekthesis, stichometry etc.) much investigated or generally well-understood, a technical genre of text presents a new challenge. Moreover, while there has been much interest in the intended audience of literary texts (e.g. whether they are those of (or for) school pupils, more advanced students, scholars, critical editions or just 'coffee table' presentation editions), and this has had a bearing on our understanding of what it was important for the average or well-educated citizen or the scholar to know, a similar process for this genre can now be adopted. Indeed it is possible to make three broad divisions.

The simplest to identify and indeed by far the most frequently found type of mathematics on papyrus is not exactly 'text',[15] but in fact tables of numbers, of fractions, additions, multiplications etc.[16] By their nature these are highly formulaic and not of particular significance in this enquiry. One can simply note that they are apparently of two kinds (and any or all of the different sorts of table can appear in either kind): those of traders for use in business (the calculation of prices, weights, etc.),[17] and those copies made apparently by school pupils in the process of learning their 'tables'.[18] In no case should these elementary tables be confused with the highly sophisticated astronomical tables of (predicted) movements of heavenly bodies, which are also found (albeit less frequently) in and after the Ptolemaic period.

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[15] I.e. proper words.

[16] The tables of numbers, however, are not included in the catalogue, on account of (a) the number of examples and (b) the relatively limited amount of information to be gained from them as far as geometry is concerned. A valuable up-to-date catalogue of these (which appears to be comprehensive) can be found in Fowler (1999: 268-76) and Fowler (1995).

[17] E.g., PMich III.146, which from its size (the roll was unusually narrow, about 10cm in height) seems to be a pocket manual.

[18] E.g., PMich III.147 where inaccuracies and features of the palaeography suggest that this was a school copying exercise.

Fowler (1999) draws some useful conclusions from this group of texts. They do provide information for the study of education, the development of arithmetic processes and the notation of numbers. (Beyond this, they may also enable us to see a contrast between the relative practicality, e.g., of the Greek and Roman systems; this might also be seen in the light of evidence about the abacus, e.g. Herodotus II.36.4, Diogenes Laertius I.59, Polybius, *Histories* V.26.13.)

A second broad grouping contains what can more properly be termed ‘texts’, specifically exercises to be solved in a range of mathematical disciplines. Some examples of these consist of tables accompanied by exercises to be solved by the (correct) use of these tables (e.g. PMich III.145). In other cases, exercises survive without tables, for which, however, these specific ‘table exercises’ provide a neat parallel. One example of this (PSI VII.763) is particularly interesting because it is almost exactly a direct translation of Horace, *Ars Poetica* 327ff. and almost contemporary with that text: methods of teaching were evidently remarkably similar across a wide geographical area.

Other arithmetical and ‘algebraic’[19] exercises are found which are not connected implicitly or explicitly with such tables (these are mainly very late, but early examples include, e.g., PLond II.265 (i cent.) & PMich III.144 (early ii cent.)); again these are of limited relevance here. In this group of practice exercises, the ones on which we shall concentrate are those with geometric content. Some fragmentary pieces are hard to classify beyond the fact that they are certainly geometry being applied to particular problems, but others seem to be divisible into two groups: texts copied by school pupils and texts with similar content but not apparently copied by them.

Among texts copied by pupils we find PHarris 50, with diagram, though it has not been possible to establish its exact nature: it may be some kind of areal or lineal calculation. Among texts copied for pupils we find PGen 124 (ii cent.) a basic textbook introduction to Pythagoras' theorem: this is a neatly set out explanation of the use of the theorem, illustrated with labelled diagrams, but not, of course, very advanced material: presumably it was intended as a textbook for learning, rather than for reference purposes. Similarly, the first-century PRain I.1 is a series of exercises; it may also have been for reference purposes but its intended audience is not clear. It is a substantial body of problems which would provide a good training in contemporary methods for finding volumes of solids, perhaps for trade or

[19] *Algebra*: formal algebra under this name was not introduced to Europe until the early ninth century (from al-Khwarizmi's *Hisab al-jabr wa'l-muqabalah*, which was based in some part on Diophantus of Alexandria's *Arithmetica* (mid third century) concerning general solutions to arithmetical problems by the use of symbols).

engineering. It is remarkable for its size and coverage. While demanding extensive study in itself, two important points are worth noting. First, this would have been an expensive text to produce, given its size and detail (with many diagrams); the contemporary importance of the content is evident from and emphasised by this. Although very advanced geometry was available at the time, the everyday need, as demonstrated by all (but in particular by these) exercises was the measurement or calculation of lines, areas and volumes. Second, this concentration contrasts with the broad range of Euclid's theoretical work. Exercises are generally confined to repeated practice in narrow areas; theory can be much more broad-ranging and this is because of its more general nature.

At this point, it is perhaps also worth recognising that these types of texts are frequently found on materials other than papyrus. In later centuries, these are more common and reflect the use of cheaper materials for everyday note taking by pupils in school, though papyrus is still used, notably for texts to be copied and preserved.

Our third type of texts is the most interesting and important group for our understanding of the pure science: treatises. These are still few in number but are identified by the way that they deal with general cases rather than specific values. These texts are usually found on papyrus, unlike the exercises, reflecting their more abstract reference nature.[20] The contents can be classified by where in Euclid's *Elements* the topics are covered; this shows us clearly how comprehensive or at least broad-ranging his collection was, since none of these sources contains anything which is not covered by his *Elements*. They range from the definitions in book I (PMich III.143) via topics found in books I (PBerl inv. 17469), II (POxy I.29) and III (POxy inv. 75/13(a)) to the construction of the regular solids found in book XIII (OBerl inv. 12609 etc., and POxy inv. 105/24). The complete absence of material concerning the topics of books IV to XII is interesting. Although in such a small sample we might not expect to find all

[20] Note, however, the Berlin ostraka (OBerl inv. 12609, etc.) which have an advanced text but are ‘das billigste Schreibmaterial’, and the editors' comment that ‘ein Buch aus Tonscherben ist undenkbar’ (OBerl inv. 12609, etc.: 9). The use of this cheaper medium for a scholar's practice notes to reproduce from memory the method of constructing regular solids is not surprising, but the completeness of the text should be observed. These do not seem to be notes made while going through a textbook, a view supported by the use of iota on the diagrams which Euclid avoided (see below, p. 35).

parts represented (and, of course, these are concerned more with arithmetic rather than geometry),[21] the concentration on those containing theorems about basic plane figures (i.e. books I to III) rather than proportion (V & VI), rational arithmetic (VII to IX) or irrationals (X) suggests that the first were the core areas of education; in contrast, book XIII concerns fairly advanced solid geometry (depending to some degree on the irrationals in book X), but may have either (a) formed a vital part of what a more advanced scholar or tutor might be expected to have mastered, but which he needed in writing for reference; or (b) had some particular application (whether of practical or perhaps even religious significance) which remains obscure.

With these types of source in mind, let us turn to a examination of some features of the material. Because the treatises are continuous, it is worth comparing their structure with the structure of Euclid's proofs as set out by Proclus:[22] the parts of his propositions are divided between the general and the quasi-specific, as follows. The πρότασις (‘enunciation’) and the συμπέρασμα (‘conclusion’) are entirely general, making a statement about what is true in all cases where the relevant conditions are fulfilled: e.g., ‘in [any] right-angled triangle, the square on the hypotenuse is equal to the square on the other two sides’. Both these parts of the proof are obligatory and always identical to each other, save that the latter is distinguished by the logical connective ἄρα (as second word) and appended to it is ὅπερ ἔδει δεῖξαι (or ὅπερ ἔδει ποιῆσαι).

The remaining parts are ‘quasi-specific’: they refer to an example case which, because it is dealing with relationships and never absolute values, can show the validity of the general statement of the proposition. It consists of up to four parts. The ἔκθεσις (‘setting-out’) uses third person imperatives to frame the conditions which obtain in the proposition in question: e.g., ‘let there be a triangle ABC, and let AB be produced to a point D, and let AD be joined’. The διορισμός indicates how the general statement to be proved applies to this case, and is

[21] In fact, even though texts concerning arithmetic and proportion might fall outside the scope of the present study, there are no new sources which contain any reference to material covered in *Elements* IV to XII.

[22] For a fuller account, see Fraser (1972: i.394-5).

usually introduced by λέγω ὅτι: e.g., 'I say that the angle DBC is greater than angle ABC'. The κατασκευή ('construction') may be necessary if all the lines required for the proof have not yet been drawn as part of the ἔκθεσις; e.g., 'let CB be produced to a point E, and ED be joined'. The only obligatory[23] one of these quasi-specific parts, however, is the ἀπόδειξις ('proof'). This may contain a (further) construction, but almost always relies on propositions demonstrated earlier (never, of course, on ones to be shown later). It consists in the main of statements introduced by ἐπεί: e.g., 'since AD is longer than AB, the angle subtended at C by AD is greater than that by AB'.

Euclid's consistent use of this structure seems to have obliged him to reformulate many of the proofs handed down to him, since this does not seem to have been a standard method among the writers of earlier *Elements*. After him it became a standard method to construct a proof and to work synthetically, i.e. using only what is already established to prove new truths.[24] Using both these facts we can more easily identify (a) which parts of a proof we have, or may be missing; and (b) how it fits into a wider picture: if a proof is relying on a proposition *P*, consistency with the synthetic approach requires that it must logically come after that proposition.[25]

Other than the identified Euclid texts, only three other theoretical texts survive where analysis in the light of the propositional structure is useful. Each, however, is revealing because they allow application of this in different ways. The Berlin ostraka (OBerl inv. 12609 etc.) show a distinctly Euclidean approach. The longest text (OBerl inv. 12609) clearly uses this method: at the same time deviation from it can be seen in the use of ἄρα in line 6, a connective which should properly only be found in the general conclusion, but here seems to be in a quasi-specific section. The implication is that this is not exactly Euclid's text but an attempted reproduction thereof, in the appropriate style; the editors describe it on other grounds (mainly

[23] 'Obligatory' in the sense that it is found in all propositions and, in a minimal proposition, may be the only one of the quasi-specific parts included.

[24] For Euclid's own distinction between analysis and synthesis, see his 'definition' in book XIII after proposition 5, printed in Appendix I by Stamatis (1973: 198-204).

[25] For an example of looking at an unknown text in this way, see POxy inv. 75/13(a).

the material) as the ‘Übungsaufgaben eines in der Mathematik Fortgeschrittenen ..., der seinen Euklid studiert hatte und nun die Aufgabe einmal selbstständig zu lösen unternimmt’,[26] which supports this interpretation. To look at a text in this way can help to identify who might have written it and for what reason.

Structural analysis shows a different result in POxy inv. 75/13(a), which seems to conform only with difficulty to the Euclidean model: but this tells us two valuable things. First, despite the apparently Euclidean style and vocabulary, the structure is sufficiently deviant from Euclid's to make it unlikely to be by him. Second, there still seems to be an attempt to use this structure, albeit not as effectively as Euclid, and one can conclude that this was seen by others as the appropriate method to adopt for formulating geometry.

Finally, there is the prospect that this type of analysis will aid the reconstruction of fragmentary texts such as PBerl inv. 21153v. Even if these 28 fragments do not all belong to the same proposition, they can be divided into general and quasi-specific groups and a speculative ordering might be suggested. Indeed, the style of exposition used might make one more confident about supplements of items which one knows (from the structure) are generally to be found in, e.g., the construction or conclusion. This will not solve all the problems, of course, but it will be an important and helpful consideration.

A second important feature to consider is the layout of these texts on the page. The positioning of diagrams is particularly interesting: in general, where found, diagrams are at the end of the text which they illustrate. Where this does not appear to be the case, important questions are raised.

POxy I.29 is unusual in setting the diagram into the text apparently towards the start, as in modern editions of Euclid. This would be highly significant in terms of presentation, since the text flows around the diagrams, requiring that they be drawn first; this demands that some consideration be given by the scribe to where he should begin the diagram in relation to the

[26] OBerl 12609 etc., p. 9

start of the proposition. Moreover this diagram is found on the right hand side of the column, not to the left, which may be the result of several factors: e.g., the scribe's exemplar was laid out in this way (which just pushes the puzzle further back in the chain of production); or, the diagrams appeared in the right margin (as perhaps unlabelled annotations to the exemplar) but for reasons of column width and spacing, he has moved them into the column; or, this text is not in fact drawn from a complete text of the *Elements*.

Fowler (1999: 213 & 215f.) suggests initially that this piece may be part of the 'notes by someone working through' *Elements* II.[27] Clearly this papyrus is unusual, but his later suggestion is more persuasive: this is from a summary or index text of (at least) *Elements* II, in which only the enunciations themselves, together with a miniature indication of the diagram's shape, are included, much as catalogues of manuscripts and modern editions of collections of music consist of titles and *incipits* (in music, normally the first two or three bars of each piece). Most of his argument for the reconstruction of the text as the enunciations of II.4 and II.5 (delineated by *paragraphoi*) with their associated diagrams[28] seems quite right: however, the text does seem to be too systematically numbered and carefully laid-out to be merely notes. This is, therefore, surely some kind of index to the *Elements*, possibly to accompany a full edition in which the proofs are given in full with full-size labelled diagrams; it could also be used independently as a condensed reference-guide or aide-mémoire. In this case, the fact that the diagrams <u>appear</u> to be at the start of the propositions is no longer a difficulty, since they are <u>in fact</u> at the end of the (therefore complete) texts to which in both cases they refer: the absence of labels will be consistent with the absence of the text in which those labels might be used (and one working through the text would surely be expected to draw a rather larger diagram and to need to label it as he went along anyway). Moreover, the

[27] Contrast the Berlin ostraka, which must indeed be the notes of a scholar working through and trying to reproduce a text (*Elements* XIII), but which are neat and with careful labelled diagrams; the method in that text, however, is basically complete. Here the text seems not to be complete, and the diagram is not really comparable; this text looks altogether different from that confirmed example of notes, hence the need for an alternative explanation of its layout.

[28] Fowler sensibly dismisses the reconstructed diagram of lines in the *editio princeps* in favour of a reconstruction of that of II.4: the trace of a line which survives is compatible (particularly in its position in this space) with the diagram expected for II.4.

presence of the number ε′ next to the diagram as well as the text suggests this may be some kind of index.

Whether such condensed texts of other authors were widely available is not clear, because few genres would be suitable for this kind of treatment: it is not implausible, however, that ‘economy’ editions of Euclid containing just the statements of what he shows to be the case (without the proofs) might have been produced (with relatively wide margins as here for notes), since a full text edition of even one book with diagrams would be an expensive item which would have to be produced by a particularly careful and skilled scribe. Despite the limited number of sources for Euclid on papyrus, a suitable parallel for such an edition may be found in PBerl inv. 17469, which has just the enunciations of *Elements* I.8-10 with unlettered diagrams (but numbered in the same way as POxy I.29).[29] Its diagrams are more carefully ruled and the hand is slightly more formal, but both examples seem to be this type of edition.

After the layout, in which diagrams form a major part, the diagrams are themselves worthy of further examination. Of the surviving texts only a proportion have diagrams, and these fall neatly divided between the two groups of proper ‘texts’ established earlier: problems/exercises with numerical solutions and theoretical treatises without. This can be seen in both the appearance of the diagram itself and still more in its labelling, because the former group have the lengths of lines and areas of shapes indicated by numerals (i.e. letters),[30] whereas the latter have points defining lines, angles and shapes marked with unique reference letters. More will be said of these below, but first it is worth looking at this division and what else follows from it.

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[29] Fowler's comment (1999: 215) on the layout of PBerl inv. 17469 seems inexplicable and erroneous to me: ‘This is the only Greek text I know of where the figures are placed before the enunciations ...’. This statement is simply wrong: the middle figure labelled θ′ appears below the text to which it refers, and similarly the text below it refers to the diagram below that.

[30] One example is PBerl inv. 21188v which is entitled ‘Treatise on Geometry’ but clearly contains applied geometry, since $\overline{\iota\delta}$ (fr. 1, line 16) and $\overline{\kappa\delta}$ (fr. 5, line 5) are highly likely to be numbers, and in fr. 2, line 1 we have ἀρουρη: contrast PBerl inv. 21153v discussed below, p. 38.

Because the publication of these papyri does not always include exact 'diplomatic' reproductions of the diagrams (instead stylised versions are printed, or editors have seen fit to present an edited/supplemented version without making this clear), comparison of the appearance of these has not been widely considered.[31]

The geometric exercises worked through in PRain I.1 (i cent.: effectively the earliest among the group of exercises) contain a number of diagrams, typical of this type in being neither to scale nor drawn with any kind of straight edge. The lengths are marked in unspecified units, and only those lengths given in the problem are marked. Adjacent to or within some diagrams are numbers indicating either the volume, height or area which was to be calculated (depending on the type of problem).[32] Since this is a text of solid geometry, the problem of the representation of three dimensions on the two-dimensional page had to be faced. In some cases, the view is from above looking down vertically onto the solid, in others from the side: depth is sometimes produced by oblique drawing, but this is not used consistently throughout. All of this points towards the diagrams as simply further explanation or illustration of the question being posed, drawn freehand though carefully; some numbers on them remain to be explained, but this difficulty may be the result of the poor preservation of the main text in places. Nevertheless, they still serve to make the problems more understandable to us, e.g. by allowing us to explain the abbreviations and methods used.

The scribe of the Ayer papyrus (PChic 3: i-ii cent.) is criticised scathingly for 'being careless' and 'absent-minded' with his diagrams: 'No effort seems to have been made to draw them in true proportion'.[33] While indeed the parallelogram diagram in question has opposite sides equal and the sides labelled with the values required for the example, in the light of the above

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[31] In looking at these, I have been limited to those where photographs are available or 'diplomatic' reproductions printed.

[32] In the second diagram in column 13 in this text (Exercise 30), the proposed supplement for these adjacent figures does not appear to take this factor into account: the supplement proposed is an unimportant value from an intermediate stage in the calculation, which is itself a supplement in the main text too. A better supplement would be the volume of the pyramid in question, as in the diagram in the previous column (Exercise 28): this would be π[δ] which fits better with the surviving π than does π[ϲνβ] and survives in the main text.

[33] PChic 3, p. 20.

examples it is very far from apparent that the scribe was 'oblivious to the incongruity'.[34] Not only may his exemplar have contained diagrams in the proportions in which he copied them, but he may rightly have recognised the diagrams as symbolic and not intended to be in proportion. In the third diagram on this papyrus, a parallelogram, the (total) area of the figure is given to the right of it (as with the volume on some diagrams in PRain I.1, above). Both here and in PRain I.1 the numbers expressing the result of the calculation are preceded by a forward oblique stroke '/', which in texts is an abbreviation for γίνεται or ἐϲτί (i.e. 'equals, =');[35] this interpretation seems to hold good here.

Coming into the second century, we see more numerous examples; in some cases (MPER XV. 172, 173 & 178) these are drawn freehand and mainly unlabelled, while in others a straight edge has been employed (e.g. PGen III.124), suggesting care taken to produce both an aesthetically pleasing and at the same time easily intelligible edition.[36] Finally, in the fourth century, we come to the most puzzling of illustrations, in PSI III.186. The significance of this pair is altogether obscure and, although they are certainly related to the problems beneath which they are found, they are not of <u>geometrical</u> importance, lacking both labelling letters or values, and indeed any recognisable geometrical shapes.

More helpful are the diagrams of theoretical treatises; usually labelled with letters, and certainly never labelled with numerical values, these theoretical representations illustrate the texts they accompany. The proof is, of course, always the procedure explained in the text, and the diagram is merely a visual aid, which, although useful, is not intrinsically part of the proof, i.e. no single demonstration with a diagram of particular dimensions can be a general proof, particularly in the idealised Euclidean system of points-without-area and lines-without-

[34] It is altogether to be regretted that the editor, like many others, did not take the surely consequent view that a diplomatic representation of the diagrams should accompany his transcript.

[35] Cf. OBodl II.1847

[36] In this case, it appears that in the centre of the first diagram, although not printed in the transcription, the letter ϛ appears (representing 6, the area of the triangle).

thickness.[37] They allow a general representation of the relationships between the values in question (as in the *Meno* demonstration[38]) without the need for the absolute values.

The earliest diagrams of this kind on papyrus are found on PHerc 1061 (i cent. BC). One of these illustrates the bisection of an angle and is neatly drawn (although not apparently ruled) and labelled. This is a philosophical text with some geometric content, and we can note that by the date of this copy the diagrams are included with this kind of text even where it is found within a non-geometric context (diagrams were not transmitted with the geometric episodes in Plato's *Meno*). The link between the text and the diagram seems to be strong, especially because the text here is in summary form and does not give the proposition in full (without the full text the diagram is essential, whereas with it the diagram is not essential).

Among the other theoretical texts with diagrams are the two index texts. The first, POxy I.29, has a small, unlabelled, rough diagram which seems to be an incidental illustration; this may indicate a cheaper edition produced by a less skilled scribe trying to fit more into the available space. In contrast the later PBerl inv. 17469 has large, ruled (but not labelled) diagrams. The care taken here seems to indicate a more expensive reference work of some kind, produced by a professional scribe. In PFay 9 (ii cent.) we find Euclid I.39 and I.41, divided by a labelled and ruled diagram: again this is a professionally produced text. Clearly Euclid was being being copied and distributed in a variety of types of editions from the first century onwards.

In the third century there is POxy inv. 105/24, which has part of a diagram surviving: this has been neatly and carefully constructed with a straight edge, and although recognisably the right diagram, the labelling is not consistent with the usual reading of the text. It is certain that this comes from *Elements* book XIII: the diagram is thus one which accompanies a complicated

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[37] Szabó (1978: 185-199) raises an interesting question about the significance of the word δεῖξαι in ὅπερ ἔδει δεῖξαι: originally it meant physically showing something to be the case (possibly on a diagram) but by the time of Euclid at least this was no longer an acceptable method of proof, which had instead to come from deductive logical sequences.

[38] In fact Socrates uses particular lengths in his demonstration, but they seem to be representative unit lengths to show how the relationships of areas depend on relationships of lines: the absolute values are not important geometrically but are needed to make the problem accessible to the slave (and Meno).

proposition. It may be, therefore, that it has been simplified, or is a supplementary diagram intended to illustrate just the corollary to XIII.17.[39]

As noted above, letters frequently accompany diagrams to indicate points on the same. These, where present, are then used in the text to refer to the points indicated as part of the way a proof is explained. The use of letters in a general proof, of course, goes back at least to Aristotle's day in logic, but the origin of their use in geometry is uncertain. They were definitely introduced before the time of Euclid, who makes considerable use of them.[40] Earlier use is obscure, but some have claimed that Hippocrates of Chios is quoted verbatim in Simplicius, and see there the original phrasing (with prepositions) used for these letters: ἡ ἐφ' ᾗ AB ('the line on which A and B are points', which is later shortened to ἡ AB, 'the line AB'). Further, it has been claimed that the more perspicuous phrasing suggests these were an innovation in Hippocrates' day (i.e. the prepositions were needed because it was not yet obvious that these were indexical letters).[41] There may also have been a development from that time, as Hippocrates appears to have included iota in the alphabet used for this purpose, to Euclid who omitted iota. Perhaps iota was at risk of being thought to be a line on the diagram rather than a label.[42]

Euclid is absolutely dependent on these letters and they clearly represented a breakthrough in the way that increasingly complex geometric proofs *vel sim.* could be spread accurately

[39] That the labelling letters do not correspond to the usual reading of the text confirms the way that they are at particular risk in transmission. Zaitsev (1999: 525-6) makes a similar point about mediaeval manuscripts which is equally applicable to this papyrus: 'To begin with, some manuscripts ... lacked drawings entirely, a consequence of the common medieval scribal practice of copying the text first and leaving the drawings to be filled in later; sometimes the latter were completely forgotten. A second problem ... is that some of the drawings that did appear were incorrect.'

[40] These letters are present in the earliest new texts, the Berlin ostraka, OBerl inv. 12609 etc., but this group comes too late (i.e., shortly after Euclid) to provide a new *terminus post quem* for their overall introduction as a device.

[41] See Gow (1884: 169).

[42] In fact, the Berlin ostraka would then be exceptional in being a (late) example of where iota is used on a diagram. Either this omission of iota was a recent phenomenon (these letters were still developing; overlining had yet to appear) so Euclid had adopted it but not everyone had; or it was confined only to proper professionally-copied texts and not seen in the everyday workings of a scholar. This would support the view that iota was eventually omitted consistently because of the risk of its loss in transmission.

without the need for direct contact between scholars (i.e. to point out physically on a diagram exactly what is being shown). This must immediately have increased the speed with which this information could be transmitted, for once the principle of labelling is understood, scholars at a distance are no longer reliant on direct contact. This is perhaps reflected in the fact that our earliest new texts, the Berlin ostraka, were found in Elephantine, a good distance from the centre of education, Alexandria. Moreover, from the point of view of education, the amateur geometrician can purchase a text and learn without the need for constant, possibly expensive, tuition.

In terms of what might be called 'academic rigour', the development of the diagram can be ranked among the greatest achievements of ancient mathematics, because it allows the setting out of formal proofs with a distinct structure, i.e. one moving from the general to that kind of specific but idealised case from which a generalised conclusion can then be drawn. Hence we can see why the standard format of a Euclidean proof included a number of stages in which the diagram necessary for the <u>explanation</u> of the proof is constructed. Consequently, because its construction is described, the diagram is, at least from the time of Euclid, not in itself a <u>necessary</u> part of the text, and so may be omitted, especially where space is limited (due, e.g., to cost): the result is that, in texts where no diagram survives while letters in the text do, it is important not to conclude that a diagram was present just because letters are. The reader could well have been expected, especially in learning or working through the proof, to construct the diagram in order better to understand the principles. Absence of diagrams also underlines the point that a diagram *per se* is not the proof of a proposition, merely the result of a demonstration of its validity in a prototypical case (i.e., one which can be generalised).

The letters are in general palaeographically unexceptional, but conventionally distinguished from words by a line drawn above the letters. This is a convention which shows development. For some reason, in the earliest example on the Berlin ostraka, they are not distinguished in this way. Instead, spacing is used before, and sometimes after, the (groups of) letter(s) to separate them from the words written in *scriptio continua*.[43] Overlining must have begun at some time, and these ostraka are evidence that in the second half of the third century BC it was (at least) not yet completely established as an inviolable convention. In later examples, it seems always to have been used. One piece of evidence does not, of course, prove that the convention was not used at all at this time, but it does help show a transition with respect to convention between the date of these ostraka and the beginning of the first century BC.[44]

This convention seems to have developed to prevent possible confusion:[45] early on, when a diagram was customary with all texts containing such letters, they were easily identified (perhaps by spacing) as references to those letter-labelled points on that diagram; later, however, as the letters became commonplace (and this happened gradually after they were introduced) but the diagram was omitted from some copies (e.g., for reasons of space), these letters were at risk of no longer being clearly recognised as these labels (extra spacing is not a very robust way of indicating such information because it can easily be overlooked in copying or read mistakenly as punctuation) and therefore particularly at risk of copying errors. Perhaps by analogy with numbers, a more definite marking convention began to be used to highlight them: overlining. Once established, however, the connection between overlining and indexical letters meant that subsequently these letters in the body of the text were so marked even when the diagram was present (i.e. had been re-introduced, perhaps in an illustrated edition), because it was the marking that told the reader that these were such indexical letters rather than text: in support of such a view we note particularly that the letters actually <u>on</u> the diagrams are <u>not</u> usually marked in this way in any period.[46]

[43] Note that, whereas the extra spacing after each example might be explained as corresponding to punctuation expected at that point by the sense, additional space before these letters lacks any other plausible explanation, because the previous word is almost always the definite article.

[44] In PHerc 1061 (beginning of i cent. BC) the letters are distinguished in this way: although this is not clear from the drawings, the overlining can be made out on photographs. It is clearly seen on POxy inv. 75/13(a) (end of i cent. BC).

[45] These letters, not being words, are akin to the alphabetic Greek numerals in needing to be distinguished from letters forming words — hence the parallel development.

[46] I have found no examples of such letters on diagrams being overlined.

Overlining is not, of course, always sufficient for the modern scholar to be certain that these are letters of this labelling kind and not overlined numbers, but a preponderance of two-letter combinations in the text found together with the feminine definite article is strong evidence that these are referring to a line.[47]

These letters can also be used to help assign even the most fragmentary texts to the categories set up earlier, on account of their generalising force. For example, one papyrus entitled 'Geometrical Problems' (PBerl inv. 21153v: ii cent.) is almost certainly a theoretical treatise and not a collection(?) of applied problems, on a number of grounds: the vocabulary (e.g. fr. 1, lines 1 & 2: ἄκρον καὶ μέϲον [sc. λόγον]) and style are consistent with a treatise, but the clinching argument is that overlined letters found here are of this kind (e.g. fr. 2, line 4: πρὸϲ $\overline{\gamma}\overline{\alpha}$) while none appears to represent a numerical value (which would be expected in at least some cases if this were a set of geometrical problems or exercises).[48]

A brief note will suffice on how these letters were used, since they are still used in basically the same way. A single letter with the neuter article indicates a point: τὸ A (sc. ϲημεῖον). Two letters with the feminine article indicate a line: ἡ AB (sc. εὐθεῖα). The square on a line and, by extension, the square of a length are indicated by a noun phrase: τὸ ἀπὸ τῆϲ AB (sc. τετράγωνον). Polygons are referred to by the neuter article with the letters on its vertices, circles by the masculine article with the letters of points lying on it; in each case the letters are in the order on the diagram, generally anticlockwise. This does highlight one feature: that the letters are allocated in the order which they are needed in the description of the construction. Each additional point is given the next letter in alphabetical order, illustrating the primacy of the text and the ancillary nature of the diagram. Modern practice, even though the diagram is not a proof, tends to create a finished diagram and label it as far as possible sequentially so

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[47] Moreover, the most frequent type of combinations are letters from early in the alphabet which cannot be numbers: $\overline{\alpha\beta}$ is nonsense as a number — the first two-character number is ια′.

[48] Contrast this with PBerl inv. 21188v, referred to above, p. 31.

that point B is a vertex next to A, and C next to B etc.; the proof is then written using these letters, and often letters are allocated which have no actual part to play in the proof.[49]

In addition to letters, there are also important abbreviations which contribute to the formulaic technical style. These can be divided into two types. The first, which we have already seen, is the elliptical use of short phrases or single words in the place of much longer phrases. This is understandable because it eliminates words which can be readily supplied in order to promote a clearer, briefer explanation. The second type is directly parallel to what is found in modern mathematical notation, with symbols for various kinds of operation; the ancient system in fact only contained a few for the more common operations.[50] The most common is the symbol for ‘equals, =’ (in Greek, variously ἐϲτί, εἰϲί or γί(γ)νεται, γί(γ)νονται), which can take two forms used basically interchangeably. The first usually represents ἐϲτί or εἰϲί, takes the form of a backslash ‘/’ and is frequently found next to diagrams (as noted above) as well as in texts; the origin of the shape is unclear. The second represents γί(γ)νεται or γί(γ)νονται, and is clearly in the form of gamma combined with iota (crossing the horizontal of the Γ). These are both frequently found in the exercises. Also common in the exercises are abbreviations of units of measurement, especially the words ἀρτάβη, ἄρουρα and ποδῶν, which occur frequently and are parallel to modern abbreviations of units of measurement.

Important too are those abbreviations found in the Euclid fragment POxy inv. 105/24. Because this is an identified text, we know exactly what these abbreviations are to be expanded into. Some are common elsewhere, although mainly in technical literature or scholia (e.g., the monogram ϗ for καί, paralleled by κϛ in POxy IV.663, a late ii cent./early iii cent. hypothesis of a play). Others, although found elsewhere, can be seen as characteristic of the technical/formulaic nature of the text, for example α’τ’ for ἀ(πὸ) τ(ῆϲ). This is akin to our use of a superior ‘2’ to indicate the square of a value, but reminds us that in geometry this is referring to the square on the line, not just the numerical/arithmetical square of the value. One

[49] An early example of non-sequential letter-labelling on papyrus (derived from Euclid) is found on PHerc 1061 (beginning of i cent. BC).

[50] For a comprehensive survey of abbreviations, see Blanchard (1974).

new symbol is found: ʅ̲ for (πρóϲ) (= ‘is to’ or ‘:’, expressing a ratio). This word is abbreviated also in POxy VIII.1086 (i cent. BC scholia), according to the editors as a semicircle for the π surrounding a short vertical stroke (for the ρ). In POxy VI.856 (iii cent. scholia) the ρ is written on top of the π, producing a monogram. However, the use of the particular symbol in the Euclid fragment seems to be unique and may be either specifically geometric (like ‘:’ for ‘is to’) or just the idiosyncrasy of a particular scribe.

A second symbol, resembling the obelos periestigmenos (⸓), is used in this piece at the start of a line but not in the margin; this may indicate that this was a critical edition of the text which had comments elsewhere on the passage. Alternatively, the mark may serve as a cross-reference to another part of the text, since it is found at the start of a line which contains the beginning of a new sentence (a sentence which states the proved result of another proposition, XIII.14). If this were in ekthesis in the scribe's exemplar, and the sign were also used as a space-filler at line end, it is not impossible that this sign was mistakenly copied into the text with this meaning (it is unlikely to be a space-filler at line beginning).[51]

With all these features in mind it is worth considering what in general can be learnt about this genre on papyrus (and in the other ancient sources). None of these sources seems to contain material which was completely unknown when they were written. Indeed, by Euclid's day most of the groundwork for their content had been done, and his formulation of it survived in a position of prestige down to modern times: his geometric system was not seriously challenged for over a thousand years. They do underline, however, the fact that the status of the *Elements* was established in antiquity, possibly even within Euclid's lifetime. At the same time, everyday use of mathematics was understandably centred on arithmetic (in trade) and the practical application of geometry (in say, building, surveying etc.), mainly in terms of calculating lengths, areas and volumes. The conclusions to be drawn from the evidence when

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[51] For this sign as a reference mark for marginal comments, cf. POxy I.16, XV.1797 and XXI.2306. For this sign in mathematical texts, cf. MPER XV.173 (ii cent.), where it also occurs at line beginning but is supposed to be an abbreviation for ἄρουρα: clearly this is not the case here. Similarly at MPER XV.177 it may represent ἀρτάβη or εἰϲίν. On this sign and its use for reference, see McNamee (1977: 110; also 1992).

taken as a whole are in fact thus limited, mainly because the evidence is still not extensive; but it is certainly growing, and there will be advances as connections are made. However, for the present one should be content with two points which have been brought out here. First, on a small scale, reassessing the details of particular sources can allow us to see them in a new light and explain apparently anomalous features (such as POxy I.29, PRain I.1 etc.). Second, the wider context can be informed by these discoveries. Geometry had spread widely from a concentrated focus in Alexandria and was being practised at a whole range of levels across a good part of the Greek-speaking world, e.g. from basic education and handbooks of exercises to scholars' treatises. While much of the use was practical, a certain amount of theory was learned and in the form of exercises it constituted a key part of general education. It will be interesting to see what continued study will be able to discover about this somewhat neglected area.