Testing for rational bubbles in a co-explosive vector autoregression

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Reproducing some of empirical results

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The results were produced in OxMetrics using category `model for time series data` model class `multiple-equation dynamic modelling`.
Use data base `USstockdated.in7`

1. **Unrestricted vector autoregression:** Let $X_t = (P_t, D_t)'$. Fit the model (2.6):
$$X_t = A_1 X_{t-1} + A_2 X_{t-2} + \mu + \epsilon_t.$$
OxMetrics gets:

```
SYS( 1) Estimating the system by OLS
        The dataset is: C:\USstockDated.in7
        The estimation sample is: 1976 - 2000

URF equation for: P
                  Coefficient  Std.Error  t-value  t-prob
P_1                  0.983136     0.2065     4.76  0.0001
P_2                  0.350667     0.3059     1.15  0.2653
D_1                   1.90361      23.96   0.0795  0.9375
D_2                  -4.44279      23.06   -0.193  0.8492
Constant            -0.339314      1.105   -0.307  0.7619

sigma = 0.355313   RSS = 2.524948157

URF equation for: D
                  Coefficient  Std.Error  t-value  t-prob
P_1                0.00437328   0.001564     2.80  0.0111
P_2               -0.00428609   0.002317    -1.85  0.0792
D_1                   1.32880     0.1815     7.32  0.0000
D_2                 -0.560458     0.1747    -3.21  0.0044
```

```
Constant              0.0186366   0.008366     2.23  0.0375

sigma = 0.00269116   RSS = 0.0001448468787

log-likelihood     108.709071  -T/2log|Omega|     179.655997
```

2. **Specification analysis:** Use `test menu` then `test summary` to reproduce Table 1. Note that portmanteau test not valid in explosive case, see Nielsen (2006a). Preliminary calculations suggest that hetero test and reset test not valid either.

3. **Cointegration analysis:** Use the standard options: `test` then `dynamic analysis and cointegration` then `I(1) cointegration analysis`. The results match those of Table 2. The reported asymptotic p-values apply in explosive case.

```
I(1) cointegration analysis, 1976 - 2000
  eigenvalue    loglik for rank
                  98.20514   0
     0.45990      105.9052   1
     0.20093      108.7091   2

  H0:rank<=  Trace test [ Prob]
      0         21.008 [0.038] *
      1         5.6077 [0.231]
```

4. **Characteristic roots:** Use the standard options: `test` then `dynamic analysis and cointegration` then `roots of companion matrix`.

```
Eigenvalues of companion matrix:
       real       imag    modulus
      1.258     0.0000      1.258
     0.6747     0.3538     0.7619
     0.6747    -0.3538     0.7619
    -0.2953     0.0000     0.2953
```

5. **Estimating the model H_1 with cointegrating rank 1:** In model formulation remember to restrict constant to cointegrating space, that

is, turn flag `U` off. In estimation, choose model type `cointegrated VAR` then set rank to 1. Note that likelihood value is the same as found in item 2.

```
SYS( 2) Cointegrated VAR

Cointegrated VAR (2) in:
[0] = P
[1] = D
Restricted variables:
[0] = Constant
Number of lags used in the analysis: 2

beta
P                    1.0000
D                    29.296
Constant            -4.0079

alpha
P            0.27333
D         -0.0010627

Standard errors of alpha
P           0.070029
D         0.00057071

......

log-likelihood      105.905205  -T/2log|Omega|     176.852132
no. of observations         25  no. of parameters           8
rank of long-run matrix      1  no. long-run restrictions   0
beta is not identified
No restrictions imposed
```

6. **Updated characteristic roots:**

```
Eigenvalues of companion matrix:
       real       imag    modulus
```

1.223	0.0000	1.223
1.000	0.0000	1.000
0.4820	0.0000	0.4820
-0.2388	0.0000	0.2388

7. **Estimating the model** M_{1D}**:** Reformulate model on the form (2.8) so

$$\Delta_1\Delta_\rho X_t = \alpha_1\beta_1^{*\prime}\Delta_\rho X_{t-1}^* + \alpha_\rho\Delta_1 D_{t-1} + \epsilon_t$$

Run the algebra code:

```
DD  = diff(D,1);
DrP = P - 1.223*lag(P,1);
DrD = D - 1.223*lag(D,1);
```

Then fit a cointegrated model. In the model specification choose the variables

```
Y DrP
  Constant
  DrP_1
Y DrD
  DrD_1
U DD_1
```

which gives the output

```
SYS( 3) Cointegrated VAR

Cointegrated VAR (1) in:
[0] = DrP
[1] = DrD
Unrestricted variables:
[0] = DD_1
Restricted variables:
[0] = Constant
Number of lags used in the analysis: 1
```

```
beta
DrP                    1.0000
DrD                    29.857
Constant              0.90319

alpha
DrP          -1.2208
DrD        0.0047249

....

log-likelihood     105.905044  -T/2log|Omega|     176.851971
no. of observations        25  no. of parameters           6
rank of long-run matrix     1  no. long-run restrictions   0
beta is not identified
No restrictions imposed
```

The likelihood is reduced from `105.905205` to `105.905044`. There are two reasons to this reduction: a restriction has been imposed, and the estimator for ρ is not maximum likelihood under the restriction. Therefore, re-run the algebra above for different values of ρ, repeat the estimation and collect the results. Note that the model settings are the same as before. The only difference is that the variables are changed in the data base. That is, it suffices to (1) find the old algebra code in the results file, modify, highlight, press `ctrl+a`, (2) press the `estimate` button. An example is given in Table 1. Note that the profile likelihood is nearly quadratic

8. **Estimating the model M_{1DS}:** Run the algebra code:

```
DD  = diff(D,1);
DrP = P - 1.224*lag(P,1);
DrD = D - 1.224*lag(D,1);
```

Then run the same cointegrated system as before. Impose `general restrictions` corresponding to $\beta_1 = (1, -1/R)'$. In OxMetrics notation that is the restrictions

ρ	log likelihood
1.204	105.877009
1.214	105.897914
1.222	105.904824
1.223	105.905044
1.2235	105.905099
1.224*	105.905118*
1.2245	105.9051
1.225	105.905044
1.234	105.897584
1.244	105.874337

Table 1: Profile likelihood under M_{1D}

ρ	log likelihood
1.224	105.274514
1.253	105.711723
1.262	105.745204
1.263*	105.745513*
1.264	105.745121
1.273	105.709305

Table 2: Profile likelihood under M_{1DS}

```
&2=1; &3=-1/0.224;
```

Table 2 gives the profile likelihood.

9. **Estimating the model** M_{1DSB}**:** Fit the equations (2.11) and (2.12) by two separate OLS regressions Run the algebra code:

```
DD  = diff(D,1);
DrP = P - 1.263*lag(P,1);
DrD = D - 1.263*lag(D,1);
DrS = DrP - 1/0.263*DrD;
M   = DrP + D;
```

It is not possible to compute the likelihood for M directly in OxMetrics because it includes no regressors. Several ways around the problem. For instance: (1) Estimate residual variance for M equation by

$$\begin{aligned}
\hat{\sigma}_M^2 &= \frac{1}{T}\sum_{t=1}^{T} M_t^2 \\
&= \frac{1}{T}\sum_{t=1}^{T}(M_t - \overline{M})^2 + \overline{M}^2 \\
&= \frac{T-1}{T}\{\frac{1}{T-1}\sum_{t=1}^{T}(M_t - \overline{M})^2\} + \overline{M}^2
\end{aligned}$$

and use sample variance and sample mean estimated from descriptive statistics in OxMetrics. (2) Compute the likelihood value as

$$\ell_M = -\frac{T}{2}\log(2\pi e\hat{\sigma}_M^2).$$

To find the standard error for M compute descriptive statistics. Choose category "Other models" and model class "Descriptive statistics using PcGive" to get, for $\rho = 1.156$ and $T = 25$.

```
Means, standard deviations and correlations
The dataset is: C:\USstockDated.in7
The sample is: 1976 - 2000
Means
            M
    -0.055595
Standard deviations (using T-1)
            M
      0.35713
Correlation matrix:
                  M
M           1.0000
```

For $\rho = 1.156$ and $T = 25$ the standard error is

$$\begin{aligned}
\sigma_M &= \sqrt{\frac{T-1}{T}(0.35713)^2 + (-0.055595)^2\}} \\
&= 0.35430350,
\end{aligned}$$

which gives the likelihood values

$$\ell_M = -\frac{T}{2}\log(2\pi e\hat{\sigma}^2_M) = -9.53342862.$$

To get the likelihood for $\Delta_1 D_t$ fit a unrestricted model

```
Y DrD
  DrS_1
  DrD_1
  M
```

Table 3 gives the profile likelihood. A 95% confidence band for ρ is given by (1.109,1.200) in that the twice the likelihood is 3.84 lower here than at the maximum.

The final estimated model, with $\rho = 1.156$ is

```
SYS( 4) Estimating the system by OLS

                  Coefficient  Std.Error  t-value  t-prob Part.R^2
DD_1                 0.505582     0.1719     2.94  0.0076   0.2822
DrS_1              0.00317733   0.001471     2.16  0.0419   0.1750
M                 -0.00205475   0.001683    -1.22  0.2351   0.0634

sigma = 0.00295447   RSS = 0.0001920352953

log-likelihood     111.735377  -T/2log|Omega|      147.20884
|Omega|        7.68141181e-006  log|Y'Y/T|        -11.3255408
R^2(LR)              0.363115  R^2(LM)             0.363115
no. of observations        25  no. of parameters          3
mean(DD)           0.00097317  se(DD)            0.00340249

F-test on regressors except unrestricted: F(3,22) = 4.18104 [0.0174] *
F-tests on retained regressors, F(1,22) =
        DD_1       8.64965 [0.008]**       DrS_1       4.66715 [0.042]*
           M       1.49001 [0.235]

correlation of URF residuals (standard deviations on diagonal)
```

ρ	s_M	$\overline{M}$	$\hat{\sigma}_M$	ℓ_M	$\ell_{\Delta_1 D}$	ℓ
1.144	0.36374	-0.022926	0.35712760	-9.73190988	111.792437	102.060527
1.154	0.35817	-0.050151	0.35449886	-9.54720961	111.743911	102.196701
1.155	0.35765	-0.052873	0.35439037	-9.53955750	111.739589	102.200032
1.156	0.35713	-0.055595	0.35430350	-9.53342862	111.735377	102.201948*
1.157	0.35662	-0.058318	0.35424808	-9.52951782	111.731278	102.201760
1.158	0.35611	-0.061040	0.35421412	-9.52712108	111.727294	102.200173
1.263	0.34034	-0.34690	0.48118362	-17.1858049	111.856032	94.6702271

Table 3: Profile likelihood under M_{1DSB}

```
                 DD
DD       0.0029545
correlation between actual and fitted
           DD
      0.59858

AR 1-2 test:      F(2,20)   =   3.2394 [0.0604]
ARCH 1-1 test:    F(1,23)   =  0.67284 [0.4205]
Normality test:   Chi^2(2)  =  0.37549 [0.8288]
Hetero test:      F(6,18)   = 0.082138 [0.9973]
Hetero-X test:    F(9,15)   =  0.12651 [0.9982]
RESET23 test:     F(2,20)   =  0.35042 [0.7086]
```

10. **Estimating the model M_{1DSB} as a system using constrained optimization:** As an alternative, the model M_{1DSB} can be estimated directly as system by fitting the equation (2.8) subject to the constraints (2.9), (2.10). Run the algebra code:

```
DD  = diff(D,1);
DrP = P - 1.157*lag(P,1);
DrD = D - 1.157*lag(D,1);
DrS = DrP - 1/0.157*DrD;
DDrP= diff(DrP,1);
DDrD= diff(DrD,1);
```

To get the joint likelihood set up the model

```
Y DDrP
Y DDrD
  DrS_1
  DD_1
```

Estimate it through constrained simultaneous equations estimation. Keeping all variables then OxMetrics has the following notation for the model:

```
DDrP = &0*DrS_1+&1*DD_1;
DDrD = &2*DrS_1+&3*DD_1;
```

Then impose the restrictions

```
&2=-1-&0;
&3=-1.157*1.157/0.157-&1;
```

The output is then

```
MOD( 1) Estimating the model by CFIML

Equation for: DDrP
                  Coefficient  Std.Error  t-value  t-prob
DrS_1                -1.00317   0.001405    -714.  0.0000
DD_1                 -7.87512     0.1633    -48.2  0.0000

sigma = 0.362308

Equation for: DDrD
                  Coefficient  Std.Error  t-value  t-prob
DrS_1              0.00316692        ---
DD_1                -0.651303        ---

sigma = 0.00292624

log-likelihood     102.202027  -T/2log|Omega|     173.148953
no. of observations        25  no. of parameters           2
```

ρ	log likelihood
1.154	102.196561
1.155	102.200207
1.156*	102.202029*
1.157	102.202027
1.158	102.200202
1.159	102.196557

Table 4: Profile likelihood under M_{1DSB}

```
LR test of over-identifying restrictions: Chi^2(2)  =  0.44411 [0.8009]
BFGS using analytical derivatives (eps1=0.0001; eps2=0.005):
Strong convergence

Constraints:
&2=-1-&0;
&3=-1.157*1.157/0.157-&1;
```

Varying the value of $\rho = 1 + R$ gives the results reported in Table 4. There is a slight deviation as compared with the previous results in Table 3. This must be due to numerical differences. In the paper we judged that the results from this second procedure are numerically more reliable. In any case the difference is not that big.

11. **Estimating intermediate model between M_{1DS} and M_{1DSB}:** The intermediate model M_{1DSB-} without eliminating the restricted constant parameter, $\mu = \alpha_1\zeta_1$, i.e. without setting $\zeta_1 = 0$ can be estimated directly as system by fitting the equation (2.8) subject to the constraints (2.9), (2.10), albeit without restricting ζ_1. Run the algebra code:

```
DD  = diff(D,1);
DrP = P - 1.244*lag(P,1);
DrD = D - 1.244*lag(D,1);
DrS = DrP - 1/0.244*DrD;
DDrP= diff(DrP,1);
DDrD= diff(DrD,1);
```

To get the joint likelihood set up the model

```
Y DDrP
Y DDrD
  DrS_1
  DD_1
  Constant
```

Estimate it through constrained simultaneous equations estimation. Keeping all variables then OxMetrics has the following notation for the model:

```
DDrP = &0*DrS_1+&1*DD_1+&2*Constant;
DDrD = &3*DrS_1+&4*DD_1+&5*Constant;
```

Then impose the restrictions[1]

```
&3=-1-&0;
&4=-1.244*1.244/0.244-&1;
&5= -&2*(1+1/&0);
```

The last of these restrictions restricts the constant so $\mu = \alpha_1\zeta_1$. In principle it could also be coded as `&5=&3*&2/&0`, but that is more demanding on the constrained optimization algorithm. The output is then

```
MOD( 2) Estimating the model by CFIML

Equation for: DDrP
                  Coefficient  Std.Error  t-value  t-prob
DrS_1                -1.00415   0.001532    -655.  0.0000
DD_1                 -5.59237     0.1587    -35.2  0.0000
Constant            -0.323579    0.06336    -5.11  0.0000

sigma = 0.338885

Equation for: DDrD
```

[1]Due to the precedence order of the operators then one has to be careful at this point. The codes `&4=-1.244*1.244/0.244-&1;` and `&4=-(1.244^2)/0.244-&1;` give same result, whereas `&4=-1.244^2/0.244-&1;` is interpreted as `&4=(-1.244^2)/0.244-&1;`

ρ	log likelihood
1.22	104.498942
1.224	104.544638
1.23	104.597634
1.24	104.643984
1.243	104.647575
1.244	104.64771*
1.245	104.647314
1.25	104.637362
1.26	104.577664

Table 5: Profile likelihood under M_{1DSB-}

```
                   Coefficient  Std.Error  t-value  t-prob
DrS_1               0.00414731        ---
DD_1                 -0.749990        ---
Constant            0.00133644        ---

sigma = 0.0027565

log-likelihood      104.64771  -T/2log|Omega|     175.594637
no. of observations        25  no. of parameters           3

LR test of over-identifying restrictions: Chi^2(3)  =   3.2332 [0.3570]
BFGS using analytical derivatives (eps1=0.0001; eps2=0.005):
Strong convergence

Constraints:
&3=-1-&0;
&4=-1.244*1.244/0.244-&1;
&5= -&2*(1+1/&0);
```

Varying the value of $\rho = 1 + R$ gives the results reported in Table 5. It seen that the likelihood values are sitting between those of the models reported in Table 2 for model M_{1DS} and Tables 3/4 for model M_{1DSB}. The estimated value of ρ is closer to those of the less restricted model.