GLOBAL UNIQUENESS AND MONEY NON-NEUTRALITY IN A WALRASIAN DYNAMICS WITHOUT RATIONAL EXPECTATIONS

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ABSTRACT.— We define a non-tâtonnement dynamics in continuous-time for pure-exchange economies with outside and inside *fiat* money. Traders are myopic, face a cash-in-advance constraint, and play dominant strategies in a short-run monetary strategic market game involving the limit-price mechanism. The profits of the Bank are redistributed to its private shareholders, but they can use them to pay their own debts only in the "next period". Provided there is enough inside money, monetary trade curves converge towards Pareto optimal allocations; money has a positive value along each trade curve, except on the optimal rest-point where it becomes a "veil" while trades vanish. Moreover, typically, given initial conditions, there is a piecewise globally unique trade-and-price curve not only in real, but also in nominal variables. Finally, money is locally neutral in the short-run and non-neutral in the long-run.

KEYWORDS. Bank; Money; Price-quantity Dynamics; Limit-price mechanism; Inside money; Outside money.

JEL Classification: D50, E40, E41, E50, E58.

1. INTRODUCTION

Most macroeconomic models reduce the aggregate economy to manageable proportions, and frequently a common simplification is the representation of each sector by agents which behave identically; consequently, they are presented in "representative agent" format. On the other hand, standard general equilibrium with heterogenous agents quickly becomes intractable. Closed form solutions typically cannot be derived, and their results are often not robust. The main impediment lies on the multiplicity of equilibria. A second one is the static viewpoint. Even though most economies under study are intertemporal ones, the solution concept at hand, namely that of a general equilibrium (with rational expectations), is essentially static. In most cases, theory is unable to describe in a sensible way what happens *out of equilibrium.*

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In the present paper, while maintaining the basic ingredients of equilibrium analysis, namely market clearing and agent optimization, we offer a dynamic continuous-time non-tâtonnement process for monetary economies with heterogenous agents. By considering myopic households, we deviate from the rational expectations hypothesis. By building upon the strategic market games framework, and embedding the limit-price mechanism of Mertens (2003), we maintain heterogeneity, and we manage to build a model with a monetary sector where money has non-neutral effects. In addition, the state of an economy always converges towards some Pareto-optimal allocation provided there is enough money, and *both real and nominal variables* are generically piecewise unique.

1.1. **Monetary short-run linear economies.** More precisely, we consider an exchange economy whose dynamics is driven, at each time instant, by a monetary short-run (linear) economy in which agents maximize the first-order approximation of their current utilities subject to a cash-in-advance constraint *à la* Clower (1967).[1] Hence, our approach can be viewed alternatively as the monetary counterpart of Giraud (2004), which was itself a game-theoretic rewriting of Champsaur & Cornet (1990). Giraud (2004) demonstrated in a barter exchange economy existence, convergence to Pareto-optimal allocations and, typically, piecewise uniqueness of trade curves.

In the monetary setting of the present paper, we postulate a cash-in-advance constraint whereby receipts from commodity sales cannot be used contemporaneously for purchases. Therefore, traders borrow inside money from a loan market in anticipation of future income which is used to pay back their loans. Thus, agents are not only endowed with commodities, but also with monetary endowments which are owned free and clear of any debt. The aggregate of all private monetary holdings is the *outside* money of the economy. The government Bank loans money to agents who, in turn, repay after they have received income from the trades of their commodities. Hence, for government Bank issued money, there exists an offsetting liability, which induces its exit from the economy. This money is called *inside*.[2]

We partly follow the monetary paradigm as set out by Dubey & Geanakoplos (1992, 2003a,b) by considering households endowed with outside money together with a government Bank injecting inside money. But we depart rom their framework in as much as we allow agents to send limit-orders (and not just market orders) to the market, and profits of the Central Bank from period t are redistributed to private shareholders in the "next" period $t + dt$ (cf. Shubik & Tsomocos (1992)). Drèze & Polemarchakis (1999, 2000, 2001) in a related monetary framework assume, in addition, that the Bank distributes its profits to private shareholders. Since they are in a static one-shot world, the shareholders can use them to pay their own debts to the Bank. As a consequence, there is no outside money in their model, and nominal indeterminacy of the static equilibria obtains. The exception is when the government budget constraint is violated, in which case Drèze & Polemarchakis'model would also induce nominal determinacy. Violation of the government budget constraint is equivalent to existence of outside money. Notice

[1] See also Grandmont & Younès (1972).

[2] The distinction between outside and inside money has been introduced by Gurley & Shaw (1960).

that, in the present paper, the governement violates its budget constraint only during a nano-second. This suffices to recover determinacy in our dynamical set-up. Here, indeed, when a shareholder receives dividends at the end of time t, she can only use them as cash (outside money) at time $t + dt$. The consequence is that, at variance with Dubey & Geanakoplos (2003a,b), the exit of the quantity, $\overline{m}$, of outside money is incipient because this money (being equal, along each trade curve, to the Bank's instantaneous profit) is reinjected in the economy one nano-second later. But, unlike Drèze & Polemarchakis (2001, strong indeterminacy) and Dubey & Geanakoplos (2003b, generic local uniqueness), we get (piecewise, generic) *global* uniqueness of the outcome.

The advantage of the formulation we adopt in this paper is that it enables us to introduce a Central Bank providing inside money, keeps a cash-in-advance constraint, and formulate all these ingredients within a dynamics which is driven *via* a full-blown local game associated to each short-run economy. We show that our dynamic process of infinitesimal trades continues till the resulting system of differential equation converges to a rest-point. If there is enough inside money, this rest-point is a Pareto-optimal point. Otherwise, the economy remains stuck in a "liquidity trap", at a non-efficient point. Notice that, a Pareto-optimal rest-point of our dynamics is a no-trade static equilibrium of the economy starting at this very point. However, as it is to be expected from a non-tâtonnement approach,[3] and even if there is no money, there is no reason, in general, for such a rest-point to be an equilibrium of the economy starting at some other state along the trade-curve. Thus, even in its non-monetary form, our model is not a dynamical selection procedure of the Walrasian correspondence.

Intraperiod nominal interest rates are endogenously determined at each instant, and serve as market-clearing prices between bonds and inside money. The feasibility constraint on each money-selling order is tantamount to a cash-in-advance constraint. We show that the "classical dichotomy" holds in the short-run, provided there is enough inside money. Indeed, provided there are enough gains to trade,[4], Fisher's quantity theory of money holds in the short-run. One can separate the real and nominal sides of the economy, solving the real side for relative prices, and fixing their levels by the stock of nominal money. But this holds only *in the short-run* because the amount of inside money *must* change over time in order to maintain enough gains to trade each point of time. How it changes will then necessarily affect both nominal and real variables along every trade curve. Moreover, the (implicit) price of money (to be distinguished both from the interest rate) is always equal to 1, and its velocity is a decreasing function of time (and of interest rates), bounded from above by 1. Finally, not only does money have value in our model, its value is determinate. For generic economies, the dynamics of interest rates, price levels and commodity allocations are uniquely defined (within a certain time interval). Monetary policy being non-neutral in the long-run, its effects can in principle be tracked because of the global uniqueness of solution paths to our dynamics.

[3]See, e.g., Smale (1976).

[4]The importance of the "gains to trade" hypothesis is discussed in Dubey & Geanakoplos (1992, 2003a), who first introduced it.

1.2. **A central clearing house for inside and outside money.** We elaborate our own definition of a short-run outcome on Dubey & Geanakoplos' (2003a) definition of a monetary equilibrium. As every Walrasian-like equilibrium concept, this monetary equilibrium does neither guarantee, in general, existence and global uniqueness of its corresponding allocation, even when restricted to linear short-run economies.[5] On the other hand, it heavily rests on the perfect foresight assumption. Indeed, each "period" t is divided in three subperiods t_α, t_β and t_γ. At time t_α, agents borrow money from a Central Bank by selling bonds; at t_β, they exchange commodities for money; at t_γ, they repay their loans to the Central Bank. When selling bonds in the first subperiod t_α, households need to perfectly anticipate both t_β-commodity prices and interest rates (the latter being charged at time t_γ) in order not to default. And from a game-theoretic viewpoint, this no-default constraint also induces the use of a generalized game, and not a full-blown normal-form game, or the introduction of penalty rules in order to guarantee that, in a Nash equilibrium, nobody will go bankrupt.

In this paper, we keep the spirit (originating in Shubik & Wilson (1977)) of introducing money in general equilibrium theory *via* a central Bank. However, we drop the rational expectations hypothesis, which, though commonly used in the general equilibrium literature, would be at odds with the postulated myopia of our boundedly rational agents. The key insight of our analysis consists in extending (along the lines of Mertens (2003)) the paradigm introduced by Dubey & Geanakoplos (2003a) to limit-price orders — i.e., orders that are conditional on the realization of prices (or interest rates). This enables to relax the rational expectations hypothesis, to end-up with a classical one-shot normal-form game (repeatedly played in each period t), and to recover the uniqueness of the short-run outcome associated to each state of the economy (equivalently, to get a *vector* field, and not just a *cone* field).

For this purpose, we equip each individual with a certain amount of bonds with which she starts afresh at each time t. Before entering the market for commodities, traders can trade their bonds against inside money. In stage t_β, they then play a dominant strategy in the limit-price mechanism, taking into account the fact that, afterwards, they will have to fully deliver on their bonds.

Fiat money in our model corresponds to the paper used as cash in everyday life. In our model as in Dubey & Geanakoplos (2003a), the ratio of outside to inside money plays a critical role in determining the endogenous real and monetary variables. Hence, it is pivotal in the determination of the whole dynamics. It is the interplay between today's outside and inside money (or, equivalently, yesterday's profits of the Central Bank and today's inside money) that is responsible for the results obtained in our model. In the absence of a Central Bank providing liquidity, there would be no inside money in our model — which at least contradicts the fact that *fiat* money is a creature of the State, and immediately prompts the question where outside money comes from. Moreover, in a model with just outside or just inside money used as a medium of exchange, nominal prices would be indeterminate (see below Remark (3.2.1) for details).

[5] Unless preferences are artificially restricted to be strictly monotone and to verify a strong irreducibility restriction. These restrictions cannot be adopted in our model because utilities are of course not strictly increasing with respect to *fiat* money.

Finally, we organize trades not according to the decentralized trading post market structure *à la* Shapley & Shubik (1973) (called TP-mechanisms in the sequel), but according to Shapley's windows model.[6] In the Shapley's model, all pairwise markets exist and all commodities can be exchanged against each other and money. A central clearing house links all trades simultaneously and determines prices that clear all pairwise markets. Infinitesimal trades at a given state of the long-run economy correspond commodities acquired from trades in the associated short-run economy minus commodities sold in all pairwise markets.

The next section presents the model in details. Section 3 is devoted to our main results. A last section offers some concluding remarks. Some technical definitions are gathered in an Appendix.

2. The model

2.1. **The fundamentals.** We first depict the long-run economy $\mathcal{E}$, then move to the construction of the short-run economy associated with each state of $\mathcal{E}$, and describe the rules of the game according to which trades occur in the short-run economy. This enables us to finally define the dynamics followed by the state of $\mathcal{E}$.

Commodities

The long-run economy $\mathcal{E}$ is defined by $C \geq 1$ consumption goods $c \in \{1, ..., C\}$ and $I \geq 1$ types of households $i = 1, ..., N$, characterized by $(u_i, \omega_i)_i$.[7] Each type i is represented by a *continuum* of clones, say $[0, 1]$, together with the (restriction of the) Lebesgue measure. Thereafter, we shall consider strategies of one (negligible) clone in each type, so that prices can be taken fixed and the spirit of perfect competition be preserved (see Aumann (1984)).

For each i, $\mathbb{R}^C_+$ represents type i's trading set. The function $u_i : \mathbb{R}^C_+ \to \mathbb{R}$ is the (long-term) utility function of type i. The vector $\omega_i \in \mathbb{R}^C_+ \setminus \{0\}$, is her/his initial endowment.

Definition 2.1.1. (i) An **allocation** is an integrable, type-symmetric map $x : [0, 1]^N \to \mathbb{R}^{C+1}$ belonging to the *feasible set* τ:

$$\tau := \Big\{x \in L^1([0, 1]^N, \mathbb{R}^C_{++}) \ : \ \int_0^1 x_i di = \overline{\omega} := \int_0^1 \omega_i di = \frac{1}{N} \sum_i \omega_i \Big\}.$$

(ii) An allocation x is **individually rational** whenever $u_i(x_i) \geq u_i(\omega_i)$ for a.e. $i \in [0, 1]^N$.

Because of type-symmetry of allocations, we identify τ with:

$$\tau := \{x \in \left(\mathbb{R}^C_+\right)^N \mid \sum_i (x_i - \omega_i) = 0\}.$$

[6]Mertens' limit-price mechanism is compatible with both the TP market structure and the windows model. See Giraud (2003) for an extensive discussion of these strategic market games.

[7]Throughout the paper, $\mathbb{N}_C$ designates the finite set $\{1, ..., C\}$, and $\nabla u(x)$ is the gradient of u at x.

For every household i, $\hat{X}_i$ is the subset of consumption bundles $x_i >> 0$ such that $x_i \leq \overline{\omega} := \sum_h \omega_h$ (resp. $\hat{X}_i^*$ is the subset of feasible and individually rational points, i.e., of x_i in $\hat{X}_i$ such that $u_i(x_i) \geq u_i(\omega_i)$).

Assumption (C)(i) For each i, the restriction of u_i to $\hat{X}_i^*$ is $\mathcal{C}^1$, verifies $\nabla u_i(x_i) > 0$, and is quasi-concave.

(ii) For each i and every $x_i \in \partial X_i \cap \hat{X}_i^*$, $\nabla u_i(x_i) \cdot x_i > 0$. Moreover, the restriction of u_i to $\hat{X}_i^*$ is strictly quasi-concave.

This assumption is weak in the sense that it makes no use of any boundary condition in order to keep away the dynamics from the boundary $\partial\tau$ of the feasible set. Similarly, preferences are not assumed to be strictly monotone.

A map $x : [0,1]^N \to \mathbb{R}^C$ is said to be *type-symmetric* whenever each restriction, $x_{[0,1]}$, on the i^{th} element of the Cartesian product $[0,1]^N$ is a.e. constant for all $i = 1, ..., N$. We denote by $x_i \in \mathbb{R}^C$, the equivalence class of this restriction. Hence, we use the notation x_i in three different ways: As the vector in $\mathbb{R}^C$, which is the common individual allocation attributed to each household of type i, as the constant function which maps each agent in $[0,1]$ to the vector x_i, and describes the symmetric allocation-selection of households of type i, and as the integral of this constant function on the unit interval $[0,1]$, which gives the aggregate allocation of agents of type i. The way it is used will be clear from the context.

Money

Fiat money is present in the economy: At time $t = 0$, each type i has a private endowment of outside money $m_i(0) \geq 0$ and of bonds $b_i > 0$ and a stock $M = M(0) > 0$ of money (inside money) is held by the Central Bank. Outside money is owned by households free and clear of debt. Inside money is always accompanied by debt when it comes into households' hands. Bonds can be thought of as IOU notes held by households, and used to borrow money. They are returned to households when they repay their loans. The quantities $m = (m_i(0))_i, b = (b_i)_i$ and M are exogenously fixed, and $\overline{m} = \sum_i m_i(0)$ (resp. $\overline{b} = \sum_i b_i$) represents the aggregate stock of outside money (resp. bonds) held by the agents at the beginning of time. (Since b will remain constant across time, there is no need for indexing it with respect to time.) The monetary long-run economy is defined by $(\mathcal{E}, m, b, M)$ and the private sector is defined by $(\mathcal{E}, m) \equiv (u_i, \omega_i, m_i, b_i)_i$.

Let $p_c(t) > 0$ denote the price of good c in terms of money and $r(t) \geq 0$ denote the money rate of interest on the Bank loan. The vector $(p(t), r(t)) \in \mathbb{R}_+^C \times \mathbb{R}_+$ denotes the market prices and interest rate at time t.

The configuration space of our dynamics is the set of *states*, *i.e.* of feasible allocations in commodities and stocks of money $(x, m, M) \in \tau \times \mathbb{R}_+^{N+1}$ with $\sum_i m_i = \overline{m}$. Trades occur in continuous time. For the sake of simplicity, the stock of bonds $b = (b_i)_i$ is constant across time because we assume that, at each time t, each individual i starts afresh with the same stock b_i. At each time t, the profits of the Central Bank will be equal to $r(t)M(t)$ (where $r(t)$ is the current intra-period interest rate). It is distributed to its private shareholders according to some fixed ownership structure $\nu : i \mapsto \nu_i \in [0,1]$ such that $\sum_i \nu_i = 1$. However, shareholder i can use only at time $t+dt$, the cash received as dividend at time t. A general (static) formulation where shares of ownership are endogenously determined can be found in Shubik & Tsomocos (1992). However, in our in our stripped-down dynamic model, there is no loss of generality in postulating that, for every i, $\nu_i > 0$. Indeed, myopic households make no expectations about the future, hence have no precautionary demand for money. As a consequence, they spend their whole cash at time t in order to repay their loans. Thus, an agent receiving no dividend from the Central Bank at time t, will enter the markets at time $t + dt$ with no outside money.[8]

2.2. **The short-run economy.** At each time t, the state is $(x(t), m(t), M(t), b)$; the *short-run* economy $T_{x(t),m(t),M(t),b}\mathcal{E}$ is a monetary linear economy defined by the same characteristics as $\mathcal{E}$ except that:

> $\triangleright$ The set of infinitesimal trades of agent i in $T_{x(t),m(t)}\mathcal{E}$ is the shifted cone $-(x_i(t), m_i(t)) + \mathbb{R}_+^{C+1}$
>
> $\triangleright$ Initial endowments of consumption goods (resp. outside money) are replaced by 0. Current allocations $(x_i(t))_i$ (resp. current endowment $(m_i(t))_i$) play the role of short-sale constraints.
>
> $\triangleright$ household's short-run preferences are given by the linear utilities over commodities induced by their current gradients.[9] In other words, agent's i short-run utility $v_i : \mathbb{R}^C \to \mathbb{R}$, is:

$$v_i(\dot{x}_i(t)) := \frac{\nabla u_i(x_i(t))}{\|\nabla u_i(x_i)\|} \cdot \dot{x}_i(t) := g_i(x_i(t)) \cdot \dot{x}_i(t). \tag{2.2.1}$$

At each state, the motion of $\mathcal{E}$ is dictated by some type-symmetric strategy-proof equilibrium of a strategic market game $G[T_{x(t),m(t),b(t),M(t)}\mathcal{E}]$ associated to the short-run economy $T_{x(t),m(t),b(t),M(t)}\mathcal{E}$. This captures the myopia of consumers; They are not sophisticated enough to solve an intertemporal optimization programme (involving the heroic solution of, say, the associated Hamilton-Jacobi-Bellmann partial differential equation). However, they are nevertheless more rational than in most of the evolutionary game theory. They try to trade in the direction of the steepest increase of their current utility.

2.3. **The monetary short-run game.** Each period $t \in \mathbb{R}_+$ is divided into three subperiods. At the first subperiod t_α, agents borrow money from the Bank by selling bonds ; at time t_β, they sell commodities for money and buy goods with money ; eventually, at time t_γ, Bank's loans are repaid with money. Agents enter the commodity market with the money they got from the loan market *plus* their initial holding in outside money.

In stage t_β, agents meet in a monetary version of the strategic market game induced by Merten's (2003) limit price mechanism. This mechanism itself can be viewed alternatively as the multi-item extension of double auctions, or as the extension of Shapley's windows model (see Sahi & Yao (1989)) to limit-price orders.[10] Money is denoted by m and bonds by b. The time period t is fixed.

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[8] Note that an agent without outside money will not be able to trade, and, therefore, could be disregarded altogether.

[9] Money and bonds are of no intrinsic value.

[10] See Giraud (2003) for an introduction to strategic market games, and a discussion of this point.

Market orders

In order to describe the action of a player, let us begin with market orders. The vector q_i of i's market order has $2C+1$ components:

- q_i^{bm} is the quantity of bonds sold by i to the Bank for money at time t_α;
- q_i^{mc} is the money spent by i to purchase c at time t_β, $c = 1, ..., C$;
- q_i^{cm} is the quantity of commodity c sold by i to acquire money at t_β, $c = 1, ..., C$;
- $q_i^{cc'}$ is the quantity of commodity c sold by i for commodity c' at time t_β, $c \neq c' = 1, ..., C$.

Alternatively, a trader i's signal has two components: the first one consists in q_i^{bm} and is sent in stage t_α, the second is a $(C+1)\times(C+1)$ matrix whose k, ℓ-entry $q_i^{k\ell}$ indicates the amount of item k s/he is offering in exchange for item ℓ. Prices are then computed according to the following set of equations:

$$1 + r = \frac{\sum_i q_i^{bm}}{M}, \tag{2.3.1}$$

$$\sum_{k=1}^{C+1} \Big(\sum_{i=1}^{N} q_i^{k\ell} p_k \Big) = p_\ell \sum_{k=1}^{C+1} \Big(\sum_{i=1}^{N} q_i^{\ell k} \Big) \quad \ell = 1, ..., C+1 \tag{2.3.2}$$

where the "price" of money is normalized to 1. Finally, commodities are redistributed in such a way that i's final allocation is

$$x_\ell^i := \omega_\ell^i + \frac{1}{p_\ell} \sum_{k=1}^{C+1} p_k q_i^{k\ell} - \sum_{k=1}^{C+1} q_i^{\ell k}. \tag{2.3.3}$$

In other words, prices form so as to clear all markets; all of inside money M is disbursed to households in proportion to their bonds; the interest rate r forms as a market-clearing price on the market for bonds against inside money; and at each commodity-money market, all the money (or commodity) received is disbursed to households in proportion to the commodity (or money) sent by them.

Given prices (p, r), the (competitive) budget set $B(p, r, x_i(t), m_i)$ of household i is the set of all market orders and final allocations $(q_i, \dot{x}_i(t)) \in \mathbb{R}^{2C+1} \times \mathbb{R}_+^C$ that satisfy the constraints below for all $c \in C$ and all $t \geq 0$:

$$\sum_{c=1}^{C} q_i^{mc} \leq m_i + \frac{q_i^{bm}}{1+r} \tag{2.3.4}$$

$$q_i^{cc'} + q_i^{cm} \leq \dot{x}_i^c(t) \tag{2.3.5}$$

$$q_i^{bm} \leq m_i + \frac{q_i^{bm}}{1+r} - \sum_{c=1}^{C} q_i^{mc} + \sum_{c=1}^{C} p_c q_i^{cm} \tag{2.3.6}$$

$$\dot{x}_i^c(t) \leq q_i^{cm} - x_i^c(t) + \frac{q_i^{mc}}{p_c} \tag{2.3.7}$$

(2.3.4) is the cash-in-advance constraint faced by each trader every period. A common criticism is that such cash-in-advance constraints are *ad hoc*, and do not adequately capture liquidity. However, in any strategic market game or in a monetary economy, they emerge naturally. Their main intuition is that the different instruments and commodities of the economy are not equally liquid. As long as there exist different liquidity parameters which are less than 1 for the endowment vector, money demand is positive to bridge the gap between expenditures and receipts. Otherwise, the budget constraints collapse to the standard Arrow-Debreu constraints.

(2.3.5) precludes commodity short sales, and (2.3.6) specifies loan repayment in the final subperiod from money carried over from the first subperiod and receipts from commodity sales. Finally, (2.3.7) describes the final allocations.

Limit-orders

We now supplement the preceding market structure by allowing traders to send limit-price orders to the market. For various reasons (that are spelled out in Mertens (2003)) only *selling* orders are allowed (but this implies no loss of generality). If a player wants to buy a commodity, he just has to sell some money.

Definition 2.3.1. A **limit-order** to sell item ℓ in exchange for item c[11] gives a quantity $q_{\ell c}$ to be sold, and a relative price $\frac{p_\ell^+}{p_c^+}$. The order is to sell up to $q_{\ell c}$ units of item ℓ in exchange for item c if the relative price $\frac{p_\ell}{p_c}$ is greater than, or equal to, $\frac{p_\ell^+}{p_c^+}$. When $\frac{p_\ell^+}{p_c^+} = 0$, one gets a familiar *market order.*

Remark 2.3.2. A limit-order to "sell" ℓ against m at relative prices $p_m^+ = 0, p_\ell^+ > 0$ is, in fact, an order *not* to buy money.

Every trader i in $T_{x(t),m(t),b(t),M(t)}\mathcal{E}$ is allowed to send as many (market and/or limit) orders as he wants. Nevertheless, due to the fact that a short-run economy is linear, we shall see that every player has at her disposal a unique dominant strategy on the commodity market, which in addition involves at most $C+1$ limit-price orders (whose limit-prices will exactly coincide with this agent's current marginal rates of substitution among commodities and money).[12]

In order to prepare for the (next) definition of a *monetary order book*, observe that, given some collection of orders, one can define a *fictitious linear monetary* economy as follows:

Definition 2.3.3. A **fictitious linear monetary** economy $\mathcal{L} = \langle I, \mathcal{I}, \mu, b, e, M\rangle$, is defined by a positive, bounded measure space $(I, \mathcal{I}, \mu)$ of traders, and measurable functions $b, e : I \to \mathbb{R}_+^{C+3}$, e being integrable.

In such a linear economy $\mathcal{L}$, there are $C + 1$ objects of exchange: C consumption commodities and money. Every "trader's" $i \in I$ consumption set is $\mathbb{R}_+^{C+1}$. "Trader's" i utility for x^i is $b^i \cdot x^i$, and $e^i = (e_1^i, ..., e_{C+2}^i, e_{C+1}^i)$ is her initial endowment, with its last component being i's current holding of money. We designate the

[11]Here, an *item* may designate a consumption commodity as well as money or a bond.

[12]See Giraud (2004a) for details.

set of "agents" of a linear economy by an abstract measure space $(I, \mathcal{I}, \mu)$ because we will need later on to interpret it as a set of limit orders. Fortunately, I will rapidly turn out to be equal to $[0,1]^N$ (equipped with the product of Lebesgue measures) in most of the situations of interest for us.

▷ I is, now, the set of **orders**;
▷ For each fictitious "agent" i (i.e., for each order), its linear "utility" is given by $b^i := (p_1^+, ..., p_C^+, 0)$;
▷ Its "initial endowment" is defined as $e^i := (q_i^1, ..., q_i^C, q_i^m)$.

Monetary order books

The timing of trades in each short-run economy is such that households still have to deliver on their bonds in the last subperiod t_γ of each period t. As shown by Dubey & Geanakoplos (2003a, Lemma 1), the multiple constraints of each stage t_α, t_β can be summarized in a unique (non-linear) constraint where revenues from sales are discounted by the interest rate. In a sense, the banking system extracts (inside) money every time a household i purchases beyond its outside financial wealth m_i.[13] We capture this property in our short-run game $G[T_{x,m,b,M}\mathcal{E}]$ as follows.

• Suppose first that $r = 0$. Any order on the commodity-money market of stage t_β, consisting of a linear "utility" u, together with an "endowment" e, is equivalent to a set of $C+1$ separate orders, the c^{th} of them selling an amount e^c (resp. e^m) of commodity c (resp. money) with the utility u. Therefore, we concentrate on sell-orders of a single item — say c_0. Since money is worthless, every truth-telling sell-order will involve a zero utility for money. But, being negligible, it is a dominant strategy for each player to "reveal the truth" when sending sell-orders to the market. Hence, a "utility function" will typically have the form:

$$v_i(\dot{x}) = \sum_{c=1}^{C} g_i^c(x_i)\dot{x}^c, \tag{2.3.8}$$

where $g_i^c(x_i)$ is the personal relative price of c for agent i (cf. (2.2.1)). This corresponds to a sell-order of commodity c_0 against any other commodity according to which one will yield the most value in terms of the personal relative price system $(g_i^1(x_i), ..., g_i^C(x_i))$. Formally, if p is the emergent price vector, this order will be executed as an order to sell e^{c_0} against $\dot{x}_{c^*} := \frac{p_{c^*}}{p_{c_0}} e^{c_0}$ for c^* in

$$\text{Argmax } \{v_i(\dot{x}_c) \; c = 1, ..., C\},$$

provided that there is at least some commodity c such that $\frac{p_{c_0}}{p_c} \geq \frac{g_i^{c_0}(x_i)}{g_i^c(x_i)}$. If this last condition is not satisfied, then the order is not executed (and automatically disappears from the order book). If this last condition is satisfied at most as an equality, then the order may be only partially executed (and the unexecuted part automatically disappears from the order book). If this condition is satisfied as a strict inequality, then the order is fully executed.

[13]This viewpoint is developed by Dubey & Geanakoplos (1992), and is consistent with Dubey & Shubik's (1978) seminal approach.

• Next, if $r > 0$, the same logic applies, except that, when a player is selling commodities, her revenue is rescaled at the ratio $\frac{1}{1+r}$ in order to take account of the dissipation of money in the system — which is but the cost to pay for the fact that inside money facilitates trades. (Recall, nevertheless, that the money that exited the economy at the end of time t is immediately reinjected into the system at the beginning of time $t+dt$ through the dividends.) This is equivalent to requiring that, when a player announces (2.3.8) to the market and sells commodity c_0 (against money), an outcome will be computed for the modified economy where (2.3.8) has been replaced by:

$$v_i(\dot{x}) = (1+r)g_i^{c_0}(x_i) + \sum_{c=1, c\neq c_0}^{C} g_i^c(x_i)\dot{x}^c. \tag{2.3.9}$$

On the other hand, when she is sending an order to sell money (i.e., to *buy* some commodity), this player's announcement is not modified. The fact that such a modified order is equivalent to a standard one, when the discount factor r is taken into account, can be best viewed as follows. As long as a player is using her own outside money in order to finance her purchase, she incurs no discount rate. This is reflected by the fact that an order to sell money (against commodity c) stays unmodified. On the contrary, as soon as a player spends some inside money in order to finance additional purchases, then she has to incorporate this cost in her budget constraint. Everything then goes as if the price at which she will be ready to sell endowment in commodity c_0 was not p_{c_0} but $\frac{1}{1+r}p_{c_0} < p_{c_0}$. Thus, the corresponding order should not be executed as long as

$$\frac{1}{1+r}\frac{p_{c_0}}{p_c} < \frac{g_i^{c_0}(x_i)}{g_i^c(x_i)}.$$

But this is equivalent to modifying the "utility function" associated with the corresponding limit-price order according to (2.3.9).

Let us call r-**monetary order** such a limit-price order where revenues from sales are discounted by the interest rate r according to (2.3.9). Note that the discount imposed by the market makers is similar to some transaction cost that introduces a wedge between buying and selling prices. Since it affects only revenues from *sales*, it works like a bid-ask spread.

Given some $r > 0$, a r-**monetary order book** in $G(T_{x,m,b,M}\mathcal{E})$ is a fictitious linear monetary economy $\mathcal{O} = \big(I, \mathcal{I}, \mu, \mathbf{b}, \mathbf{e}, M\big)$ such that each "agent's utility" verifies (2.3.9).

2.4. **Monetary infinitesimal trades.** We now define the short-term *outcome* that will be induced by a collection of r-monetary order books sent by players in the short-term economy $T_{x(t),m(t),b(t),M(t)}\mathcal{E}$. Such an outcome will provide the direction in which the state of the underlying economy $\mathcal{E}$ moves (i.e., the infinitesimal trades, $(\dot{x}(t), \dot{m}(t), \dot{b}(t))$, occurring) at time t.

Consider a fictitious linear monetary economy $\mathcal{L} = \big(I, \mathcal{I}, \mu, b, e, M\big)$. Select first a proportional pseudo-equilibrium of $\mathcal{L}$, in the sense of Mertens (2003) (see the Appendix, subsection 5.4. for a refresher). Recall that a pseudo-equilibrium is an extension of the standard Walrasian concept, tailored-made in order to take account of the problems arising on the boundary of the utility space for linear economies.

(Indeed, it is well-known that, whenever agents do not have strictly increasing linear preferences, Walrasian equilibria may fail to exist; by contrast, Mertens' pseudo-equilibria exist as soon as linear preferences are weakly monotone.) Such pseudo-equilibria are proportional whenever they verify a certain proportionality requirement (see subsection 5.4) that will serve to provide the global uniqueness of the Mertens (2003) solution concept for linear economies. Intuitively, an equilibrium of a linear economy is then defined by Mertens (2003) by means of the following algorithm: first, one partitions the set of commodities into two subgroups at every step of the algorithm. One looks for a proportional pseudo-outcome of the economy restricted to one of the two subgroups while the price of the other commodities is provisionally normalized to 0. Since every pseudo-outcome price is positive, by backward induction, no price to which the algorithm eventually converges will be equal to zero at the end. Mertens (2003) shows that, regardless of the partition at every step, the algorithm produces the same final price and allocation.

Remark 2.4.1. In this paper, we prefer not to use the terminology introduced in Mertens (2003): first, because in Mertens (2003), a final "equilibrium" is called an "optimal allocation" (hence could be mistakenly confused with a Pareto-optimal allocation of the long-run economy $\mathcal{E}$). Second, because even the word "equilibrium" is misleading in our context, as it my refer to the outcome of an optimization process, and eventually confused with a rest-point of our dynamics. Thus, we call the unique "optimal allocation" (Mertens' terminology) of a linear short-run economy, its *short-run outcome*. Similarly, we call one of its "pseudo-equilibria" (Mertens' terminology) a *pseudo-outcome*.

$P(\mathcal{L})$ will denote the set of pseudo-outcome prices of the linear economy $\mathcal{L}$, and for all $p \in P(\mathcal{L})$, $X_p(\mathcal{L})$ the corresponding set of pseudo-outcome allocations.

2.5. **Strategy-proof trade curves.** Our short-run strategic market game is feasible: at the start of period t, the Bank holds $M(t)$ and households hold $\overline{m}(t)$ of money. Money market clearing (2.3.1) in stage t_α guarantees that the Bank stock $M(t)$ flows to traders at the end of t_α. When sending orders to the central clearing house in stage t_β, everything occurs as if players would not use inside money, but only outside money. The use of outside money is implicit in the fact that they can send orders to sell commodities against commodities (but with the specific cost (2.3.9) described in the preceding section). Thus, the commodities to be traded in stage t_β are the consumption commodities *plus* outside money.

Commodity market clearing in stage t_β is guaranteed by the fact that Mertens' (2003) mechanism is balanced (cf. (2.3.3), see also Lemma 1 (a), p. 448 if needed). Consequently, the total stock of commodities and outside money is preserved and redistributed among the households during the second stage. At the end of the two first stages, all of $(M + \overline{m})(t)$ is with households.

The no-default constraint (2.3.6) is always satisfied because of (2.3.9), and implies that the total bonds sold by households do not exceed $(M + \overline{m})(t)$. At the end of stage t_γ, the Bank holds $(1 + r(t))M(t) \leq M(t) + \overline{m}(t)$, and households hold the balance $\overline{m}(t) - r(t)M(t)$. The profit of the Bank is therefore $r(t)M(t)$, and it is redistributed to its shareholders at the beginning of period $t + dt$ as a new endowment of outside money. Hence, no money disappears from the system. The initial endowment, $m_i(t + dt) = m_i(t) + \dot{m}_i(t)$, in outside money of household i at the beginning of time $t + dt$ will be the amount of outside money she was left at the end of t_γ (i.e., the difference between the right-hand side and the left-hand side of (2.3.6)) *plus* her dividend:

$$m_i(t) + \dot{m}_i(t) = m_i(t) + \tfrac{q_i^{bm}(p(t))}{1+r} - \textstyle\sum_{c=1}^{C} q_i^{mc}(p(t))$$

$$+ \sum_{c=1}^{C} p_c(t) q_i^{cm}(p(t)) - q_i^{bm}(p(t)) + \nu_i r(t) M(t).$$

A **strategy** s_i of player i in the game $G(T_{x,m,b,M}\mathcal{E})$ consists in sending an order to buy inside money in subperiod t_α and a limit-price order for each pair (k, k') of items in period t_β. Players have no memory, and cannot condition their current behavior the past. Let us denote by $\varphi_i(\mathbf{s})$ the final allocation received by player i whenever the strategy profile $\mathbf{s} := (s_h)_h$ is played. Having defined the "rules of the game", it remains to characterize the players' behavior. We shall consider only weakly dominant strategies.

Definition 2.5.1. A **strategy-proof profile** is a strategy profile $\underline{\mathbf{s}}$ such that a.e. player i plays a weakly dominant strategy in the short-run game, taking $m_i(t)$ as hiher current initial endowment in outside money, i.e., for a.e. player $\tau \in [0,1]^N$ one has:

$$g_i(x_i) \cdot \varphi_i(\underline{\mathbf{s}}) \geq g_i(x_i) \cdot \varphi_i(\underline{\mathbf{s}}^{-i})$$

where $\underline{\mathbf{s}}^{-i}$ is the strategy profile obtained by replacing $\underline{\mathbf{s}}^{i}$ with some arbitrary strategy.

We are eventually ready to define the dynamics of the *Limit-Price exchange Process* (LPP). For every strategy profile $\mathbf{s}$, we denote by $\pi(\mathbf{s}) \in \mathbb{R}_{++}^{C+1}$ (resp. $\dot{x}(\mathbf{s})$) the set of short-term outcome prices (resp. trades) induced by $\mathbf{s}$ in $T_{x(t),m(t),b(t),M(t)}\mathcal{E}$.

Definition 2.5.2. A **monetary strategy-proof trajectory** is a "solution" of the following differential inclusion:

$(x(0), m(0)) = (\omega, m(0))$ and

$$\dot{y}(t) = \varphi\Big(\mathbf{s}\Big[G[T_{x,m,b,M}\mathcal{E}]\Big]\Big) \text{ and } p(t) \in \Pi\Big(\mathbf{s}\Big[G[T_{x,m,b,M}\mathcal{E}]\Big]\Big). \tag{2.5.1}$$

where $\forall t \geq 0, \mathbf{s}\Big[G[T_{x,m,b,M}\mathcal{E}]\Big]$ is a strategy-proof profile of $G[T_{x,m,b,M}\mathcal{E}]$.

Here, "solution" has to be understood in the sense of Filippov (see the Appendix, subsection 5.3).

3. Uniqueness and non-neutrality

3.1. **Global nominal uniqueness.**

Definition 3.1.1. (Mertens (2003)) A market order to sell commodity k for commodity j is **non-executable** if there exists a partition of $\mathbb{N}_L$ into $A \cup B$ such that $j \in A, k \in B$, and for every "agent" i,

(α) either $e_a^i = 0\ \forall a \in A$, ($\beta$) or $b_b^i = 0\ \forall b \in B$.

Definition 3.1.2. (i) A linear economy $\mathcal{L} = \big(I, \mathcal{I}, \mu, b, e\big)$ is *weakly reducible* if there exists a partition $A \cup B = \mathbb{N}_L$ such that for each "agent" i, either $b^i_b = 0 \ \forall b \in B$, or $e^i_a = 0 \ \forall a \in A$, and there exists some triple (i_0, b, a) with $e^{i_0}_b > 0, b^{i_0}_b = 0$ and $b^{i_0}_a > 0$.

(ii) $\mathcal{L}$ is *weakly irreducible* if it is not weakly reducible, i.e., if it admits no inexecutable order.

We shall need the following weak assumption:[14]

Assumption (I) $\mathcal{E}$ is **dynamically weakly irreducible**, that is, for every $x \in \tau^*$, the short-term economy $T_{x,b,m,M}\mathcal{E}$ is weakly irreducible.

A *strict* trade $\dot{x}$ in some linear economy $\mathcal{L}$ is such that either $\dot{x}_i \geq 0$ or $b_i \cdot (\dot{x}_i - e_i) > 0$ for a.e. "agent" i. Mertens (2003) proves that every short-run outcome is Pareto optimal when optimality is checked only with respect to strict trades.

A feasible allocation (x, m) is infinitesimally Pareto-optimal if there does not exist any path $\phi : [a, b) \to \tau$ such that $\phi(a) = (x, m)$ and $\nabla u_i(x_i) \cdot \phi'(x) \geq 0$ for every i, with at least one strict inequality. It is **infinitesimally optimal in strict trades** if there does not exist any path $\phi : [a, b) \to \tau$ such that $\phi(a) = (x, m)$, $\phi'(x)$ involves only strict trades in $T_{x,m,b,M}\mathcal{E}$, and $\nabla u_i(x_i) \cdot \phi'(x) \geq 0$ for every i, with at least one strict inequality. We denote by $\overline{\theta}$ (resp. Θ) the set of such infinitesimally optimal allocations (resp. in strict trades). Clearly, $(x, m) \in \overline{\theta}$ (resp. Θ) iff (x, m) is Pareto-optimal (resp. Pareto-optimal when only strict trades are allowed) in $T_{x,m,b,M}\mathcal{E}$. Finally, θ is the relative interior of $\overline{\theta}$.

Finally, for a given $r \geq 0$, a trade $\dot{x}$ is **r-infinitesimally optimal in strict trades** if there does not exist any path $\phi : [a, b) \to \tau$ such that $\phi(a) = (x, m)$, $\phi'(x)$ involves only strict trades in $T_{x,b,m}\mathcal{E}$, and $\nabla \tilde{u}^r_i(x_i) \cdot \phi'(x) \geq 0$ for every i, with at least one strict inequality, where $\tilde{u}^r_i$ is the "modified" utility function defined as follows.[15]

Let $z^i \in \mathbb{R}^C$ be a trade vector of i (with positive components representing purchases and negative ones representing sales). For any scalar $\gamma > -1$, define:

$$\tau^i_c(\gamma) := \ \min \big\{ z^i_c, \frac{1}{1+\gamma} z^i_c \big\}. \tag{3.1.1}$$

$z^i_c(\gamma)$ entails a diminution of purchases in z^i by the fraction $\frac{1}{1+\gamma}$. The (continuous and concave) utility function $\tilde{u}^r_i(\cdot)$ is given by:

$$\tilde{u}^r_i(x) = u_i(e^i + (x - e_i)(r)). \tag{3.1.2}$$

Needless to say, if $r = 0$, a feasible allocation x is r-infinitesimally optimal in strict trades if, and only if, it belongs to Θ. We denote by Θ_r the subset of r-infinitesimally optimal allocations in strict trades.

Of course, due to the transaction cost induced by r, the short-term outcome of a short-run economy need not be Pareto-optimal in the corresponding linear economy, even when optimality is checked with respect to strict allocations. (It would be so for sure if $M = +\infty$, i.e., $r = 0$.) Nevertheless, the next theorem says that the

[14] See Giraud (2004) for a discussion of this assumption.

[15] See Dubey & Geanakoplos (2003a) for details.

possibility of retrading in continuous time ensures that all r-gains to strict trades will be exhausted at the end of a monetary trade curve.

THEOREM.— *Under (C)(i), for* $b > \overline{m} + M$*, these three parameters being fixed,*

(i) every short-run economy $\mathcal{L} = T_{x,m,b,M}\mathcal{E}$ *admits a globally unique short-term outcome* $(\dot{x}, \dot{m}, p, r)$*.*

(ii) Moreover, $r = \frac{\overline{m}}{M}$*. Except when* $x \in \Theta_r$*, the corresponding short-term price* $\pi = (p, r)$ *is unique, i.e.* $P(\mathcal{L})$ *reduces to a singleton.*

(iii) Monetary trade curves exist.

(iv) Provided $M, m > 0$ *and, if in addition, (C)(ii) and (I) hold, all monetary trade curves converge to* $\overline{\theta}_r$*.*

Remark 3.1.3. The need for distinguishing between (i) and (ii) in the above Theorem arises from the fact that the short-term price of a short-term economy, under the above mentioned assumptions, is unique except if the state corresponding to this short-run economy belongs to Θ_r.

Proof. (i) and (ii). If, at some time t, $b(t)$ is sufficiently large (e.g., $b(t) > \overline{m}(t) + M(t)$), we can be sure that, for every i, the feasibility constraint on the market for bonds, (2.3.6), will be binding:

$$q_i^{bm}(t) = m_i(t) + \frac{q_i^{bm}(t)}{1 + r(t)} - \sum_{c=1}^{C} q_i^{mc}(t) + \sum_{c=1}^{C} p_c(t) q_i^{cm}(t). \tag{3.1.3}$$

Indeed, each player i must return $q_i^{bm}(t)$ to the Central Bank in stage t_γ, and there is at most $\overline{m}(t) + M(t)$ units of money in the economy. Since players are negligible and play a dominant strategy, there is no loss of generality in assuming that, at the end of t_γ, after repaying the Bank, no player will be left with cash. Otherwise, she should have spent more money earlier in order to purchase commodities, or else curtailed her sale of commodities, improving her short-run utility.[16] Hence, exactly $\overline{m}(t) + M(t)$ is owed to the Bank, so that $(1 + r(t))M(t) = M(t) + \overline{m}(t)$, i.e., $r(t) = \frac{\overline{m}(t)}{M(t)}$. As a consequence, exactly $r(t)M(t) = \overline{m}(t) = \overline{m}$ is redistributed to the households at the end of time t, so that i will start at time $t + dt$ with $\nu_i \overline{m}$ units of outside money in her pocket. From now on, we therefore consider the quantity $m_i(t)$ of outside money held by household i as a constant (independent from t for every $t > 0$).

On the other hand, even if they are informed of the r-manipulation operated by the market-makers on their commodity-sell orders, negligible players have no interest to manipulate their preferences. Manipulating their announcements has no effect on the emerging sort-run price p, while the very definition of a short-term outcome implies that the induced outcome will solve:

$$\text{Max } \nabla u^i(x_i) \cdot \dot{x}^i(r),$$

under the constraints: $p_c = 0 \Rightarrow \dot{x}^i = 0$ and $p \cdot \dot{x} \leq 0$ and $\dot{x} \geq -(x_i, m_i)$, on the economy restricted to the commodities belonging to the support of p.

[16]Which is strictly monotone with respect to commodities.

Thus, when analyzing a local game $G[T_{x,m,M,b}\mathcal{E}]$, one can restrict attention to the r-monetary linear economy $T_{x,m,r}\mathcal{E}$ obtained from $T_{x,m,M,b}\mathcal{E}$ by the same transformation as the one used to go from (2.3.8) to (2.3.9), and with $r := \frac{\overline{m}}{M}$.

Let us denote by $\big(\pi(T_{x,m,r}\mathcal{E}), \dot{x}(T_{x,m,r}\mathcal{E})\big)$ the unique short-term outcome associated to $T_{x,m,b,M}\mathcal{E}$. Existence and uniqueness results of "optimal allocations" (called short-term outcomes here) follow from Mertens (2003). There, uniqueness in price is understood up to a normalization. Here, as we impose that money's price be equal to 1, prices are automatically normalized (hence unique in nominal terms).

(iii) Existence of monetary trade curves then follows from standard arguments (see Giraud (2004) for details). We denote by $V : \tau \times T_\tau \times \mathbb{R}^{C+1}$ the cone field associating to each state the set of infinitesimal trades in commodities and money $(\dot{x}, \dot{m})$ induced by our dynamics. Except on the subset Θ_r, this cone field reduces to a vector field. As is clear from Giraud (2004), this vector field is discontinuous in general. Filippov (1988), however, ensures that the differential equation with discontinuous right-hand associated to this vector field can be translated into a differential inclusion that is upper semi-continuous, non-empty,convex and compact-valued. Existence of monetary trade curves then follows from standard existence results for differential inclusions, see, e.g., Aubin & Cellina (1984) and Giraud (2004, Theorem 4.1.1.).

(iv) We first remark that, under (C)(ii), every individually rational trade $\dot{x}$ in every short-run monetary economy $T_{x,b,m,M}\mathcal{E}$ at some state such that $x \in \tau^*$, must be strict. Indeed, $\nabla u_i(x_i) \cdot \dot{x}_i \geq \nabla u_i(x_i) \cdot x_i > 0$. Thus, Θ_r reduces to $\overline{\theta}_r$. The rest of the proof follows the standard Lyapounov argument, see Giraud (2004, Theorem 4.2.1).

□

The next result states that, given aggregate initial endowments $(\overline{\omega}, \overline{m}, M)$, and for a dense subclass of monetary economies of particular interest, namely finitely sub-analytic (see Giraud (2004)) economies, the vector field associated to our dynamics is smooth on an open and dense subset of the feasible set. The Cauchy-Lipschitz theory of smooth differential equations implies that, when restricted to this generic subset, the Cauchy problem induced by our dynamics admits a (piecewise) unique solution path not only in real but also in nominal terms.

Proposition 1.— *For any finitely subanalytic economy $\mathcal{E}$ satisfying (C)(i) and (ii), then, for every fixed r, the feasible set τ can be partitioned as*

$$\tau = \mathcal{R} \cup \mathcal{C}$$

where both $\mathcal{R}$ and $\mathcal{C}$ are finitely subanalytic subsets, the latter being closed, of dimension strictly less than $CN - C =$dim$\mathcal{R}$, and containing $\overline{\theta}_r$. Moreover, the restriction of V to the (open and dense) subset $\mathcal{R}$ is a real-analytic, hence smooth, vector field.

Proof. We first need to prove that θ_r is of measure zero in τ. It follows from the standard argument involving the strict quasi-concavity of preferences[17] and from Dubey & Geanakoplos (2003a, Lemma 2) that a point $x \in \tau_r$ belongs to θ_r iff it is

[17]See Giraud (2004, Lemma 3.1.1.).

Pareto-optimal (in the usual sense) for the auxiliary economy $\mathcal{E}_r$, which is defined as follows. Each household i's utility u_i is changed into $\tilde{u}_i^r$ as defined above by (3.1.2) (with the help of (3.1.1)). Since u_i is strictly quasi-concave and increasing, so is $\tilde{u}_i^r$. Thus, the set of Pareto points in $\mathcal{E}_r$ is homeomorphic to the $(N-1)$ unit simplex (cf. Mas-Colell (1985), Prop. 4.6.2, p. 155). So is therefore θ_r. As a consequence, it is negligible in τ.

Thus, we can perturb our generalized vector field in a way analogous to the one followed in Giraud (2004, Theorem 4.3.1) in order to be able to apply Filippov's theory. From there, the conclusion follows from classical properties of finitely subanalytic vector fields (cf. Theorem 4.3.1 in Giraud (2004) for details).

□

Mathematically, the proof looks the same as in Giraud (2004). However, from an economic viewpoint, there is a big difference in the way prices have been normalized. In Giraud (2004), prices are normalized *a priori* in the unit simplex, because the whole real dynamics is homogeneous of degree zero with respect to prices. Here, prices are endogenously normalized by equation (3.2.3). This generic global uniqueness result has to be contrasted with the generic local uniqueness of monetary equilibria obtained in Dubey & Geanakoplos (2003a, Theorem 3).

The set $\mathcal{C}$ of critical economies being finitely subanalytic, it is the finite, disjoint union of smooth submanifolds, all of them of dimension less than $CN - C$. The picture that can be derived from the previous theorem is therefore the following: τ can be partitioned into finitely many open, disjoint subsets, separated by smooth submanifolds, such that the union of these open subsets ($=\mathcal{R}$) is dense in the feasible set, and the restriction of our vector field to each open subset is smooth.

The key insight of our argument is that myopic behavior coupled with the limit-price mechanism results into piecewise globally unique trajectories.

3.2. **Long-run non-neutrality of money.** Is money neutral in our model ? It is clear that if *both* m and M are multiplied by some constant $\lambda > 0$, then nothing changes in the analysis. This means that there is no money illusion. However, if m and M are changed disproportionately, then there will be, typically, a change in the long-run real variables characterizing the monetary trade curves of the economy. We show in this subsection how to characterize the short-run and long-run impact of such a monetary change on the real sector. Unless otherwise specified, we assume hereafter that $\overline{m} > 0$.

Let us start by stressing that a short-term outcome of the short-run economy $T_{x,m,b,M}\mathcal{E}$ does not coincide, in general, with a monetary equilibrium (with rational expectations) in the sense given to this word by Dubey & Geanakoplos (2003a), and whose definition is recalled in the Appendix (subsection 5.1). Agents have zero endowment in bonds, and therefore no feasibility constraint is put on bond-selling market-orders. This is at variance with Mertens' (2003) non-monetary mechanism, but in accordance with the traditional modelling of financial assets in perfectly competitive general equilibrium theory and with, e.g., Peck & Shell (1991). Consequently, there is only a no-default constraint (5.1.3). This (infinitesimal budget) constraint can be written in a more compact way:

$$q_i^{bm} \leq \Delta(5.1.4) + \sum_c p_c q_i^{cm},$$

where $\Delta(5.1.4)$ is the difference between the right-and the left-hand side of (5.1.1). Due to this constraint, we do not end up with a full-blown game, on account of the fact that no player can default. But, as already remarked by Dubey & Geanakoplos (2003a, footnote 14), this is not a real issue. By adding sufficiently harsh default penalties, one could get a classical game, and still guarantee that, at least in (strategic) equilibrium, nobody goes bankrupt.

PROPOSITION 2.— *(i) Under (C), for b sufficiently large, if $t \mapsto M(t)$ grows sufficiently rapidly, so that*

$$M(t) > \frac{\overline{m}}{\gamma(x(t))}, \tag{3.2.1}$$

for every t, then every short-run outcome is a monetary equilibrium of the corresponding short-run economy. Provided that each u_i is strictly quasi-concave, every monetary trade curve converges to some point $x^* \in \Theta$.

(ii) On the contrary, if at some point x, $\gamma(x) < \frac{\overline{m}}{M}$, then the short-run outcome of $T_{x,m,b,M}\mathcal{E}$ coincides with no-trade, and x is a rest-point of the dynamics.

Proof.

(i) In view of the Theorem and Proposition 1, it suffices to show that, if

$$M(t) > \frac{\overline{m}}{\gamma(x(t))},$$

then $T_{x(t),m(t),b,M(t)}\mathcal{E}$ admits a monetary equilibrium which is also its unique short-run outcome. For this purpose, observe that all the assumptions of Theorem 2 in Dubey & Geanakoplos (2003a) are verified by $T_{x(t),m(t),b,M(t)}\mathcal{E}$. Take therefore a monetary equilibrium of $T_{x(t),m(t),b,M(t)}\mathcal{E}$. By Lemma 1 of Dubey & Geanakoplos (2003a), it is such that every agent is maximizing her short-run utility over the non-linear budget set induced by the condition:

$$C(r, \dot{x}_i - e_i) \leq m_i,$$

where the function $C(\cdot)$ has been introduced by Dubey & Geanakoplos (2003a), and is defined by:

$$C(r, \dot{x}_i - e_i) := \sum_{c \in \mathbb{N}_C} \frac{p_c}{1+r}[\dot{x}_i^c - e_i^c]^+ + \sum_{c \in \mathbb{N}_C} p_c[\dot{x}_i^c - e_i^c]^-,$$

with $[x]^+ := \max\{x, 0\}$ and $[x]^- := \min\,\{x, 0\}$.

Equivalently, i is maximizing her modified short-run utility $\tilde{v}_i^r$ over her "true" competitive constraint $p \cdot \dot{x} \leq p \cdot 0$. Thus, the only property of a short-term outcome that this ME could fail to verify is proportionality. But the proportionality rule comes into play only if two orders are sent to the market with the same limit-price, i.e., when two players of the short-run economy $T_{x,m,b,M}\mathcal{E}$ have marginal rates of substitution that converge to each other. In this last case, there is a continuum of possible infinitesimal trades that are compatible with the definition of a ME, and the proportionality rule just chooses one of them. In all the other cases, the proportionality rule does not come into play, so that the monetary equilibrium is actually proportional, hence is the short-run outcome.

The rest follows from the results obtained by Dubey & Geanakoplos (2003a) on ME (applied to each linear short-run economy). Thus, as time grows, $r(t)$ must decline as rapidly as $\gamma(x(t))$, and converge to 0 (as $x(t)$ converges to Θ).

(ii) Now, if $\gamma(x) = \gamma(e) < \frac{\overline{m}}{M} = r$, then the linear short-run economy at x admits *no* monetary equilibrium. Indeed, according to Theorem 6 in Dubey & Geanakoplos (2003a), if

$$\frac{\overline{m}}{M} > \Gamma(x),$$

then no monetary equilibrium exists — where $\Gamma(x) := \sup_y \gamma(y)$, the supremum being taken over all the y that are feasible and individually rational with respect to x. But, in a short-term *linear* economy, $\Gamma(e) = \gamma(e)$. Hence, if $\gamma(x) = \gamma(e) < \frac{\overline{m}}{M} = r$, then obviously our short-run outcome cannot coincide with any monetary equilibrium, because the later fails to exist. This implies that the short-run price must arise from a combination of several pseudo-equilibrium prices. Suppose that the short-run outcome does not reduce to no-trade. In view of (i) in this proof, there must be some non-trivial partition of the set of commodities and outside money, $\{1, ..., C+1\} = \cup_k C_k$, such that, when restricted to the economy with C_k as commodities, the short-run price is a monetary equilibrium price. Let us denote by $T_{x,b,m,M}\mathcal{E}_k$ the linear economy obtained by restricting oneself to the commodities in C_k, and let $x_i \in \left(\mathbb{R}^k\right)^N$ be the corresponding truncated allocation. Evidently,

$$\gamma(e_k) = \gamma(x_k) < \gamma(x) = \gamma(e),$$

so that $T_{x,b,m,M}\mathcal{E}_k$ also fails to admit a monetary equilibrium. Thus, our short-run outcome must induce no-trade.

$\square$

When a Pareto-optimal point is reached, there are no more gains to trade, hence money becomes a veil. Interest rate is then equal to zero. If there is not enough inside money along a monetary trade curve, then the economy still converges to some rest-point (by compactness of the feasible set) but not necessarily to some infinitesimal Pareto-point. According to Theorem 1, it converges to some point x^* in Θ_r. This means that, in order to go away from x^*, every Pareto-improving path $\phi : [a, b) \to \tau$ with $\phi(a) = x^*$ in strict trades requires more inside money than there actually is. Thus, the economy can remain stuck at some inefficient state due to the lack of liquidity. Of course, such monetary trade curves are incomplete in the sense of Smale (1976).

To summarize, our theory provides a minimal growth rate of inside money for the dynamic analogue of the First Welfare Theorem to hold true. This minimal growth is given by (3.2.1), and can be rewritten as:

$$\dot{M}(t) > \frac{\overline{m}}{\gamma(x(t))} - M(t), \quad \forall t$$

or, equivalently,

$$M(t) > \frac{\overline{m}}{\gamma(\omega)} e^{\int_0^t \frac{r(s) - \gamma(x(s))}{\gamma(x(s))} ds} \quad \forall t$$

On the other hand, when the economy stops at some point in $x^* \in \Theta_r$, then it falls in a **liquidity trap**. Indeed, a small change in r will not suffice to move the economy out of x^* provided the change is sufficiently small so as to still verify:

$$\gamma(x^*) < r. \tag{3.2.2}$$

In this case, a small monetary change has no real effect because real trades still collapse, and the state of the economy remains constant.

Most textbooks devoted to monetary theories with rational expectations conclude that money is non-neutral in the short-run, but neutral in the long-run. Here, we get the opposite conclusion. This paradox can be explained as follows: in the short-run, if $\gamma(x) \neq r$, a sufficiently small change in r will not affect the direction in which the state of the long-run economy moves. Indeed, either $\gamma(x) < r$, in which case there is no trade; or $\gamma(x) > r$, in which case the long-run economy still moves in the direction of a Walrasian equilibrium of its linear short-run approximation. In this narrow sense, money can be said to be **locally neutral in the short-run** — "locally" because the preceding argument holds only for "small" changes in the monetary variables. Observe, nevertheless, that if $\gamma(x) = r$ (a non-generic event), then the slightest change of r will have a real effect, *even in the short-run*.

Now, in the long-run, different amounts of inside or outside money will induce different trade curves in real terms. Indeed, if r is fixed, then the trajectory followed by the long-run economy will stop at some point $x \in \tau$ where $\gamma(x) = r$. If $r \neq r'$, then $x \neq x'$. As a consequence, money is non-neutral in the long-run.

Observe that, in most of the literature derived from Lucas (1972), information is asymmetric, and it is the fact that a change in the money level is unanticipated that makes the money non-neutral. On the other hand, in such models, there is usually no outside money. As a consequence, money non-neutrality reduces to money illusion in this approach when information becomes symmetric. By contrast, here, the presence of both outside and inside money enables to combine no money illusion (the whole dynamics is 0-homogeneous with respect to (m, M)) with the non-neutrality of money. Finally, in the literature just alluded to, one often asks whether money is super-neutral, meaning that a change in the *growth* of the level of money would affect real variables. It should be clear from the preceding discussion that money is not super-neutral in our model since there is a minimal rate of growth for inside money, below which the economy remains traped in a liquidity trap before having reached an optimum.

Remark 3.2.1. In our model, an economy with just outside money used as a medium of exchange would exhibit indeterminate nominal prices. To realize this, just consider Mertens' (2003) limit-price mechanism associated to a graph of trades that is star-shaped with respect to (and only to) some worthless *numéraire* (called outside money). Then, nominal prices would be indeterminate (as they are in standard GET). By contrast, in a situation involving only inside money, our model reduces to the Walrasian (i.e., non-monetary) one introduced in Giraud (2004a). As a consequence, real trades are determinate, but there is nominal indeterminacy in prices (and $r = 0$).

Finally, co-existence of inside and outside money does not suffice *per se*, however, to drive our results. Suppose, indeed, that at each time t, inside money can be exchanged against outside money, but that outside infinitesimal trades are performed *via* to the classical Shapley-Shubik (1977) model of trading-posts, and that there are separate trading-posts for each type of money. The variable r is the relative price of outside to inside money. Then, as soon as $M > m$, all the individuals will sell their whole endowment of outside money against inside money, and perform all their trades in commodities solely with outside money. Player i will end up with $\frac{Mm_i}{\sum_h m_h} = \frac{m_i}{m} M$, and will spend this amount of cash to buy commodities. The quantity theory of money will then look like

$$M = \sum_i \sum_c p_c \dot{x}_c^i, \tag{3.2.3}$$

and the endogenous variables p_c of the real sector of the economy will again be indeterminate (and depend exclusively on M). Only r will depend both on M and m. Consequently, a slight change in M, sufficiently small to keep $M > m$, will not affect the real terms of trade, so that money will again be essentially neutral. Thus, the last ingredient for our recipe to hold water is to organize trades not according to the decentralized trading-posts *à la* Shapley-Shubik but according to Shapley's windows model.

3.3. **An example.** A simple example will clarify the picture. In order to facilitate comparisons, we adopt a linearized version of Dubey & Geanakoplos (2003a)'s example (section 6).

Suppose $N = C = 2, e^1 = e^2 = (50, 50), m^1 = m^2 = 5, M = 90$, and $v^1(\dot{x}_1^1, \dot{x}_2^1) = \frac{10}{75}\dot{x}_1^1 + \frac{3}{25}\dot{x}_2^1, v^2(\dot{x}_1^2, \dot{x}_2^2) = \frac{3}{25}\dot{x}_1^2 + \frac{10}{75}\dot{x}_2^2$. In this (very exceptional) situation, at the unique short-run outcome, household 2 sells part of its endowment of commodity 1, and buys commodity 2; household 1 sells part of its endowment of commodity 2 and buys 1; both agents borrow money from the central Bank. In short-run outcome, $p_1 = p_2 = 2, p_m = 1, r = \frac{1}{9}, \dot{x}^1 + e^1 = (75, 25), \dot{x}^2 + e^2 = (25, 75)$. Agent 1 spends his €5 and buys 2.5 units of good 1. She also borrows €45 from the Bank, promising to repay €50. This loan is spent to buy 22.5 additional units of good 1. Finally, agent 1 sells 25 units of good 2 to agent 2, and is able to repay the Bank. Traders' final gradients are not parallel, because:

$$\frac{\frac{\partial v^1}{\partial x_1^1}}{p_1} = (1 + r)\frac{\frac{\partial v^1}{\partial x_2^1}}{p_2}$$

$$\frac{\frac{\partial v^2}{\partial x_2^2}}{p_2} = (1 + r)\frac{\frac{\partial v^2}{\partial x_1^2}}{p_1}.$$

This misalignment confirms that a short-run outcome may fail to be Pareto-optimal in the short-run economy, and is clearly due to the transaction cost r.

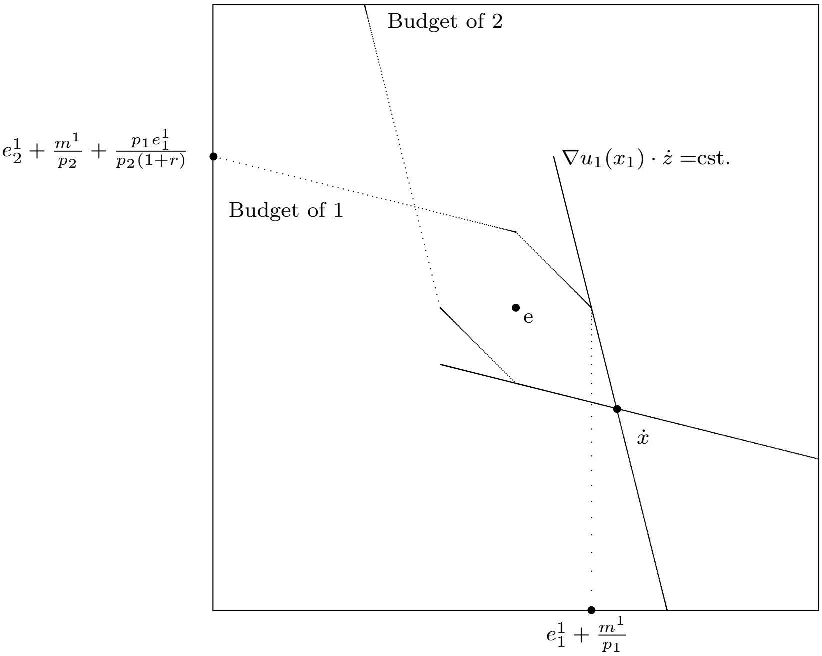

Figure 1. An interior short-run monetary equilibrium

What happens if, everything else being kept fixed, M decreases ? Then r increases above $\gamma(x) = \gamma(e) = \frac{1}{9}$, and the unique short-run outcome is no-trade. On the contrary, when M increases, r decreases below $\gamma(e)$. If the resulting short-term outcome $\dot{x}$ was still an interior monetary equilibrium, then we should have $0 < \gamma(\dot{x}) \leq r < \frac{1}{9}$ (Theorem 4 in Dubey & Geanakoplos (2003a)). But due to the linearity of short-term preferences, $\gamma(\dot{x})$ is constant in the interior of the Edgeworth box. Hence, the short-term outcome must lie on the boundary $\partial\tau$. Thus, the unique short-term outcome then coincides with the unique Walras equilibrium of this economy (which is also the unique short-term outcome of this linear economy when there is no money at all): $\dot{x}_1 = (100, 0)$ and $\dot{x}_2 = (0, 100)$.

How do prices evolve as M increases ? For a given $r < \frac{1}{9}$, one gets: $p_1 = p_2 = \frac{1+r}{10r}$. Therefore, as soon as there is enough grease to turn the wheels of commerce, i.e., as soon as $M > \frac{\gamma(e)}{m}$, then the "classical dichotomy" holds in the short-run: an increase of inside money just increases prices proportionally and decreases the interest rate without affecting real trades. Notice that, in this example, when M increases from $\frac{\gamma(e)}{\overline{m}} = \frac{1}{90}$ to any higher value, the resulting direction in which the economy moves is the same (namely the left-bottom angle of the Edgeworth box). Only the speed at which the economy moves is modified: the state moves more slowly when $M = \frac{\gamma(e)}{\overline{m}}$, than when $M > \frac{\gamma(e)}{\overline{m}}$. Thus, above a certain threshold, an increase of inside money has no impact but nominal inflation.

Suppose, now, that M is fixed. What happens as m varies (proportionally for each household) ? As long as $m > M\gamma(e)$, no-trade is the unique outcome and prices are indeterminate. Whenever $m = M\gamma(e)$, the economy starts moving; it is actually driven by the unique interior monetary equilibrium of its short-run economy. When

m further decreases, the economy moves slightly more rapidly in the same direction, r decreases and prices decrease as well. As $m \to 0^+$, the interest rate r goes to zero, and at the limit, the unique short-term outcome converges again to the unique Walras equilibrium of the short-run economy with prices equal to $p = (\frac{1}{10}, \frac{1}{10})$.

Thus, one can summarize the short-run effects of (i) monetary policy (M varies) and (ii) non-discriminatory fiscal policy (m varies proportionally for each household) by means of the following two diagrams:

Figure 2. m fixed

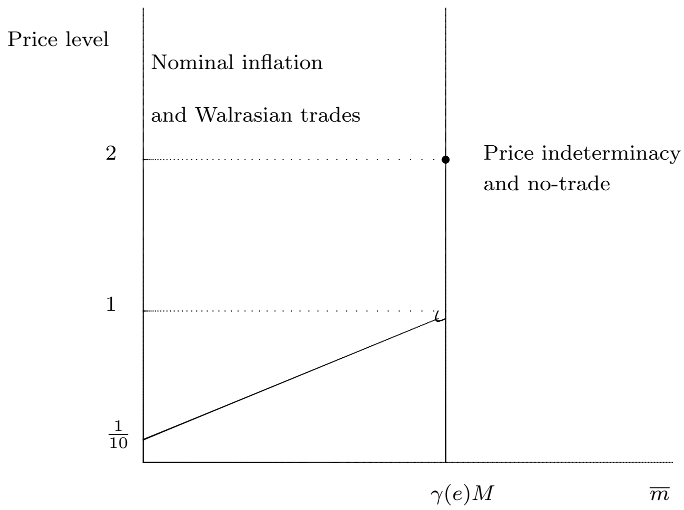

Figure 3. M fixed

When compared with the Figure 6 in Dubey & Geanakoplos (2003a), notice that, here, there is no "hyperinflation phenomenon": as M decreases to $\frac{\overline{m}}{\gamma(e)}$ ($\overline{m}$ being fixed), prices converge to 1. At the moment where $M = \frac{\overline{m}}{\gamma(e)}$, prices jump to 2. Similarly, when M is fixed, as $\overline{m}$ increases towards $\gamma(e)M$, prices converge, and then jump.[18] Looking now at the dynamic picture, one sees that the trade curve followed by our long-run economy depends upon the quantity of circulating money in the following way:

a) either $\overline{m} > 0$ and there is enough inside money throughout, in which case the economy follows a unique trade curve ϕ (which coincides with the non-monetary "Walrasian" trade curve as studied in Giraud (2004)); in particular, it converges to some Pareto-optimal point, $r(t) \to 0^+$ and prices remain bounded ;

(b) or $\overline{m} = 0$, in which case, whatever being the amount $M(t) > 0$ of inside money, the economy follows the same trade curve ϕ;

(c) or $\overline{m} > 0$ and, at some point t, there is not enough inside money, i.e., $0 < M(t) < \frac{\overline{m}}{\gamma(x)}$, in which case the economy stops at x (even though $x \notin \Theta$), with $r(t) > 0$.

4. Concluding Remarks

In order to focus on the essentials, we restricted ourselves to a finite-dimensional economy populated by finitely many types of agents. Using the same technique as in Giraud (2004), one could partially drop this restriction by assuming that there are only finitely many types of preferences but that the endowment map $i \mapsto \omega_i$ can be any integrable map.

Some final remarks are offered below:

A. A quantitative analysis of the long-run impact of money will be performed later, taking advantage of the global nominal uniqueness of trade curves in our dynamics, and of the fact that this dynamics is numerically computable (see Giraud (2004)).

B. Everyday experience on the interbank market suggests that (at least in Europe) this is usually a highly imperfectly competitive market, where a few "big" atomic players interact strategically. Thus, this first study calls for an analogous analysis within an imperfectly framework. This implies studying Mertens' limit-price mechanism with finitely many players. A first step in this direction has been made by Weyers (2003).

C. The presence of outside money in our model, although it plays a crucial role in order to get (global) uniqueness results, remains questionable from an economic point of view. We plan to explore in a subsequent work the impact of allowing for a certain amount of default along trade curves on the determinacy of such curves, taking inspiration from Tsomocos (2003). Default, indeed, is known to be able to play a role analogous to outside money in the analysis of money in a general equilibrium setting. Default and different lending and deposit rates as in Goodhart, Sunirand & Tsomocos (2006) allow for analyzing credit spreads.

D. In a companion paper, Giraud & Rochon (2004) have extended the basic framework underlying the present work to economies with (possibly non-convex) production. By combining production and money, we should be able to explore a dynamic version of IS-LM within our general equilibrium set-up. Finally, since our monetary

[18] Of course, our linear economy can be approximated by a strictly concave one by replacing each linear short-run preference v_i by $v_i + \varepsilon \sum_c \sqrt{x_c^i}$. One then sees that our diagrams are degenerate limits of figures 6 and 7 of Dubey & Geanakoplos (2003a).

setting enables to endogenously normalize prices (or, equivalently, to endogenously fix the nominal level of prices), it should be instrumental in order to provide a dynamic solution of the celebrated price normalization problem when defining a firm's objective.

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5. Appendix

5.1. **Monetary equilibria (Dubey & Geanakoplos (2003a)).** We recall the definition of a monetary equilibrium in the sense of Dubey & Geanaoplos (2003a). Given prices (p, r), the (competitive) budget set $B(p, r, x_i(t), m_i)$ of type i is the set of all market orders and final allocations $(q_i, \dot{x}_i(t)) \in \mathbb{R}^{2C+1} \times \mathbb{R}^C_+$ that satisfy the constraints below for all $c \in C$ and all $t \geq 0$:

$$\sum_{c=1}^{C} q_i^{mc} \leq m_i + \frac{q_i^{bm}}{1+r} \tag{5.1.1}$$

$$q_i^{cm} \leq x_i^c(t) \tag{5.1.2}$$

$$q_i^{bm} \leq m_i + \frac{q_i^{bm}}{1+r} - \sum_{c=1}^{C} q_i^{mc} + \sum_{c=1}^{C} p_c q_i^{cm} \tag{5.1.3}$$

$$\dot{x}_i^c(t) \leq q_i^{cm} - x_i^c(t) + \frac{q_i^{mc}}{p_c} \tag{5.1.4}$$

A vector $(p(t), r(t), (q_i, \dot{x}_i(t))_i) \in \mathbb{R}^C_{++} \times \mathbb{R}_+ \times (\mathbb{R}^{2C+1}_+ \times \mathbb{R}^C_+)^I$ is a **monetary equilibrium** (in the sense of Dubey & Geanakoplos (2003a) adapted to our linear/short-run setting) of $T_{x(t),m(t),b,M(t)}\mathcal{E}$ if all agents' market orders are in their competitive budget sets:

$$(q_i, x_i(t)) \in B(p(t), r(t), x_i(t), m_i), \tag{5.1.5}$$

demand equals supply for the loan market and for all good markets:

$$\sum_i \frac{q_i^{bm}}{1+r(t)} = M(t) \tag{5.1.6}$$

$$\sum_i \frac{q_i^{mc}}{p_c(t)} = \sum_i q_i^{cm}, \quad c \in C \tag{5.1.7}$$

and each agent optimizes

(5.1.8) $v_i(\dot{x}_i(t)) \geq v_i(\dot{\underline{x}}_i(t))$ for all $(\underline{q}_i, \dot{\underline{x}}_i(t)) \in B(p(t), r(t), x_i(t), m_i)$.

5.2. **Gains to trade (Dubey & Geanakoplos (1992)).** We now recall the definition of the measure of *gains to trade* $\gamma(x)$, as introduced by Dubey & Geanakoplos (1992). For any $\gamma \geq 0$, we say that $x = (x_i)_i \in \left(\mathbb{R}^C_{++}\right)^N$ is *not* γ*-Pareto-optimal* if there exist feasible trades $z = (z_i)_i \in \left(\mathbb{R}^C\right)^N$ such that $\sum_i z_i = 0; x_i + z_i >> 0$ and $u_i(x_i[\gamma z_i]) > u_i(x_i)$ for all i,[19] where $x_i^c[\gamma z_i] := x_i^c + \min\{z_i^c, \frac{z_i^c}{1+\gamma}\}$ for every $c = 1, ..., C$.

Thus, the feasible trades contemplated to γ-Pareto improve x involve a tax of $\frac{\gamma}{1+\gamma}$ on trades. If, at the allocation x, we can find some price vector $p \in \mathbb{R}^C_+$ such that

$$p \cdot z_i \leq 0 \Rightarrow u_i(x_i[\gamma z_i]) \leq u_i(x_i) \ \forall i,$$

then, x is γ-Pareto optimal. Of course, 0-Pareto optimality coincides with the standard notion of Pareto optimality. Finally, the gains to trade $\gamma(x)$ at $x \in \tau$ is defined as the supremum of all γ for which x is not γ-Pareto optimal.

5.3. **Filippov's solutions.** Let

$$\dot{x}(t) \in f(x(t)), \tag{5.3.1}$$

where $f : \mathbb{R}^m \hookrightarrow \mathbb{R}^m$ is a possibly discontinuous generalized vector field.

Definition 5.3.1. (Filippov (1988))

A **Filippov solution** of (5.3.1), is an absolutely continuous trajectory $\phi : [a, b) \to \tau$ such that, for a.e. $t \in [a, b)$,

$$\dot{\phi}(t) \in G_f(\phi(t)) := \cap_{\varepsilon>0} \cap_{A\in\mathcal{N}} \overline{\text{co}}\{y \mid d(y, f(\phi(t)))) < \varepsilon, y \notin A\}. \tag{5.3.2}$$

where $\mathcal{N}$:= family of sets $A \subset \mathbb{R}^m$ of (Lebesgue) measure zero.

5.4. **"Optimal allocations" (Mertens (2003)).** We recast in our notations Mertens (2003) definition of an optimal allocation (called short-run outcome in the present paper[20]) for linear economies obtained by considering money as a $C + 1^{\text{th}}$ commodity with respect to which each agent's linear utility is zero.

[19] Since preferences are strictly increasing, there is no need to distinguish weak from strict γ-Pareto optimality.

[20] See Remark (2.4.1) *supra*.

Definition 5.4.1. A **pseudo-equilibrium** of $\mathcal{L}$ is a price system $p \in \mathbb{R}_+^{C+1} \setminus \{0\}$ and an infinitesimal trade $(\dot{x}, \dot{m}) \in \Big(\mathbb{R}_+^{C+1}\Big)$ verifying $p_{C+1} = 1$ and

(i) For every "agent" $i \in I$, $\quad p \cdot (b^i, 0) = 0$ implies $(\dot{x}^i, \dot{m}^i) = 0_{\mathbb{R}^{C+1}}$.

(ii) For every $i \in I$, $\quad (\dot{x}^i, \dot{m}^i)$ maximizes $(b^i, 0) \cdot (\dot{x}, \dot{m})$ subject to the (infinitesimal) budget constraints:

$$(\dot{x}, \dot{m}) \geq -(x_i, m_i), \quad p \cdot (\dot{x}, \dot{m}) \leq 0 \tag{5.4.1}$$

$$\text{and } \big(p^k = 0 \Rightarrow \dot{x}^i_k = 0\big).$$

(iii) For every item c, $p^c = 0$ implies that, for μ-a.e. i, $\Big(p \cdot e^i > 0 \Rightarrow b^i(c) = 0\Big)$.

Definition 5.4.2. (Mertens (2003)) (i) A "pseudo-equilibrium" [21] is **proportional** if all buyers who quoted the market price as limit price get their orders executed in the same proportion, and similarly for sellers:

> For every pair of items $(c, c') \in \mathbb{N}_{C+1}$, with non-zero prices, there exists $m_{cc'} \geq 0$ s.t.
> a) $m_{cc'} + m_{c'c} > 0$;
> b) $m_{c_1c_2}m_{c_2c_3}m_{c_3c_4} = m_{c_1c_3}m_{c_3c_2}m_{c_2c_1}$ (consistency);
> c) all "agents" i with non-zero utility whose demand set $\delta^i_p \ni \{c, c'\}$ receive them in quantities proportional to $m_{cc'}$ and $m_{c'c}$,

where the **demand set** of i at price p is

$$\delta^i_p := \Big\{\ell \in \mathbb{N}_{C+1} \mid p_\ell \leq r(b_i, \ell, k)p_k, \quad \forall k \in \mathbb{N}_{C+1}\Big\},$$

with $r(b_i, \ell, k) := \frac{b^i_\ell}{b^i_k}$ denoting the marginal rate of substitution between ℓ and k (with the convention $\frac{b}{0} := 0$).

(ii) An **"optimal allocation"**[22] of $\mathcal{L}$ is defined by the following algorithm: pick any proportional pseudo-outcome, settle the corresponding trades, and repeat the procedure with the linear sub-economy $\mathcal{L}'$ restricted to the commodities that had zero price. Until the algorithm ends.

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[21] Called *pseudo-outcome* in the body of this paper, see Remark (2.4.1).

[22] Called *short-run outcome* in this paper, *ibidem*.