Price, Trade Size, and Information Revelation in Multi-Period Securities Markets

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Abstract. We study price formation in securities markets, using the sequential trade framework of Glosten and Milgrom [7]. This paper makes one basic methodological advance over previous research on sequential securities trading: we allow traders to choose from n trade sizes in a multi-period market, where n can be arbitrarily large. We examine how trade size multiplicity affects the intertemporal dynamics of trading strategies, bid-ask spreads, and information revelation. We show that price impact, as a function of trade size, is increasing and exhibits (discrete) concavity.

Key Words: Market microstructure; Glosten-Milgrom; Price formation; Sequential trade; Asymmetric information; Trade size; Bid-ask spreads
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1 Introduction

Market microstructure studies the price formation process, and how this process is affected by the organization of the market. The main objective of this paper is to understand how trade sizes affect the price formation process dynamically within an environment where traders can choose from multiple trade sizes.

There are two standard reference frameworks in the market microstructure theory. One is the continuous auction framework, first developed by Kyle [13]. The other is the sequential trade framework, introduced by Copeland and Galai [4] and Glosten and Milgrom [7]. In the Kyle framework the asset orders are submitted first then the asset prices are set and made public whereas in the sequential trade framework the prices are announced before the orders are submitted. Both frameworks are sufficiently simple and well-behaved that they easily lend themselves to analysis of policy issues and empirical tests.[1] Although most markets are organized as in the sequential trade models, these models tend to be less tractable than the Kyle model as Back and Baruch [2] point out.[2]

In this paper, we adopt the sequential trade framework to study the relationship between price, trade size, and information. Sequential trade models consider markets where a risky asset is traded between a market maker, strategic traders and liquidity traders. First, the market maker, who is not informed of the risky asset payoff, quotes the bid and ask price. Then either a strategic trader or a liquidity trader arrives at the market in a random manner. The liquidity trader's trading motive is not related to the risky asset payoff at all. Whereas the strategic trader has information on the risky asset payoff, hence her trades reveal information. In the model of Copeland and Galai [4], the risky asset payoff becomes public information after each trade. In the Glosten and Milgrom [7] model, trading goes on for many rounds before the risky asset payoff is made public. Therefore, the latter allows us to see how price compounds information over time. Glosten and Milgrom [7] also show that the bid-ask spread declines in expectation, and that the spread eventually vanishes almost surely as the number of trading rounds tends to infinity.

One of the simplified assumptions in Glosten and Milgrom [7] is that traders can only trade one share at any given period. Easley and O'Hara [6] extend the Glosten-Milgrom model by allowing for two trade sizes: one small and one large. By doing so, they theoretically justify the empirically observed phenomenon that block trades are made at "worse" prices than small trades. However, Easley and O'Hara [6] mostly focus on the static characterization of equilibrium prices and spreads.[3]

Our analysis extends the analyses by Glosten and Milgrom [7] and Easley and O'Hara [6] in two directions: time and trade size. We extend Glosten and Milgrom's [7] model by allowing for multiple

[1] See Madhavan [14] and Biais, Glosten and Spatt [1] for extensive surveys of the literature.

[2] In a continuous time setup, Back and Baruch [2] show that the equilibrium of the Glosten-Milgrom model is approximately the same as the equilibrium of the Kyle mode, when the trade size is small and uninformed trades arrive frequently.

[3] Having said that, Easley and O'Hara [6] employ a model with a richer information structure (compared to the Glosten-Milgrom model and ours) which makes the analysis of intertemporal equilibrium dynamics more difficult.

trade sizes for traders to choose from. Also, in comparison to Easley and O'Hara [6], we are not confined in our analysis to two trade sizes and we focus more on the intertemporal equilibrium dynamics. In our model, both trade sizes and trading rounds can be arbitrarily many.

The main contribution of this paper is to examine how trade size (i.e. trading volume) affects the intertemporal dynamics of trading strategies, bid-ask spreads and information revelation. In particular, we establish the following results. In each period there is a positive cutoff trade size for the informed trader who observes that the risky asset payoff is high. She assigns no probability to purchasing amounts below this trade size while assigning positive probability to each trade size above the cutoff. The situation is symmetric for the informed trader who observes that the risky asset payoff is low: there is a positive least amount that she sells with positive probability, and she assigns positive probability to selling each allowed amount greater than this cutoff. The bid-ask spreads exist only in the trade sizes where informed trading is considered probable by the market maker. The cut-off trade sizes for the informed traders can decrease over time, and small trade sizes initially with zero bid-ask spreads can later have positive spreads.

Moreover, we prove a couple of asymptotic results comparable to those of Glosten and Milgrom [7]: the market maker learns the true risky asset payoff almost surely as the number of trading rounds tends to infinity, and the bid-ask spreads vanish in the limit.

Our work also yields results which provide testable hypotheses. We show that large trades cause bid-ask spreads to widen and that this widening in the spreads is temporary. We further show that the price impact,[4] as a function of trade size, is increasing and exhibits (discrete) concavity. Both of these results are supported by empirical evidence.

The organization of our paper is as follows. Section 2 presents the model and the equilibrium concept. In Section 3, we present the equilibrium analysis and our results. Section 4 concludes. Most of the proofs are delegated to the appendix.

2 The Model

We consider a market in which potential buyers and sellers trade a risky asset with a competitive market maker. The economy lasts for $T+2$ many periods. The periods are indexed by $t = 0, 1, ..., T, T+1$. Trade takes place in periods $t = 1, ..., T$ and consumption of a single good in period $T+1$. The risky asset pays off in period $T+1$. The risky asset payoff $\tilde{v}$ takes values from the set $\{0, V\}$ with the prior probability $\Pr(\tilde{v} = 0) = \delta$. We assume that $V > 0$ and $0 < \delta < 1$.

There are three types of agents in the economy: informed traders, liquidity traders and a competitive market maker. Informed traders are risk neutral, and they try to maximize their expected profits by trading. Informed traders also know the realization of the risky asset payoff $\tilde{v}$. Liquidity traders trade according to

[4] Price impact is the absolute value of the price change caused by the latest trade.

their liquidity needs, which are exogenous to the model. The competitive market maker supplies against the demands of informed traders and liquidity traders.

Traders can choose from multiple trade sizes when they are trading the risky asset. In particular, they can trade the risky asset in the trade sizes which are elements of the set $\Omega_n := \{-n, ..., -1, 0, 1, ..., n\}$. In our notation, k and $-k$ represent the purchase and the sale of k units of the risky asset, respectively. $\Omega_n^+ := \{1, ..., n\}$ denotes the set of possible purchase trade sizes while $\Omega_n^- := \{-n, ..., -1\}$ denotes the set of possible sales trade sizes. 0 represents no trade.

The timing of events in our model is as follows:

1. In period 0, nature chooses the realization $v \in \{0, V\}$ of the risky asset payoff $\tilde{v}$. Informed traders observe v.

2. In successive periods, indexed by $t = 1, ..., T$, the events realize in the following order:

   · Having observed the realized trades in periods $1, ..., t-1$, the competitive market maker posts a price for each trade size in Ω_n.

   · A new trader (either an informed trader or a liquidity trader) arrives at the market and learns market maker's price quote for each trade size.

   · If the trader is informed, she takes the profit-maximizing quote. If the trader is a liquidity trader, she trades in the trade size determined by her liquidity needs.

3. In period $T + 1$, the realization of $\tilde{v}$ is publicly disclosed, and consumption takes place.

The type of the trader arriving in period t is determined by the random variable $\tilde{\theta}_t$ which takes values from the set $\{i_v, l\}$. The letters, i_v and l, denote the informed type and the liquidity type, respectively. The random variables $\{\tilde{\theta}_t : t = 1, ...T\}$ are i.i.d. across the periods $1, ..., T$ and satisfy $\Pr(\tilde{\theta}_t = i_v) = \mu$. If the trader type in period t is l, then the demand at that period is determined by the random variable $\tilde{L}_t$ which takes values from Ω_n. The random variables $\{\tilde{L}_t : t = 1, ..., T\}$ are i.i.d. and satisfy $\Pr(\tilde{L}_t = q) = \gamma(q) > 0$. Also, for any given period t, the random variables $\tilde{\theta}_t$, $\tilde{L}_t$, $\tilde{v}$ are mutually independent. We assume that informed traders, who trade once, gets the chance to re-trade with probability 0. Thus, informed traders behave myopically and they (rationally) ignore the effect of their trades on future periods. The market maker is risk-neutral and her price quotes make her zero expected profit in each period,[5] i.e. in period t, $t = 1, ..., T$, she chooses the price of each trade size $q \in \Omega_n$ equal to the expected value of the risky asset payoff conditional on her information at period t and the trade realization being equal to

[5] A Bertrand competition among market makers is the standard assumption to have zero expected profit for the (competitive) market maker.

q. Informed traders and market maker correctly anticipate each other's trading and pricing strategies. The structure of the economy, described so far, is common knowledge.

Next we describe the details with regard to market maker's pricing strategy and informed traders' trading strategy. To that end, we first need to introduce some notation. Let q_t denote the trade size that the market maker receives in period t, i.e. q_t is the realized trade size for period t. A *period-t history* $h_t := (q_1, ..., q_t)$ is the sequence of realized trade sizes for periods up until $t+1$. The space of all possible period-t histories, $t \geq 1$, is denoted by $\Omega_n^t := \prod_{\tau=1}^{t} \Omega_n$, and h_t is taken to be the generic element of Ω_n^t. $h_T \in \Omega_n^T$ is called a *complete history*. h_t is said to be *consistent* with $h_T = (q_1, ..., q_T) \in \Omega_n^T$ if $h_T = (h_t, q_{t+1}, ..., q_T)$. For notational convenience, we let $h_0 = \emptyset$. Also, we let $\pi_t : \Omega_n^{t-1} \times \Omega_n \to \mathbb{R}$ represent the market maker's pricing strategy function (i.e. her price menu for all trade sizes) so that $\pi_t(h_{t-1}, q)$ is the market maker's price quote for trade size q given history h_{t-1}.

Since there is a price quote for each trade size, it is possible for informed traders to obtain the same profit from two or more different trade sizes. In such cases, informed traders assign positive probabilities to those trade sizes that yield equal profit when they determine their demands. We formalize this as follows: In our model, a trading strategy is a probability function $\psi : \Omega_n \to [0,1]$ such that $\sum_{q \in \Omega_n} \psi(q) = 1$. The support of ψ is given by $\mathrm{supp}(\psi) := \left\{ q \in \Omega_n \middle| \psi(q) \neq 0 \right\}$. We let $\Delta(\Omega_n) := \left\{ \psi : \Omega_n \to [0,1] \middle| \sum_{q \in \Omega_n} \psi(q) = 1 \right\}$ denote the set of all possible trading strategies. *Informed trader's trading strategy for price menu π_t prescribes a probability distribution $\psi_t(v, h_{t-1}, \pi_t) \in \Delta(\Omega_n)$ over trade sizes in Ω_n for each $v \in \{0, V\}$ and history $h_{t-1} \in \Omega_n^{t-1}$.* We let $\psi_t(q|v, h_{t-1}, \pi_t)$ denote the probability assigned to trade size q by the probability distribution $\psi_t(v, h_{t-1}, \pi_t)$. Among all trading strategies, the informed trader chooses the strategy which maximizes her expected profit given the market maker's price menu. Therefore, *informed trader's optimal trading strategy for price menu π_t prescribes the probability distribution $\psi_t^*(v, h_{t-1}, \pi_t) \in \Delta(\Omega_n)$ over trade sizes in Ω_n for each $v \in \{0, V\}$ and history $h_{t-1} \in \Omega_n^{t-1}$ such that*

$$\psi_t^*(v, h_{t-1}, \pi_t) \in \arg \max_{\psi \in \Delta(\Omega_n)} \sum_{q \in \Omega_n} \psi(q) \left[q \left(v - \pi_t(h_{t-1}, q) \right) \right].$$

The market maker is Bayesian. She updates her belief about the risky asset payoff in each period after having observed the realized trade size for that period. Formally, $\delta_t(h_{t-1}, q)$ is the probability assigned by the market maker to the risky payoff being equal to 0 given that realized history is $h_{t-1} \in \Omega_n^{t-1}$ and the realized trade size in period-t is going to be q. As a notational convenience, we let $\delta_0 = \delta$. Bayesian updating dictates

$$\begin{aligned} \delta_t(h_{t-1}, q) &:= \Pr(\tilde{v} = 0 | h_{t-1}, q) \\ &= \frac{\Pr(\tilde{v} = 0 | h_{t-1}) \left[\mu \, \psi_t(q|0, h_{t-1}, \pi_t) + (1-\mu)\, \gamma(q) \right]}{\sum_{v \in \{0,V\}} \Pr(\tilde{v} = v | h_{t-1}) \, \mu \, \psi_t(q|v, h_{t-1}, \pi_t) + (1-\mu)\, \gamma(q)} \end{aligned} \tag{1}$$

if the market maker believes that informed trader is employing trading strategy ψ_t in period-t. As the market maker makes zero profit from her price quotes, her price menu π_t satisfies

$$\pi_t(h_{t-1}, q) = (1 - \delta_t(h_{t-1}, q))\, V, \quad \forall h_{t-1} \in \Omega_n^{t-1},\ \forall q \in \Omega_n. \tag{2}$$

We say that $\pi_t(h_{t-1}, q)$ *satisfies the zero-profit condition if equation (2) holds.*

Next we define the equilibrium for our economy:

Definition 1 *An equilibrium consists of market maker's price menus $\{\pi_t^* : t = 1, ..., T\}$, informed trader's trading strategies $\{\psi_t^* : t = 1, 2, ..., T\}$, and posterior beliefs $\{\delta_t^* : t = 1, 2, ..., T\}$ such that for all $t \in \{1, \cdots, T\}$ and for all $h_{t-1} \in \Omega_n^{t-1}$*

(P1) *$\pi_t^*(h_{t-1}, q)$ satisfies the zero-profit condition (2) given the posterior belief $\delta_t^*(h_{t-1}, q)$ for all $q \in \Omega_n$,*

(P2) *$\psi_t^*(v, h_{t-1}, \pi_t^*)$ is informed trader's optimal trading strategy for price menu π_t^* for all $v \in \{0, V\}$,*

(B) *for all $q \in \Omega_n$,*

$$\delta_t^*(h_{t-1}, q) \;=\; \frac{\Pr(\tilde{v} = 0 | h_{t-1})\ [\mu\, \psi_t^*(q|0, h_{t-1}, \pi_t^*) + (1-\mu)\, \gamma(q)]}{\sum_{v \in \{0,V\}} \Pr(\tilde{v} = v | h_{t-1})\, \mu\, \psi_t^*(q|v, h_{t-1}, \pi_t^*) + (1-\mu)\, \gamma(q)}.$$

Condition (B) specifies the equilibrium belief. It essentially reflects two critical assumptions of our model: first, market maker is Bayesian; second, informed traders and market maker correctly anticipate each other's trading and pricing strategies.[6]

Finally, we define the bid-ask spread for trade size q: *the period-t bid-ask spread for history $h_{t-1} \in \Omega_n^{t-1}$ and trade size $q \in \{1, ..., n\}$ is given by*

$$S_t(h_{t-1}, q) := \pi_t(h_{t-1}, q) - \pi_t(h_{t-1}, -q), \tag{3}$$

where π_t is the market maker's price menu.

3 Sequential Trades with Multiple Trade Sizes

Our analysis makes one basic methodological advance over previous research on sequential trade models: we let traders choose from multiple trade sizes. This allows us to see the impact of trade size on price menus, trading strategies, and information revelation in a multi-period economy.

[6] In other words, informed traders and market maker have rational expectations about each other's strategies.

3.1 Informed Traders' Equilibrium Trading Strategies

We first examine the impact of trade size on trading strategies. To that end, we analyze informed traders' equilibrium trading strategies. The following proposition lists some of the basic properties of the equilibrium trading strategies of informed traders:

Proposition 1 *If $\left\{(\pi_t^*, \psi_t^*, \delta_t^*) : t = 1, ..., T\right\}$ is an equilibrium, then for all $h_{t-1} \in \Omega_n^{t-1}$ and $t \in \{1, ..., T\}$ informed traders' equilibrium trading strategy ψ_t^* satisfies the following:*

(a) $\psi_t^*(0|v, h_{t-1}, \pi_t^*) \ = \ 0$ *for all* $v \in \{0, V\}$,

(b) $\psi_t^*(q|V, h_{t-1}, \pi_t^*) \ = \ 0$ *for all* $q \in \Omega_n^-$,

(c) $\psi_t^*(q|0, h_{t-1}, \pi_t^*) \ = \ 0$ *for all* $q \in \Omega_n^+$.

Part (a) of Proposition 1 states that informed traders always trade in non-zero quantities. This is simply a consequence of the information asymmetry between informed traders and the market maker. Since the market maker can never fully infer the risky payoff realization v at any given period $t < \infty$ due to the presence of liquidity traders, informed traders, who know v, are better off trading non-zero quantities of the risky asset as they can make non-zero profits by doing so. Parts (b) and (c) say that informed traders sell when $v = 0$ and buy when $v = V$, respectively. Since the market maker always quotes a price strictly between 0 and V (due to her uncertainty about the risky asset payoff), informed traders are better off selling when they know $v = 0$ and they are better off buying when they know $v = V$.

The next result shows that informed traders' equilibrium trading strategies satisfy a special condition: given any history and period, there is a cut-off trade size above which informed traders buy with positive probabilities if $v = V$ and another cut-off size above which informed traders sell with positive probabilities if $v = 0$. Formally:

Theorem 2 *If $\left\{(\pi_t^*, \psi_t^*, \delta_t^*) : t = 1, ..., T\right\}$ is an equilibrium, then for all $h_{t-1} \in \Omega_n^{t-1}$ and $t \in \{1, ..., T\}$ there exist cut-off trade sizes $k_t^+(h_{t-1}) \geq 1$ and $k_t^-(h_{t-1}) \geq 1$ such that*

$$\begin{aligned}
\mathrm{supp}\{\psi_t^*(V, h_{t-1}, \pi_t^*)\} &= \{k_t^+(h_{t-1}), \cdots, n\}, \quad \textit{and} \\
\mathrm{supp}\{\psi_t^*(0, h_{t-1}, \pi_t^*)\} &= \{-n, \cdots, -k_t^-(h_{t-1})\}.
\end{aligned}$$

Theorem 2 essentially says the following: if informed traders assign positive probability to trade size q in their equilibrium trading strategy, then they also assign positive probabilities to trade sizes larger than q. The key to this result lies in the fact that informed traders' equilibrium trading strategies affect the market maker's price menu. If there were a uniform price per share, then an informed trader, who makes a profit by trading q shares, would make higher profits by trading more than q shares. However, in our differential pricing setup, when informed trader wants to buy more than q shares, say $q+x$, $x \geq 1$, she is given a "worse" price quote by the market maker. How much worse the price quote will be is determined by the market maker's belief on how likely she thinks $q+x$ shares will be traded by an informed trader. Therefore, informed traders would like to assign a positive probability to the larger trade size, $q+x$, so that the profit they make by trading on the larger trade size, $q+x$, at a worse price would equal the profit they make by trading on the smaller trade size, q, at a better price. Existence of such a positive probability essentially implies that both q and $q+x$ are in the support of informed traders' equilibrium trading strategy, and Theorem 2 proves that this positive probability exists.

Also, Theorem 2 lets us use a simple classification system for informed traders' equilibrium trading strategies. Let $\left\{(\pi_t^*, \psi_t^*, \delta_t^*) : t = 1, ..., T\right\}$ be an equilibrium. The equilibrium trading strategy of informed traders, ψ_t^*, is said to be

- *k partially pooling on the long side for history h_{t-1}* if $0 < \psi_t^*(q|V, h_{t-1}, \pi_t^*) \leq 1$ for all $q \in \{k, k+1, ..., n\}$ and $\psi_t^*(q|V, h_{t-1}, \pi_t^*) = 0$ for all $q \in \{0, ..., k-1\}$,
- *k partially pooling on the short side for history h_{t-1}* if $0 < \psi_t^*(q|0, h_{t-1}, \pi_t^*) \leq 1$ for all $q \in \{-n, -n-1, ..., -k\}$ and $\psi_t^*(q|0, h_{t-1}, \pi_t^*) = 0$ for all $q \in \{0, ..., -k+1\}$.

According to this simple classification, Theorem 2 implies that there exist k_t^+ and k_t^- such that informed traders' equilibrium trading strategy is k_t^+ partially pooling on the long side and k_t^- partially pooling on the short side. For convenience, we also employ the following terminology: we say ψ_t^* is

- *separating on the long side (short side) for history h_{t-1}* if ψ_t^* is n partially pooling on the long side (short side) for history h_{t-1},
- *completely pooling on the long side (short side) for history h_{t-1}* if ψ_t^* is 1 partially pooling on the long side (short side) for history h_{t-1}.

We now turn our attention to the necessary and sufficient conditions for informed traders' equilibrium trading strategies to be k partially pooling, $1 \leq k \leq n$.

Proposition 3 *Let $\left\{(\pi_t^*, \psi_t^*, \delta_t^*) : t = 1, ..., T\right\}$ be an equilibrium. The equilibrium trading strategy of informed traders, ψ_t^*, is*

(a) k_t^+ partially pooling on the long side for history h_{t-1} if and only if

$$(1-\mu)\sum_{i=k_t^+}^{n}\left(1-\frac{i}{k_t^+}\right)\gamma(i) + (1-\delta_{t-1}^*(h_{t-1}))\mu \quad > \quad 0, \quad \textit{and} \tag{4a}$$

$$(1-\mu)\sum_{i=k_t^+-1}^{n}\left(1-\frac{i}{k_t^+-1}\right)\gamma(i) + (1-\delta_{t-1}^*(h_{t-1}))\mu \quad \leq \quad 0, \tag{4b}$$

(b) k_t^- *partially pooling on the short side for history* h_{t-1} *if and only if*

$$(1-\mu)\sum_{i=k_t^-}^{n}\left(1-\frac{i}{k_t^-}\right)\gamma(-i)+\delta_{t-1}^*(h_{t-1})\mu \quad > \quad 0, \quad \textit{and} \tag{4c}$$

$$(1-\mu)\sum_{i=k_t^- -1}^{n}\left(1-\frac{i}{k_t^- -1}\right)\gamma(-i)+\delta_{t-1}^*(h_{t-1})\,\mu \quad \leq \quad 0. \tag{4d}$$

To better understand the implications of Proposition 3, we examine the necessary and sufficient conditions for the two special cases of k partially pooling trading strategies: separating and completely pooling. We have following result for separating strategies:

Corollary 4 *Let* $\{(\pi_t^*, \psi_t^*, \delta_t^*)\ :\ t = 1, ..., T\}$ *be an equilibrium. The equilibrium trading strategy of informed traders,* ψ_t^**, is*

(a) separating on the long side for history h_{t-1} *if and only if*

$$\frac{n}{n-1} \geq 1+\frac{(1-\delta_{t-1}^*(h_{t-1}))\mu}{\gamma(n)(1-\mu)}, \tag{5a}$$

(b) separating on the short side for history h_{t-1} *if and only if*

$$\frac{n}{n-1} \geq 1+\frac{\delta_{t-1}^*(h_{t-1})\,\mu}{\gamma(-n)(1-\mu)}. \tag{5b}$$

In the case of separating trading strategies, traders trade only in the largest trade size, namely n. Observe that, for separating trading strategies, conditions (4a) and (4c) in Proposition 3 become redundant as, for $k_t^+ = k_t^- = n$, these conditions reduce to $(1-\delta_{t-1}^*(h_{t-1}))\,\mu > 0$ and $\delta_{t-1}^*(h_{t-1})\,\mu > 0$, respectively, which necessarily hold since the market maker can never fully infer the risky payoff realization v at any given period t or history h_{t-1} due to the presence of liquidity traders (meaning that $0 < \delta_{t-1}^*(h_{t-1}) < 1$). This observation and straightforward manipulations on (4b) and (4d) prove Corollary 4.

Now let us focus on the implication of this result. Corollary 4 implies that as the probability μ of informed trading goes up an equilibrium trading strategy becomes less likely to be separating. The intuition is straightforward: Following Theorem 2, informed traders always assign positive probability to the largest trade size n in their equilibrium trading strategies. If the probability of informed trading is high, then the market maker posts a large bid-ask spread for trade size n to avoid loss inflicted by informed traders. This makes trading in the largest trade size less attractive for informed traders, hence they decrease their likelihood of trading in size n by assigning positive probabilities to smaller trade sizes in their trading strategies. Therefore, an equilibrium trading strategy is less likely to be separating if the probability of informed trading is high.

The following result provides the necessary and sufficient conditions for equilibrium trading strategies to be completely pooling:

Corollary 5 *Let* $\left\{(\pi_t^*, \psi_t^*, \delta_t^*) \,:\, t = 1, ..., T\right\}$ *be an equilibrium. The equilibrium trading strategy of informed traders,* ψ_t^**, is*

(a) completely pooling on the long side for history h_{t-1} *if and only if*

$$(1-\mu)\sum_{i=1}^{n}(1-i)\gamma(i) + (1-\delta_{t-1}^*(h_{t-1}))\mu \quad > \quad 0, \tag{6a}$$

(b) completely pooling on the short side for history h_{t-1} *if and only if*

$$(1-\mu)\sum_{i=1}^{n}(1-i)\gamma(-i) + \delta_{t-1}^*(h_{t-1})\,\mu \quad > \quad 0. \tag{6b}$$

In the case of completely pooling trading strategies, informed traders trade in all possible trade sizes with positive probabilities. Therefore, conditions (4b) and (4d) in Proposition 3 are redundant, and the result above is obtained as a straightforward corollary of Proposition 3.

Note the following implication of Corollary 5: as the probability of informed trading μ increases informed traders' equilibrium trading strategy becomes more likely to be completely pooling. The intuition behind this result is very much in line with the intuition we gave for Corollary 4. Following Theorem 2, the market maker knows that informed traders are more likely to trade in large trade sizes and as a consequence she posts large bid-ask spreads for these large sizes. This makes informed traders to assign positive probabilities to smaller trade sizes so that they can enjoy "better" price quotes. Essentially, increased pooling gives informed traders increased coverage by liquidity traders against the market maker. Therefore, if the probability of informed trading is sufficiently high, the equilibrium trading strategy of informed traders is completely pooling.

3.2 Existence and Uniqueness of Equilibrium

In the standard sequential trade models, traders, who trade once, get the chance to re-trade with probability zero. This is also the case in our model. Therefore, informed traders' time scope for portfolio decisions is confined to one period. As a consequence, the only link between consecutive periods is the market maker's belief on the risky asset payoff. That is, δ_{t-1}^* is the only parameter from period-$(t-1)$, which the period-t equilibrium strategies π_t^*, ψ_t^* and the period-t equilibrium belief δ_t^* depend on. Let us demonstrate this in detail:

Take a complete history $h_T \in \Omega_n^T$, and let $\{h_t : t = 1, ..., T\}$ be the sequence of histories consistent with h_T. Fix period $t \in \{1, ..., T\}$ and history h_{t-1}. Recall from equilibrium condition (B) that

$$\delta_t^*(h_{t-1}, q) \;=\; \frac{\delta_{t-1}^*(h_{t-1})\,[\mu\,\psi_t^*(q|0,h_{t-1},\pi_t^*)+(1-\mu)\,\gamma(q)]}{\delta_{t-1}^*(h_{t-1})\,\mu\,\psi_t^*(q|0,h_{t-1},\pi_t^*)+(1-\delta_{t-1}^*(h_{t-1}))\,\mu\,\psi_t^*(q|V,h_{t-1},\pi_t^*)+(1-\mu)\,\gamma(q)}. \tag{7}$$

Equation (7) shows the functional relation between the period-t equilibrium belief $\delta_t^*(h_{t-1},\cdot):\Omega_n\to\mathbb{R}$ and the period-$(t-1)$ equilibrium belief $\delta_{t-1}^*(h_{t-1})$. Also, the zero-profit condition of the equilibrium, namely (P1), dictates that the market maker's price menu is of the form

$$\begin{aligned}\pi_t^*(h_{t-1},q) &= (1-\delta_t^*(h_{t-1},q))\,V \\ &= \frac{(1-\delta_{t-1}^*(h_{t-1}))\,[\mu\,\psi_t^*(q|V,h_{t-1},\pi_t^*)+(1-\mu)\,\gamma(q)]\,V}{\delta_{t-1}^*(h_{t-1})\,\mu\,\psi_t^*(q|0,h_{t-1},\pi_t^*)+(1-\delta_{t-1}^*(h_{t-1}))\,\mu\,\psi_t^*(q|V,h_{t-1},\pi_t^*)+(1-\mu)\,\gamma(q)}. \qquad (8)\end{aligned}$$

Equation (8) shows the functional relation between the period-t price menu $\pi_t^*(h_{t-1},\cdot):\Omega_n\to\mathbb{R}$ and $\delta_{t-1}^*(h_{t-1})$. Finally, the functional relation between informed traders' period-t equilibrium trading strategies $\psi_t^*(\cdot,h_{t-1},\pi_t^*):\{0,V\}\to\mathbb{R}^{|\Omega_n|}$ and $\delta_{t-1}^*(h_{t-1})$ is derived in Lemma 1.[7] It states that if ψ_t^* is k_t^+ partially pooling on the long side and k_t^- partially pooling on the short side for history h_{t-1}, then

$$\psi_t^*(q|V,h_{t-1},\pi_t^*)=\begin{cases}0 & : \quad q\in\{-n,\cdots,k_t^+-1\}\\ \dfrac{(1-\mu)\sum_{i=k_t^+}^n\left(1-\frac{i}{q}\right)\gamma(i)+\left(1-\delta_{t-1}^*(h_{t-1})\right)\mu}{\left(1-\delta_{t-1}^*(h_{t-1})\right)\mu\sum_{i=k_t^+}^n\frac{i\,\gamma(i)}{q\,\gamma(q)}} & : \quad q\in\{k_t^+,\cdots,n\};\end{cases} \qquad (9a)$$

$$\psi_t^*(q|0,h_{t-1},\pi_t^*)=\begin{cases}0 & : \quad q\in\{-k_t^-+1,\cdots,n\}\\ \dfrac{(1-\mu)\sum_{i=k_t^-}^n\left(1-\frac{i}{|q|}\right)\gamma(-i)+\delta_{t-1}^*(h_{t-1})\mu}{\delta_{t-1}^*(h_{t-1})\mu\sum_{i=k_t^-}^n\frac{i\,\gamma(-i)}{|q|\,\gamma(q)}} & : \quad q\in\{-n,\cdots,-k_t^-\}.\end{cases} \qquad (9b)$$

So far, we have been able to derive the period-t equilibrium strategies and belief solely as a function of $\delta_{t-1}^*(h_{t-1})$. It is easy to check that $\delta_{t-1}^*(h_{t-1})\in\mathbb{R}$ satisfies the inequalities (4a)-(4d) for some $k_t^+,k_t^-\in\Omega_n^+$ as these inequalities span the whole space of real numbers. Also, the inequalities (4a)-(4d) are mutually exclusive across different values of k_t^+ and k_t^-. All these imply that, given $\delta_{t-1}^*(h_{t-1})$, there exist period-t equilibrium strategies π_t^*,ψ_t^* and period-t equilibrium belief δ_t^*, and they are uniquely identified by the equations (7), (8), (9a)-(9b). Note that the same argument applies to all periods $t\in\{1,...,T\}$ and histories h_{t-1}, and since $\delta_0=\delta$ is an exogenous parameter of the model the overall equilibrium of the economy can be derived in a recursive fashion, using (7), (8), (9a)-(9b). In summary we have proven the following result:

Proposition 6 *There exists a unique equilibrium $\left\{(\pi_t^*,\psi_t^*,\delta_t^*):t=1,....,T\right\}$. That is, given a complete history $h_T\in\Omega_n^T$, the sequence of histories $\{h_t:t=1,...,T\}$ consistent with h_T, and $t\in\{1,...,T\}$, the equilibrium price menu $\pi_t^*(h_{t-1},\cdot):\Omega_n\to\mathbb{R}$, the equilibrium trading strategy of informed traders $\psi_t^*(\cdot,h_{t-1},\pi_t^*):\{0,V\}\to\mathbb{R}^{|\Omega_n|}$, and the equilibrium posterior belief $\delta_t^*(h_{t-1},\cdot):\Omega_n\to\mathbb{R}$ uniquely exist.*

[7] See Appendix.

3.3 Equilibrium Dynamics

In this section we turn our attention to the equilibrium dynamics. First, we examine the dynamics of the equilibrium trading strategies of informed traders. As trades unfold over time, the market maker updates her belief on the risky asset payoff. Consequently, she also updates the price menu, and in turn informed traders revise their trading strategies. Given a k partially pooling trading strategy in period-t, the revision of the trading strategy in period-$(t+1)$ can take place in two ways: (1) informed traders can maintain k as the cut-off size and just change the probabilities they assign to trade sizes over k, or (2) they can alter the cut-off size, hence change the support of their trading strategy. Naturally, the latter implies a significant change in the trading behavior of informed traders, and that is what we are after: we would like to see if informed traders ever change the support of their trading strategies over time. The following result sheds light on this.[8]

Proposition 7 *Let* $\left\{(\pi_t^*, \psi_t^*, \delta_t^*) : t = 1, ..., T\right\}$ *be an equilibrium and* $h_t = (h_{t-1}, q_t) \in \Omega_n^t$. *Let* ψ_t^* *be* k_t^+ *partially pooling on the long side and* k_t^- *partially pooling on the short side for history* h_{t-1}. *Also, let* ψ_{t+1}^* *be* k_{t+1}^+ *partially pooling on the long side and* k_{t+1}^- *partially pooling on the short side for history* h_t.

(a) $k_{t+1}^+ < k_t^+$ *if* $q_t \in \{k_t^+, ..., n\}$ *and*

$$\delta_{t-1}^* \left(\frac{(1-\mu)\sum_{i=k_t^+}^n \left(1-\frac{i}{q_t}\right)\gamma(i)+(1-\delta_{t-1}^*(h_{t-1}))\mu}{(1-\mu)\sum_{i=k_t^+}^n \gamma(i)+(1-\delta_{t-1}^*(h_{t-1}))\mu} \right) \geq \frac{(1-\mu)}{\mu\, k_t^+ (k_t^+ - 1)} \sum_{i=k_t^+}^n i\,\gamma(i). \tag{10a}$$

(b) $k_{t+1}^- < k_t^-$ *if* $q_t \in \{-n, ..., -k_t^-\}$ *and*

$$\left(1-\delta_{t-1}^*\right) \left(\frac{(1-\mu)\sum_{i=k_t^-}^n \left(1-\frac{i}{|q_t|}\right)\gamma(-i)+\delta_{t-1}^*(h_{t-1})\,\mu}{(1-\mu)\sum_{i=k_t^-}^n \gamma(-i)+\delta_{t-1}^*(h_{t-1})\,\mu} \right) \geq \frac{(1-\mu)}{\mu\, k_t^- (k_t^- - 1)} \sum_{i=k_t^-}^n i\,\gamma(-i). \tag{10b}$$

For the sake of exposition, we call $\{k^+, ..., n\}$ *the domain of informed purchasing*, $\{-n, ..., -k^-\}$ *the domain of informed selling*, and $\{-n, ..., -k^-\} \cup \{k^+, ..., n\}$ *the domain of informed trading* if the equilibrium trading strategy is k^+ partially pooling on the long side and k^- partially pooling on the short side for the given history and period. Proposition 7 reveals the following: (a) the domain of informed purchasing gets bigger in period-$(t+1)$ provided that the probability μ of informed trading is sufficiently high, a trade from the period-t domain of informed purchasing has occurred, and the market maker believed that the risky asset payoff was highly likely to be 0 before the purchase realization; (b) the domain of informed

[8] Note the following weaker version of Proposition 7: (a) if ψ_t^* is k_t^+ partially pooling on the long side for history h_{t-1} and $q_t > 0$, then ψ_{t+1}^* is k_{t+1}^+ partially pooling on the long side for history (h_{t-1}, q_t) with $k_{t+1}^+ \leq k_t^+$, (b) if ψ_t^* is k_t^+ partially pooling on the short side for history h_{t-1} and $q_t < 0$, then ψ_{t+1}^* is k_{t+1}^- partially pooling on the short side for history (h_{t-1}, q_t) with $k_{t+1}^- \leq k_t^-$. This result follows in a fairly straightforward manner from the updating rule (1) and Proposition 3.

selling gets bigger in period-$(t+1)$ provided that the probability μ of informed trading is sufficiently high, a trade from the period-t domain of informed selling has occurred, and the market maker believed that the risky asset payoff was highly likely to be V before the sale realization.

Part (a) and part (b) of Proposition 7 can be motivated in similar fashions. Let us consider part (a). Suppose that the probability μ of informed trading is sufficiently high, a trade from the period-t domain of informed purchasing has realized, and the market maker believed that the risky asset payoff was highly likely to be 0 before the purchase realization. An informed purchase would take place only if the risky asset payoff were V. Since the probability of informed trading is high and the market maker previously believed that the risky asset payoff was highly likely to be 0, the realized purchase leads to a significant change in her belief. If informed trading strategy were not to be revised in period-$(t+1)$, the market maker would substantially increase prices for the period-t domain of informed purchasing. Consequently, the period-t domain of informed purchasing would yield lower profits in period-$(t+1)$. Therefore, if the true risky asset payoff is indeed V, informed traders revise their trading strategy by decreasing the cut-off size. By doing so, they increase the probability of liquidity trading within the domain of informed purchasing and this allows higher probability of profit-making for the market maker. As the market maker is bound to make zero expected profit in equilibrium, the enlarged domain of informed purchasing yields more favorable price quotes for the informed traders.

As illustrated by the argument above, the dynamics of informed trading strategies and the market maker's learning process are closely related. We next tackle whether the market maker's belief on the risky asset payoff converges to the truth as the number of trading periods tends to infinity. Glosten and Milgrom [7] show that such convergence is obtained almost surely if the only available trade size is the unit trade size. In our generalized framework, where multiple trade sizes are available, the asymptotic result of Glosten and Milgrom [7] still holds.

Theorem 8 *Suppose $T=\infty$. Let $\left\{(\pi_t^*, \psi_t^*, \delta_t^*) : t = 1, ..., \infty\right\}$ be an equilibrium. Given a complete history $h_\infty \in \Omega_n^\infty$, let $(h_t : t = 1, ..., \infty)$ be the sequence of histories consistent with h_∞.*

(a) $\delta_t^*(h_t)$ converges to 0 almost surely as t tends to infinity if $v = V$,

(b) $\delta_t^*(h_t)$ converges to 1 almost surely as t tends to infinity if $v = 0$.

This result is driven by the fact that transaction prices (i.e. the prices of trade sizes that have been acted upon) form a martingale. The martingale property of prices guarantees their convergence. Of course, even if the beliefs converge, they need not converge to the truth. However, as in Glosten and Milgrom [7], after sufficiently high number of periods, the market maker observes sufficient number of informed trades, and these trades reveal the truth in the limit.

3.4 Bid-Ask Spreads

In this section, we investigate the equilibrium bid-ask spreads. The bid-ask spreads compensate the market maker for the risk of doing business with informed traders. Therefore the equilibrium bid-ask spreads depend on informed traders' equilibrium trading strategies: positive bid-ask spreads are observed only in the trade sizes which belong to the domain of informed trading. The following proposition formally states this result:

Proposition 9 *Let* $\left\{(\pi_t^*, \psi_t^*, \delta_t^*) : t = 1, ..., T\right\}$ *be an equilibrium. Also, let* ψ_t^* *be* k_t^+ *partially pooling on the long side and* k_t^- *partially pooling on the short side for history* h_{t-1}*. The equilibrium bid-ask spread* $S_t^*(h_{t-1}, \cdot) : \Omega_n^+ \to \mathbb{R}$ *satisfies the following:*

(a) $S_t^*(h_{t-1}, q) > 0$ *if and only if* $\min\{k_t^+, k_t^-\} \leq q \leq n$*,*

(b) $S_t^*(h_{t-1}, q) = 0$ *if and only if* $1 \leq q < \min\{k_t^+, k_t^-\}$*.*

Notice that small trade sizes initially with zero bid-ask spreads can later have positive spreads as the trades unfold over time. This actually follows from Proposition 9 and our discussions in Section 3.3. Section 3.3 has revealed that informed traders can enlarge the domain of informed trading over time as a remedy against the market maker getting close to the truth. In light of Proposition 9, this means that the domain of positive bid-ask spreads will get bigger over time if the domain of informed trading is indeed enlarged. Formally, we have the following result as a direct corollary of Proposition 7 and Proposition 9:

Remark 10 *Let* $\left\{(\pi_t^*, \psi_t^*, \delta_t^*) : t = 1, ..., T\right\}$ *be an equilibrium. Also, let* ψ_t^* *be* k_t^+ *partially pooling on the long side and* k_t^- *partially pooling on the short side for history* h_{t-1}*.*

(a) If $k_t^+ \leq k_t^-$*,* $q_t \geq k_t^+$*, and* $\delta_t^*(h_{t-1}, q_t)$ *satisfies (10a), then*

$$S_t^*(h_{t-1}, k_t^+ - 1) = 0 \quad \textit{while} \quad S_t^*(h_t, k_t^+ - 1) > 0.$$

(b) If $k_t^- \leq k_t^+$*,* $q_t \leq -k_t^-$*, and* $\delta_t^*(h_{t-1}, q_t)$ *satisfies (10b), then*

$$S_t^*(h_{t-1}, k_t^- - 1) = 0 \quad \textit{while} \quad S_t^*(h_t, k_t^- - 1) > 0.$$

The bid-ask spreads exist due to the asymmetric information between the market maker and informed traders. If the market maker were to learn the truth about the risky asset payoff, there would be no spreads as the price of the risky asset would be set equal to its payoff. So, following Theorem 8, we know that the bid-ask spreads vanish almost surely as the number of trading periods tends to infinity.

Proposition 11 *Suppose $T = \infty$. Let $\left\{(\pi_t^*, \psi_t^*, \delta_t^*) : t = 1, ..., \infty\right\}$ be an equilibrium. Given a complete history $h_\infty \in \Omega_n^\infty$, let $(h_t : t = 1, ..., \infty)$ be the sequence of histories consistent with h_∞. The equilibrium bid-ask spread $S_t^*(h_t)$ converges to 0 almost surely as t tends to infinity.*

Finally, we would like to examine the functional relation between bid-ask spreads and trade sizes. To that end, we first make the following mathematical definition. Let $X \in \mathbb{Z}$. We say $f : X \to \mathbb{R}$ exhibits *discrete concavity* if, for any $x, x-1, x+1 \in X$, it holds that

$$f(x+1) - f(x) \leq f(x) - f(x-1). \tag{11}$$

f is said to exhibit *strict discrete concavity* if (11) holds with strict inequality. Notice that our definition for discrete concavity essentially provides an extension of the concavity definition of continuous functions to discrete spaces. Recall that a differentiable function is concave if and only if its first order derivative is a decreasing function. In a discrete space, this corresponds to first order difference being a decreasing function, as in (11). The following proposition sheds light on the functional relation between the equilibrium bid-ask spreads and trade sizes.

Proposition 12 *Let $\left\{(\pi_t^*, \psi_t^*, \delta_t^*) : t = 1, ..., T\right\}$ be an equilibrium. Also, let ψ_t^* be k_t^+ partially pooling on the long side and k_t^- partially pooling on the short side for history h_{t-1}. The equilibrium bid-ask spread $S_t^*(h_{t-1}, q)$, as a function of $q > 0$, is strictly increasing and exhibits strict discrete concavity in the domain $\{\max\{k_t^-, k_t^+\}, ..., n\}$.*

Proposition 12 reveals that the equilibrium bid-ask spread, as a function of trade size, is strictly increasing and exhibits strict discrete concavity within the domain of trade sizes where both informed purchasing and informed selling are deemed probable by the market maker.

In Hasbrouck's [11] empirical study, it is shown that large trades cause bid-ask spread to widen. This empirical finding is consistent with our results. Theorem 2 suggests that large trade sizes are likely to be in the domain of informed trading. Also, Proposition 12 implies that the bid-ask spread, as a function of trade size, is strictly increasing within the domain where both informed purchasing and informed selling are considered probable by the market maker. Therefore, Theorem 2 and Proposition 12 together suggest that large trades are likely to be associated with a widening of the bid-ask spread. Furthermore, Hasbrouck [11] notes that the widening in the spread after a large trade is temporary. This finding can also be justified by our model. As large trades are likely to be in the domain of informed trading, they lead to the market maker updating her posterior belief. Theorem 8 shows that the market maker gets close to the truth regarding the risky asset payoff after a sufficiently number of periods. Hence, bid-ask spreads are eventually bound to vanish, as indicated by Proposition 11. Consequently, the widening in the spread can only be temporary.

3.5 Price Impact

The last notion to be examined in our equilibrium analysis is the price impact. Price impact measures the absolute impact of trade size on the risky asset price. Formally, *the period-t price impact of trade $q \in \Omega_n$ for history $h_{t-1} \in \Omega_n^{t-1}$ is given by*

$$I_t(h_{t-1}, q) \;=\; |\pi_t(h_{t-1}, q) - \pi_{t-1}(h_{t-1})|\,. \tag{12}$$

Hasbrouck's [11] estimates for a sample of NYSE suggests that price impact, as a function of trade size, is increasing and concave. Moreover, this concavity is quite typical of all the stocks in his sample. We provide a theoretical justification for Hasbrouck's [11] empirical finding in the following proposition:

Proposition 13 *Let $\left\{(\pi_t^*, \psi_t^*, \delta_t^*) : t = 1, ..., T\right\}$ be an equilibrium. Also, let ψ_t^* be k_t^+ partially pooling on the long side and k_t^- partially pooling on the short side for history h_{t-1}.*

(a) The equilibrium price impact $I_t^*(h_{t-1}, q)$, as a function of trade size $|q|$, is increasing and exhibits discrete concavity.

(b) The equilibrium price impact $I_t^*(h_{t-1}, q)$, as a function of trade size $|q|$, exhibits strict discrete concavity in the domain $\{-n, ..., -k_t^-\} \cup \{k_t^+, ..., n\}$.

This result follows from two basic equilibrium properties: (1) only a transaction in the domain of informed trading leads to a change in the market maker's posterior belief, hence a change in her price menu, (2) given the market maker's price menu, all transactions in the domain of informed trading yield the same expected profit (if a transaction yielded a lower expected profit compared to others, informed traders would not have made that transaction in equilibrium). The first equilibrium property implies that the equilibrium price impact of trade q equals zero if q is outside the domain of informed trading. The second equilibrium property implies that informed traders' expected profit, $(v - \pi_t^*(h_{t-1}, q))\, q$, is same over all trades, q, within the domain of informed trading. Consequently, period-t equilibrium price $\pi_t^*(h_{t-1}, q)$ is proportional to $-\frac{1}{q}$ within the domain of informed trading. This in turn implies that the equilibrium price impact, as a function of trade size $|q|$, is increasing and exhibits discrete concavity.

4 Concluding Remarks

To quote an old adage in the Wall Street, "it takes volume to move prices". This paper investigates the relationship between trade sizes and the dynamic process of price formation. Following Glosten and Milgrom [7] and Easley and O'Hara [6], we assume that the response of asset prices to trading activity is a consequence of asymmetric information. Our theoretical study reveals the following:

1. In each period there is a positive cutoff trade size for the informed trader who observes that the risky asset payoff is V. She assigns no probability to purchasing amounts below this trade size because, even at the price induced by the market maker's priors, such trades cannot capture her equilibrium information rents. She assigns positive probability to purchasing the cutoff trade size, by definition. In equilibrium any positive trade size that she assigns zero probability to is priced according to the market maker's priors, so she must assign positive probability to each trade size above the cutoff because otherwise purchasing the cutoff trade size would be suboptimal. The situation is symmetric for the informed trader who observes that the risky asset payoff is 0. There is a positive least amount that she sells with positive probability, and she assigns positive probability to selling each allowed amount greater than this cutoff.

2. Bid-ask spreads exist only in the trade sizes where informed trading is deemed probable by the market maker.

3. The cut-off trade sizes decrease following a trade provided that the trade leads to a substantial change in the market maker's belief. Consequently, the domain of trade sizes, where informed trading is deemed probable by the market maker, can get bigger over time. Therefore, small trade sizes initially with zero bid-ask spreads can later have positive spreads.

4. The market maker learns the true risky asset payoff almost surely as the number of trading rounds tends to infinity. Hence, the bid-ask spreads are eventually bound to vanish.

5. The bid-ask spread, as a function of trade size, is strictly increasing and exhibits strict (discrete) concavity within the domain where both informed purchasing and informed selling are deemed probable by the market maker. Also, the price impact, as a function of trade size, is increasing and exhibits (discrete) concavity. Both results are consistent with Hasbrouck's [11] empirical findings.

There are a number of directions in which our theoretical study can be furthered. One of them is to introduce price discreteness. Numerous empirical studies tackle the dynamics of discrete bid and ask quotes and investigate the impact of tick size reduction (i.e. price decimalization) on market quality.[9] Another possible extension is to allow for re-trading: as in Glosten and Milgrom [7], traders re-trade with probability zero in our current model. Chakraborty and Yilmaz [3] show that there is room for manipulation when re-trading is allowed and the information structure is enriched.

[9] See Goldstein and Kavajecz [8], Harris [9], [10], and Hasbrouck [12].

Appendix

Proof of Proposition 1. (a) Suppose $\psi_t^*(0|V, h_{t-1}, \pi_t^*) > 0$ for some $t \geq 1$ and $h_{t-1} \in \Omega_n^{t-1}$. Then we have

$$0 \;\geq\; (V - \pi_t^*(h_{t-1}, q))\, q \;=\; \delta_t^*(h_{t-1}, q)\, V\, q, \quad \forall q \in \Omega_n. \tag{13}$$

Following (1), $0 < \delta_t^*(h_{t-1}, q) < 1$ as the probability of liquidity trading is positive over all trade sizes. Therefore inequality (13) fails to hold when $q \in \Omega_n^+$. This proves that $\psi_t^*(0|V, h_{t-1}, \pi_t^*) = 0$ for all $t \geq 1$ and $h_{t-1} \in \Omega_n^{t-1}$. In a similar fashion, it can be easily proved that $\psi_t^*(0|0, h_{t-1}, \pi_t^*) = 0$ for all $t \geq 1$ and $h_{t-1} \in \Omega_n^{t-1}$.

(b) Suppose there exist $t \geq 1$ and $h_{t-1} \in \Omega_n^{t-1}$ such that $\psi_t^*(q|V, h_{t-1}, \pi_t^*) > 0$ for some $q \in \Omega_n^-$. Then it must hold that

$$(V - \pi_t^*(h_{t-1}, q))\, q \geq 0. \tag{14}$$

However, since $0 < \delta_t^*(h_{t-1}, q) < 1$ and $q \in \Omega_n^-$,

$$(V - \pi_t^*(h_{t-1}, q))\, q \;=\; \delta_t^*(h_{t-1}, q)\, V\, q \;<\; 0.$$

This contradicts with (14).

(c) The proof is similar to that of (b). $\Box$

Proof of Theorem 2. Suppose there exists an equilibrium $\{(\pi_t^*, \psi_t^*, \delta_t^*) : t = 1, ..., T\}$ such that for some $t \geq 1$, $h_{t-1} \in \Omega_n^{t-1}$, and $i, j \in \Omega_n^+$ with $i > j$

$$\psi_t^*(j|V, h_{t-1}, \pi_t^*) > 0, \qquad \psi_t^*(i|V, h_{t-1}, \pi_t^*) = 0.$$

Then it must hold that

$$(V - \pi_t^*(h_{t-1}, j))\, j \;\geq\; (V - \pi_{t-1}^*(h_{t-1}, i))\, i.$$

This in turn implies together with (1), (2), and Proposition 1 that

$$\frac{j}{i} \;\geq\; \frac{V - \pi_t^*(h_{t-1}, i)}{V - \pi_t^*(h_{t-1}, j)} \;=\; 1 + \frac{(1 - \delta_t^*(h_{t-1}, j))\, \psi_t^*(j|V, h_{t-1}, \pi_t^*)\, \mu}{(1-\mu)\gamma(j)}. \tag{15}$$

Since $\frac{j}{i} < 1$ and $\psi_t^*(j|V, h_{t-1}, \pi_t^*) > 0$ by assumption, (15) fails to hold. This proves that there exists $k_t^+(h_{t-1}) \in \Omega_n^+$ such that $\mathrm{supp}\{\psi_t^*(V, h_{t-1}, \pi_t^*)\} \;=\; \{k_t^+(h_{t-1}), \cdots, n\}$. In a similar fashion, it can be proved that $\mathrm{supp}\{\psi_t^*(0, h_{t-1}, \pi_t^*)\} \;=\; \{-n, \cdots, -k_t^-(h_{t-1})\}$ for some $k_t^-(h_{t-1}) \in \Omega_n^-$. $\Box$

Some of the proofs in the rest of the Appendix are based on the following lemma.

Lemma 1 *Let* $\left\{(\pi_t^*, \psi_t^*, \delta_t^*) : t = 1, ..., T\right\}$ *be an equilibrium. Also, let* ψ_t^* *be* k_t^+ *partially pooling on the long side and* k_t^- *partially pooling on the short side for history* h_{t-1}*. Then*

$$\psi_t^*(q|V, h_{t-1}, \pi_t^*) = \begin{cases} 0 & : \quad q \in \{1, \cdots, k_t^+ - 1\} \\ \dfrac{(1-\mu)\sum_{i=k_t^+}^n \left(1 - \frac{i}{q}\right)\gamma(i) + \left(1 - \delta_{t-1}^*(h_{t-1})\right)\mu}{\left(1 - \delta_{t-1}^*(h_{t-1})\right)\mu \sum_{i=k_t^+}^n \frac{i\gamma(i)}{q\gamma(q)}} & : \quad q \in \{k_t^+, \cdots, n\} \end{cases} \tag{16a}$$

and

$$\psi_t^*(q|0, h_{t-1}, \pi_t^*) = \begin{cases} 0 & : \quad q \in \{-k_t^- + 1, \cdots, -1\} \\ \dfrac{(1-\mu)\sum_{i=k_t^-}^n \left(1 - \frac{i}{|q|}\right)\gamma(-i) + \delta_{t-1}^*(h_{t-1})\mu}{\delta_{t-1}^*(h_{t-1})\mu \sum_{i=k_t^-}^n \frac{i\,\gamma(-i)}{|q|\,\gamma(q)}} & : \quad q \in \{-n, \cdots, k_t^-\}. \end{cases} \tag{16b}$$

Proof of Lemma 1. As ψ_t^* is k_t^+ partially pooling on the long side for history h_{t-1}, $\psi_t^*(q|V, h_{t-1}, \pi_t^*) = 0$ for $q \in \{1, \cdots, k_t^+ - 1\}$. Now let $q \in \{k_t^+, ..., n\}$. The equilibrium definition imposes that

$$q[V - \pi_t^*(h_{t-1}, q)] \;=\; i[V - \pi_t^*(h_{t-1}, i)], \quad \forall i \in \{k_t^+, ..., n\}, \tag{17a}$$

$$q[V - \pi_t^*(h_{t-1}, q)] \;\geq\; i[V - \pi_t^*(h_{t-1}, i)], \quad \forall i \in \{0, 1, \cdots, k_t^+ - 1\}. \tag{17b}$$

Following (2) and (17a),

$$\frac{i}{q} \;=\; \frac{\delta_t^*(h_{t-1}, q)}{\delta_t^*(h_{t-1}, i)}, \quad \forall i \in \{k_t^+, ..., n\}.$$

The equation above, (1), and Proposition 1 together imply that for $i \in \{k_t^+, ..., n\}$

$$(1 - \delta_{t-1}^*(h_{t-1}))\,\mu \left[\psi_t^*(q|V, h_{t-1}, \pi_t^*)\frac{i\gamma(i)}{q\gamma(q)} - \psi_t^*(i|V, h_{t-1}, \pi_t^*)\right] = (1-\mu)\left(1 - \frac{i}{q}\right)\gamma(i). \tag{18}$$

Summing left and right hand side of (18) over $i \in \{k_t^+, ..., n\} \setminus \{q\}$, we obtain

$$\begin{aligned} & (1 - \delta_{t-1}^*(h_{t-1}))\,\mu \left[\psi_t^*(q|V, h_{t-1}, \pi_t^*) \sum_{i=k_t^+, i\neq q}^n \frac{i\gamma(i)}{q\gamma(q)} - \sum_{i=k_t^+, i\neq q}^n \psi_t^*(i|V, h_{t-1}, \pi_t^*)\right] \\ = \;\; & (1-\mu) \sum_{i=k_t^+, i\neq q}^n (1 - \frac{i}{q})\gamma(i). \end{aligned} \tag{19}$$

Replacing $\sum_{i=k, i\neq q}^n \psi_t^*(i|V, h_{t-1}, \pi_t^*)$ with $1 - \psi_t^*(q|V, h_{t-1}, \pi_t^*)$ and rearranging the terms in (19) yield

$$\psi_t^*(q|V, h_{t-1}, \pi_t^*) \;=\; \frac{(1-\mu)\displaystyle\sum_{i=k_t^+}^n \left(1 - \frac{i}{q}\right)\gamma(i) + (1 - \delta_{t-1}^*(h_{t-1}))\mu}{(1 - \delta_{t-1}^*(h_{t-1}))\mu \displaystyle\sum_{i=k_t^+}^n \frac{i\gamma(i)}{q\gamma(q)}}.$$

Hence, equation (16a) is obtained. Equation (16b) can be obtained in a similar fashion. □

Proof of Proposition 3. (a) Suppose ψ_t^* is k_t^+ partially pooling on the long side for history h_{t-1}. This means $\psi_t^*(q|V, h_{t-1}, \pi_t^*) > 0$ for $q \in \{k_t^+, ..., n\}$ and $\psi_t^*(q|V, h_{t-1}, \pi_t^*) = 0$ for $q \notin \{k_t^+, ..., n\}$. Hence, following Lemma 1, the inequalities (4a) and (4b) must hold.

Now suppose the inequalities (4a) and (4b) hold. Following Theorem 2, there exists some K such that ψ_t^* is K partially pooling on the long side for history h_{t-1}. From Lemma 1, we have

$$\psi_t^*(q|V, h_{t-1}, \pi_t^*) = \begin{cases} 0 & : \quad q \in \{1, \cdots, K-1\} \\ \dfrac{(1-\mu)\sum_{i=K}^{n}\left(1-\frac{i}{q}\right)\gamma(i) + \left(1-\delta_{t-1}^*(h_{t-1})\right)\mu}{\left(1-\delta_{t-1}^*(h_{t-1})\right)\mu\sum_{i=K}^{n}\frac{i\gamma(i)}{q\gamma(q)}} & : \quad q \in \{K, \cdots, n\} \end{cases}$$

As ψ_t^* is K partially pooling on the long side for h_{t-1}, it must be true that

$$\begin{aligned} (1-\mu)\sum_{i=K}^{n}\left(1-\frac{i}{K}\right)\gamma(i) + (1-\delta_{t-1}^*(h_{t-1}))\mu &> 0, \\ (1-\mu)\sum_{i=K-1}^{n}\left(1-\frac{i}{K-1}\right)\gamma(i) + (1-\delta_{t-1}^*(h_{t-1}))\mu &\leq 0. \end{aligned}$$

Since the inequalities (4a) and (4b) hold, we have $k_t^+ = K$. This proves that ψ_t^* is k_t^+ partially pooling on the long side for history h_{t-1}.

(b) The proof is similar to that of (a). □

Proof of Proposition 7. (a) Let (10a) hold and q_t be from the domain $\{k_t^+, ..., n\}$. Assume to the contrary that $k_{t+1} \geq k_t$. Since ψ_{t+1}^* is k_{t+1} partially pooling on the long side for history $h_t = (h_{t-1}, q_t)$, from Proposition 3 we have

$$(1-\mu)\sum_{i=k_{t+1}^+-1}^{n}\left(1-\frac{i}{k_{t+1}^+-1}\right)\gamma(i) + (1-\delta_t^*(h_{t-1}, q_t))\mu \;\leq\; 0. \tag{20}$$

As $k_{t+1} \geq k_t$, from (20) we obtain

$$(1-\mu)\sum_{i=k_t^+-1}^{n}\left(1-\frac{i}{k_t^+-1}\right)\gamma(i) + (1-\delta_t^*(h_{t-1}, q_t))\mu \;\leq\; 0. \tag{21}$$

Since ψ_t^* is k_t^+ partially pooling on the long side for h_{t-1}, by Proposition 3, the inequalities (4a)- (4b) also hold. (4a) and (21) together yield

$$(1-\mu)\left[\sum_{i=k_t^+}^{n}\left(1-\frac{i}{k_t^+}\right)\gamma(i) - \sum_{i=k_t^+-1}^{n}\left(1-\frac{i}{k_t^+-1}\right)\gamma(i)\right] > (\delta_{t-1}^*(h_{t-1}) - \delta_t^*(h_{t-1}, q_t))\mu$$

Given that $q_t \in \{k_t^+, ..., n\}$, The inequality above, (1), and Proposition 1 imply

$$\begin{aligned}
\frac{(1-\mu)}{\mu\, k_t^+(k_t^+-1)} \sum_{i=k_t^+}^{n} i\,\gamma(i) &> (\delta_{t-1}^*(h_{t-1}) - \delta_t^*(h_{t-1}, q_t)) \\
&= \delta_{t-1}^* \left(\frac{(1-\mu)\sum_{i=k_t^+}^{n}\left(1-\frac{i}{q_t}\right)\gamma(i) + (1-\delta_{t-1}(h_{t-1}))\mu}{(1-\mu)\sum_{i=k_t^+}^{n}\gamma(i) + (1-\delta_{t-1}(h_{t-1}))\mu} \right). \qquad (22)
\end{aligned}$$

(22) contradicts (10a). Therefore, it must hold that $k_{t+1} < k_t$.

(b) The proof is similar to that of (a). □

Proof of Theorem 8. Let $\mathcal{H}_\infty$ denote the sigma field generated by all the possible histories h_∞. We consider the market maker's equilibrium belief as a stochastic process, which we denote by $\{\delta_t : t = 1, ..., T\}$. Following a theorem in Fristedt and Gray [5] (Theorem 3, p. 432), there exists a unique distribution over δ_t conditional on $(\delta_0, \cdots, \delta_{t-1})$. Therefore, the collection of probability distributions $\{\delta_t : t = 1, ..., T\}$ and the probability space $(\Omega_n^\infty, \mathcal{H}_\infty, \mathcal{P})$ are well-defined. Notice that δ_t is a martingale with respect to the market maker's information set. By the martingale convergence theorem, δ_t converges almost surely to a random variable $\hat{\delta}$. Next we prove that $\hat{\delta} = 0$ if $v = V$ and $\hat{\delta} = 1$ if $v = 0$. Let $v = V$ and suppose to the contrary that there exists a period τ and histories h_t' such that for all $t \geq \tau$

$$\Pr(h_t' : |\delta_t(h_t') - p| > \epsilon) = 0 \qquad (23)$$

for some $p \in (0, 1]$ and arbitrary small ϵ. Following Theorem 2, for all $t \geq \tau$ there exists some k_t^+ such that informed traders' equilibrium trading strategy ψ_t^* is k_t^+ partially pooling on the long side for history h_{t-1}'. By (1), if a trade larger than $k_{\tau+1}^+(h_\tau')$ realizes in period-$(\tau+1)$, the market maker's belief $\delta_{\tau+1}$ deviates from the interval of $[p-\epsilon, p+\epsilon]$. Formally,

$$\Pr(h_{\tau+1}' : |\delta_{\tau+1}(h_{\tau+1}') - p| > \epsilon) = \Pr(h_\tau' : |\delta_\tau(h_\tau') - p| \leq \epsilon) \left(\mu + \sum_{i=k_\tau^+(h_\tau')}^{n} \gamma(i) \right) > 0,$$

which contradicts with (23). Therefore, it must hold that $\hat{\delta} = 0$. It can be similarly shown that $\hat{\delta} = 1$ if $v = 0$. □

Proof of Proposition 9. From (1), (2), and Lemma 1, we have

$$\begin{aligned}
S_t^*(h_{t-1}, q) &= (\delta_t(h_{t-1}, -q) - \delta_t(h_{t-1}, q))\, V \\
&= \delta_{t-1}^*(h_{t-1})\, V \\
&\quad \times \left[\frac{\mu\,\psi_t^*(-q|0, h_{t-1}, \pi_t^*) + (1-\mu)\gamma(-q)}{\delta_{t-1}^*(h_{t-1})\,\mu\,\psi_t^*(-q|0, h_{t-1}, \pi_t^*) + (1-\mu)\gamma(-q)} - \frac{(1-\mu)\gamma(q)}{\mu\,(1-\delta_{t-1}^*(h_{t-1}))\,\psi_t^*(q|V, h_{t-1}, \pi_t^*) + (1-\mu)\gamma(q)} \right] \\
&= \delta_{t-1}^*(h_{t-1})\,(1 - \delta_{t-1}^*(h_{t-1}))\,\mu\, V \\
&\quad \times \frac{(1-\mu)[\psi_t^*(q|V, h_{t-1}, \pi_t^*)\gamma(-q) + \psi_t^*(-q|0, h_{t-1}, \pi_t^*)\gamma(q)] + \mu\,\psi_t^*(q|V, h_{t-1}, \pi_t^*)\,\psi_t^*(-q|0, h_{t-1}, \pi_t^*)}{\left((1-\delta_{t-1}^*(h_{t-1}))\,\mu\,\psi_t^*(q|V, h_{t-1}, \pi_t^*) + (1-\mu)\,\gamma(q)\right)\left(\delta_{t-1}^*(h_{t-1})\,\mu\,\psi_t^*(-q|0, h_{t-1}, \pi_t^*) + (1-\mu)\,\gamma(-q)\right)}. \qquad (24)
\end{aligned}$$

If $1 \leq q < \min\{k_t^-, k_t^+\}$, then $\psi_t^*(q|V, h_{t-1}, \pi_t^*) = \psi_t^*(-q|0, h_{t-1}, \pi_t^*) = 0$ and consequently (24) yields $S_t^*(h_{t-1}, q) = 0$. If $\min\{k_t^-, k_t^+\} \leq q \leq n$, then $\max\{\psi_t^*(q|V, h_{t-1}, \pi_t^*), \psi_t^*(-q|0, h_{t-1}, \pi_t^*)\} > 0$ hence (24) yields $S_t^*(h_{t-1}, q) > 0$.

On the other hand, if $S_t^*(h_{t-1}, q) = 0$, then following (24) it must hold that $\psi_t^*(q|V, h_{t-1}, \pi_t^*) = \psi_t^*(-q|0, h_{t-1}, \pi_t^*) = 0$ and this together with Lemma 1 implies $1 \leq q < \min\{k_t^-, k_t^+\}$. If $S_t^*(h_{t-1}, q) > 0$, then following (24) it must hold that $\max\{\psi_t^*(q|V, h_{t-1}, \pi_t^*), \psi_t^*(-q|0, h_{t-1}, \pi_t^*)\} > 0$, which together with Lemma 1 implies $\min\{k_t^-, k_t^+\} \leq q \leq n$. □

Proof of Proposition 11. The result immediately follows from (24) and Theorem 8. □

Proof of Proposition 12. From Lemma 1 and (24), we have

$$S_t^*(h_{t-1}, q) = \delta_{t-1}^*(h_{t-1})\, V \left[\frac{1-\delta_{t-1}^*(h_{t-1})}{\delta_{t-1}^*(h_{t-1})} \frac{(1-\mu)\sum_{i=k_t^-}^{n}\left(1-\frac{i}{q}\right)\gamma(-i)+\delta_{t-1}^*(h_{t-1})\,\mu}{(1-\mu)\sum_{i=k_t^-}^{n}\gamma(-i)+\delta_{t-1}^*(h_{t-1})\,\mu} + \frac{(1-\mu)\sum_{i=k_t^+}^{n}\left(1-\frac{i}{q}\right)\gamma(i)+(1-\delta_{t-1}^*(h_{t-1}))\,\mu}{(1-\mu)\sum_{i=k_t^+}^{n}\gamma(i)+(1-\delta_{t-1}^*(h_{t-1}))\,\mu} \right]. \tag{25}$$

if $q \geq \max\{k_t^-, k_t^+\}$. The result immediately follows from (25). □

Proof of Proposition 13. Using (1), (2), and Lemma 1, we derive the following:

- if $q \notin \{-n, ..., -k_t^-\} \cup \{k_t^+, ..., n\}$, then

$$I_t^*(h_{t-1}, q) = 0; \tag{26a}$$

- if $q \in \{k_t^+, ..., n\}$, then

$$I_t^*(h_{t-1}, q) = \delta_{t-1}^*(h_{t-1})\, V \left(1 - \frac{(1-\mu)\sum_{i=k_t^+}^{n} i\,\gamma(i)}{q\left[(1-\mu)\sum_{i=k_t^+}^{n}\gamma(i)+(1-\delta_{t-1}^*(h_{t-1}))\mu\right]}\right); \tag{26b}$$

- if $q \in \{-n, ..., -k_t^-\}$, then

$$I_t^*(h_{t-1}, q) = \delta_{t-1}^*(h_{t-1})\, V \times \left(\frac{(1-\mu)\left((1-\delta_{t-1}(h_{t-1}))\sum_{i=k_t^-}^{n}\left(1-\frac{i}{|q|}\right)\gamma(-i)+\delta_{t-1}(h_{t-1})\sum_{i=k_t^-}^{n}\gamma(-i)\right)+\delta_{t-1}(h_{t-1})\,\mu}{\delta_{t-1}(h_{t-1})\left((1-\mu)\sum_{i=k_t^-}^{n}\gamma(-i)+\delta_{t-1}(h_{t-1})\,\mu\right)} - 1 \right). \tag{26c}$$

The results immediately follow from (26a), (26b), and (26c). □

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