Amplification and Asymmetry in Crashes and Frenzies

Han N. Ozsoylev
Saïd Business School and Linacre College
University of Oxford

This version: June 2005

Abstract

We often observe disproportionate reactions to tangible information in large stock price movements. Moreover these movements feature an asymmetry: the number of crashes is more than that of frenzies in the S&P 500 index. This paper offers an explanation for these two characteristics of large movements in which hedging (portfolio insurance) causes amplified price reactions to news and liquidity shocks as well as an asymmetry biased towards crashes. Risk aversion of traders is shown to be essential for the asymmetry of price movements. Also, we show that differential information enhances both amplification and asymmetry delivered by hedging.

Keywords: Crash · Frenzy · Hedging · Portfolio insurance
JEL Classification Numbers: G11 · G12

This paper is based on part of my Ph.D. thesis submitted to the University of Minnesota. I am grateful to Andy McLennan and Jan Werner for their valuable advice and unwavering support. Also, I would like thank Mehmet Barlo, Michele Boldrin, Partha Chatterjee, Mehmet Ozhabes, Matthew Spiegel, Dimitrios Tsomocos and seminar participants at the University of Minnesota, the MEA and the MFA Meetings in St Louis for helpful comments. The usual disclaimer shields all. Financial support from William W. Stout Fellowship is gratefully acknowledged.

Han N. Ozsoylev
Saïd Business School, University of Oxford, Park End Street, Oxford, OX1 1HP, UK
E-mail: `han.ozsoylev@sbs.ox.ac.uk`

1 Introduction

Sudden and large movements in stock prices have always drawn economists' attention. We see them in the form of frenzies, when the price movement is in the positive direction, and crashes, when the direction is negative. This paper focuses on two characteristics of crashes and frenzies: *amplification* and *asymmetry*.

In many cases, there seems to be no significant events prior to large price movements. Cutler, Poterba and Summers (1989) document that for the postwar movements in the S&P 500 index. This empirical fact suggests that large price movements are most often *amplified price reactions* to comparatively insignificant information or liquidity shocks.

In addition, there is substantial difference between the number of crashes and frenzies, and this is what we mean by the *asymmetry*. Hong and Stein (2003) report that nine of the ten largest one-day price movements in the S&P 500 since 1947 were decreases. A broader look at the data also confirms the asymmetry. Boldrin and Levine's (2001) analysis of S&P 500 between 1889 and 1984 reveals that annual negative deviations[1] are, on average, larger than positive ones. We observe the following in the Boldrin and Levine data: There is one annual negative deviation with magnitude larger than 50% but no positive deviation exceeds this value. The number of annual negative deviations with magnitudes larger than 40% is 3 and that of positive deviations is none. There are 6 annual negative deviations of size exceeding 30% compared to 4 positive ones. Finally, 14 annual negative deviations have magnitudes larger than 20% and only 10 positive ones exceed this value. So the Boldrin and Levine data also implicates the asymmetry between crashes and frenzies.

This paper offers an explanation for the two characteristics of large price movements, depicted above. Our explanation involves the use of *hedging (portfolio insurance) strategies* in the stock market. Hedgers, who use these strategies, sell after the market has declined and buy after the market rises. Therefore portfolio insurance is negatively price sensitive since conventional supply schedules are increasing functions of price. Brady Commission Report (1988) provides evidence for the use of portfolio insurance strategies during the crash of 1987 and furthermore blames these negatively price sensitive strategies for deepening the decline hence perhaps causing the crash. The studies of Chicago Mercantile Exchange, Miller, Hawke, Malkiel, and Scholes (1987), Commodity Futures Trading Commission (1987), Securities and

[1] For the years 1889-1984, Boldrin and Levine (2001) report the real S&P 500 index, and the "deviation" from the difference between the log of the index value of a year and that of a subsequent year.

Exchange Commission (1987) also highlight the important role of these strategies in the 1987 crash.[2] As a possible contributing factor to the crash of 1929, we see arguments focusing on the use of stop-loss orders which are primitive portfolio insurance strategies (Gennotte and Leland (1990)). Gennotte and Leland (1990) explain the '87 crash in concordance with the findings of Brady Report by incorporating hedging (portfolio insurance) into a conventional noisy rational expectations model.

Following Gennotte and Leland (1990) we develop a static noisy rational expectations equilibrium (REE) model with hedgers using negatively price sensitive strategies in a CARA-Gaussian environment. Our results show that hedging strategies amplify the effect of news and liquidity shocks on price deviations. Convex hedging strategies cause overreaction to negative news and liquidity shocks, hence they create an asymmetry biased towards crashes. An important class of hedging functions (put-option replication strategies[3]) satisfies the convexity condition in a highly volatile market. We also examine the roles of risk aversion and asymmetric information in our analysis. In particular, we show that risk aversion is necessary for asymmetry of price deviations. Also, the asymmetric information is shown to enhance the amplification and the asymmetry delivered by hedging. Finally we analyze trading behavior of rational agents in the presence of hedgers, and question the emergence of hedging in financial markets.

The focus of our paper is characteristics of certain dynamic phenomena, namely crashes and frenzies. This might seem puzzling since we employ a static model for the analysis. However, in our static framework we can interpret comparative statics results on price as dynamic changes over time. In particular, the equilibrium price reactions to changes in the information or liquidity parameters are viewed as fluctuations over time. In the same fashion, crashes and frenzies are interpreted as high sensitivity to changes in information or liquidity parameters. That is, if we see a substantial fall in equilibrium price as a reaction to comparatively insignificant news, we call it a crash (or a frenzy in the case of a price increase) in our setup. Note that, by this interpretation, we also incorporate an observed characteristic, namely amplification, into our definition of crashes and frenzies.

As mentioned above, in our setup, hedging (portfolio insurance) is the cause of amplification and asymmetry in large price movements. Hedging strategies are naturally dynamic strategies dependent on the price trend. Before explaining how hedging strategies fit into our

[2] Shiller (1989)

[3] Put-option replication is formally defined in Section 4. See Rubinstein and Leland (1981) for a detailed exposition of the subject.

static environment, let us discuss why they would cause amplified and asymmetric deviations. For intuition, we can first look at stop-loss orders. With stop-loss orders we see sales after the market has fallen under some exercise value. The aim is to protect one's portfolio against future potential losses. Here it is easy to see how a crash can be the result of an amplified price reaction, because stop-loss itself puts a downward pressure on the price once the price begins to fall. Moreover since there is no accompanying upward pressure, we are likely to observe an asymmetry biased towards crashes in an environment where stop-loss orders prevail. In modern hedging strategies, such as put-option replication, the idea is the same, but now we have both upward and downward pressures on the price. That is, we see a buying spree from hedgers in a bull market, and sales in a bearish one; hence comes the amplified price reactions. If the downward pressure of the strategy were to be stronger than the upward one, we would observe asymmetry biased towards crashes. This summarizes most of what we are trying to formalize in §3.2.

Now we can return to the interpretation of hedging in our static environment. All hedging activity is aggregated into a deterministic supply function of price p, say $h(p)$. As we have only one trading period in our model, let us take p^* as our (hypothetical) initial price, and let $h(p^*) = 0$. A fall in the security price leads to positive hedging supply, thus for $p < p^*$, $h(p) > 0$. Similarly, we have positive hedging demand (or negative supply) with increasing price, thus $h(p) < 0$ for $p > p^*$. The more the price increases, the higher the hedging demand (and vice versa); thus we want h to be a *decreasing function* of p. In summary, we will view hedging as the change of a deterministic supply with respect to the change in price p compared to a hypothetical initial price p^*. The supply is deterministic, because with stop-loss there is a specific exercise value to strike on, and with others there are specific formulae to follow, such as Black-Scholes in the case of put-option replication.

Having made an informal introduction to the functioning of our model, we can now discuss how our results compare with others in the literature. Though there is an extensive literature on the amplification observed in crashes and frenzies, the asymmetric feature of these large movements has not been addressed until recent years. Boldrin and Levine (2001), Chalkley and Lee (1998), Hong and Stein (2003), Veldkamp (in press), and Veronesi (1999) address the asymmetry of crashes and frenzies. In Boldrin and Levine (2001) the asymmetry in large price movements is driven by the asymmetry in the underlying technology shocks that drive fundamentals. Chalkley and Lee (1998) propose a model with noise traders where risk aversion prevents agents from acting promptly on receiving good news and encourages them to act quickly on receiving bad news. Veronesi's (1999) work is similar to Chalkley and Lee (1998) in spirit as risk aversion makes asset price a convex function of beliefs and leads to underreaction to good news in bad times and overreaction to bad news in good times. Hong and Stein (2003) obtain asymmetry via short-sales constraints, which cause revelation of bad information in bad times and hidden bad information in good times. Veldkamp (2004) explains the asymmetric feature by asymmetric endogenous speed of learning: faster learning in good times causes quick reaction to bad news and hence sudden crashes.

This paper is similar to studies of Chalkley and Lee (1998) and Veronesi (1999) in that we also have risk aversion convexifying price reactions to changes in the underlying parameters, which leads to asymmetry. The difference is that our explanation stems from the use of hedging strategies, which also amplifies the price movements. As mentioned above, hedging is introduced to REE models by Gennotte and Leland (1990) for the first time. However they only focus on the cause of '87 crash (hence do not analyze the asymmetry between crashes and frenzies), and they define crash as a discontinuity in the price function. Here in this paper, following the REE model proposed by them, we offer an explanation for the asymmetry between crashes and frenzies, and we are not seeking any discontinuities in price.[4] There is another paper, Jacklin, Kleidon, and Pfleiderer (1992), which also attributes the '87 crash to hedging strategies. Following Glosten and Milgrom (1985), they model a market with bid-ask prices and sequential trading. What delivers crash is the underestimation of the extent of hedging activities. This might cause a rise in the security price due to imperfect information aggregation, and ultimately learning leads to a price correction, in this case to a fall in price. However, as in Gennotte and Leland (1990), the asymmetry is not sought in Jacklin, Kleidon, and Pfleiderer (1992) either.

Our paper is organized as follows. In Section 3.1 we develop a noisy REE model with hedgers and derive the unique equilibrium. Section 3.2 provides the results on amplification and asymmetry in price deviations. Section 3.3 checks whether the conditions for asymmetry derived in §3.2 are satisfied in practice, then we provide a numerical example demonstrating amplification and asymmetry in Section 3.4. Section 3.5 focuses on the roles of risk aversion and asymmetric information pertaining to amplification and asymmetry in crashes and frenzies. Section 3.6 deals with the effect of hedging on rational agents' trading behavior. Finally, Section 3.7 questions the emergence of hedging in the stock market.

[4] Actually we rule out discontinuities to ease the comparative statics exercises.

2 CARA-Gaussian economy

We employ a static REE model, which is a simplified version of Gennotte and Leland (1990) with one informed trader instead of many informed traders with different Gaussian information sets. We mimic the approach of Demange and Laroque (1995) to compute the equilibrium price.

2.1 The model

We assume two periods of time in our model. Economic agents, whom we will specify later, competitively trade in the first period and consume in the second. There is only one good in the economy, and there are two securities (i.e. two claims on the good): a risk-free security and a risky security with a future random payoff R, which realizes in the second period. The price and the payoff of the risk-free security are normalized to 1.

The four types of agents in our economy are as follows:

(1) *insider*,[5] who observes the risky security price p and a private random signal S on the risky security payoff X;

(2) *rational outsiders*, who observe the risky security price p;

(3) *liquidity traders*, whose function is to add noise to the economy, that is, they create an exogenously determined random net supply of the risky security;

(4) *hedgers*, who create a deterministic net supply of the risky security. This net supply, h, is a decreasing function of the risky security price p.

The informational structure in our model is as follows. The distribution of signal S is common knowledge whereas the realization of the signal is only known to the insider. Similarly, distribution of liquidity supply L is common knowledge, however neither the insider nor outsiders know the realization of L. The hedging supply function h is known to both insider and outsiders.

All random variables in our model are Gaussian. The future payoff of the risky security, X, is a normal random variable with non-zero variance. Insider's signal on X is of the form $S = X + \Omega$, where Ω is distributed with $N(0, \sigma_\Omega)$. The liquidity supply, L, is also normal with

[5] We can justify the price-taking behavior of the single insider by assuming that she represents a continuum of mass one of insiders who act competitively.

distribution $N(0, \sigma_L)$. The random variables X, Ω, and L are jointly normally distributed and independent from each other. Note that, throughout the paper, the random variables are denoted by capital letters, and realizations of them are denoted by the corresponding small letters.

Utilities of rational agents, namely the insider and outsiders, exhibit constant absolute risk aversion (CARA). The CARA-Gaussian setup allows us to aggregate outsiders into a single agent, as all outsiders share the same information. From now on we denote the insider by i, and the outsider by o. The constant Arrow-Pratt measure of absolute risk aversion of insider is a_i, and that of outsider is a_o. To be more precise, $\frac{1}{a_o}$ is the sum of all rational outsiders' measures of risk tolerance (as we are aggregating all outsiders into a single agent). We define the *aggregate Arrow-Pratt measure of absolute risk aversion* A by setting $\frac{1}{A} = \frac{1}{a_i} + \frac{1}{a_o}$. Utility functions of insider and outsider are of the form, $u^j(W_j) = -\mathrm{e}^{-a_j W_j}$, $j = i, o$, where W_j is agent j's random final wealth (which realizes in the second period). Both agents maximize expected utility of final wealth over the first period and their expectations depend on their Gaussian information. Insider and outsider are endowed with deterministic wealth (holdings of risk-free claim on the good) e_i and e_o, respectively.

In the first period, the risky security is traded on the market against the risk-free security. If agent j, $j = i, o$, purchases D_j units of the risky security at price p, j's random final wealth would be $W_j = D_j X + (e_j - pD_j)$. As the rational agent j maximizes her expected utility of consumption in the second period, she solves

$$\begin{aligned} &\max\nolimits_{D_j} \mathrm{E}[-\mathrm{e}^{-a_j W_j} | I_j] \\ \text{s. to } \; & D_j X + (e_j - pD_j) = W_j, \end{aligned} \tag{1}$$

where D_j is j's net excess demand of the risky security and I_j is j's Gaussian information. Liquidity traders and hedgers determine the total net supply of the risky security in the first period. Thus, in the first period, total supply of the risky security at price p is $l + h(p)$, where l is the realization of random liquidity supply L.

In the second period, all uncertainty is resolved, and consumption takes place without any further trade.

2.2 Equilibrium

Next we define the equilibrium price in the fashion of rational expectations equilibrium: *a rational expectations equilibrium price of the risky security is a function* $P(s, l)$ *such that, for* *any realization of signal and liquidity supply (s,l),*

$$D_i(p|s) + D_o(p|P(s,l) = p) = l + h(p),$$

where $D_i(p|s)$ solves insider's maximization problem given in (1), conditional on the observation of the price p and the signal s,[6] *and $D_o(p|P(s,l) = p)$ solves outsider's maximization problem given in (1), conditional on the observation of p and the knowledge about the price function $P(s,l)$ to update the beliefs on s.*

Note that as insider is the only informed trader in the economy, observation of risky security's price does not add any information on top of what he already has. We let Σ denote outsider's Gaussian information. From the definition above we already know Σ coincides to the knowledge of $P(s,l) = p$; however we would like to express outsider's information explicitly as a function of s and l in the equilibrium, hence we introduce this new notation. The excess demand functions of insider and outsider are given by[7]

$$D_i(p|S = s) = \frac{\mathrm{E}[X|s] - p}{a_i \mathrm{var}(X|S)}, \quad D_o(p|\Sigma = \sigma) = \frac{\mathrm{E}[X|\sigma] - p}{a_o \mathrm{var}(X|\Sigma)}. \tag{2}$$

The following notation is introduced:[8]

$$a_i^* = a_i \mathrm{var}(X|S), \quad a_o^* = a_o \mathrm{var}(X|\Sigma), \quad \frac{1}{A^*} = \frac{1}{a_i^*} + \frac{1}{a_o^*}.$$

Given joint distributions of X, S, and L, A^* is only a function of insider's risk aversion a_i, and outsider's risk aversion a_o. That is, the value of A^* does not depend on the realization of insider's signal and liquidity supply (since normal conditional variances are independent of realizations). We further assume the following:

S1. $I + A^* h$ is strictly monotone (i.e. either strictly increasing or strictly decreasing). [9]

[6]The random variables are denoted by capital letters and realizations of them are denoted by the corresponding small letters.

[7]Expressions of excess demand functions in CARA-Gaussian environments are well-known, however we still provide the derivations in (B1) of Appendix B.

[8]Note that we abuse the notation here by writing $\mathrm{var}(X|S)$ instead of $\mathrm{var}(X|s)$, i.e. we condition the variance of X on the distribution of signal rather than its realization. However normal conditional variances do not depend on realizations, thus our notation for the variance fits to this characteristic of the Gaussian environment.

[9]I denotes the identity function, i.e. $I(x) = x \ \forall x \in \mathbb{R}$.

This assumption guarantees a continuous equilibrium price function that can be used for comparative statics. Without assuming S1, the proof of the existence of an equilibrium still holds, but it leads to a price correspondence which may not be single-valued. One now has the following:

Proposition 1 (Equilibrium) *Assume S*1*. Then the unique rational expectations equilibrium price is given by*

$$\begin{aligned} P(s,l) &= f^{-1}\Big(\frac{A^*}{a_i^*}\mathrm{E}[X|s] + \frac{A^*}{a_o^*}\mathrm{E}[X|\sigma] - A^* l\Big) \\ &= f^{-1}\Big(\mathrm{E}[X|\sigma] + \frac{A^*}{a_i^*}(\sigma - \mathrm{E}[X|\sigma])\Big), \end{aligned}$$

where f^{-1} is the inverse of $f \equiv I + A^* h$, and $\sigma = \mathrm{E}[X|s] - a_i \,\mathrm{var}(X|S)\, l$ is the (realization of) outsider's information.

Proof. S1 guarantees that f^{-1} is a well-defined continuous function. Excess demand functions of insider and outsider are also well-defined since $\mathrm{var}\Omega$ and $\mathrm{var}X$ are non-zero. Hence market clearing yields

$$\left(\frac{1}{a_i \mathrm{var}(X|S)} + \frac{1}{a_o \mathrm{var}(X|\Sigma)}\right) p + h(p) = \frac{\mathrm{E}[X|s]}{a_i \mathrm{var}(X|S)} + \frac{\mathrm{E}[X|\sigma]}{a_o \mathrm{var}(X|\Sigma)} - l.$$

Outsider's information σ is revealed by the observation of price and the knowledge of price function. The price function is essentially derived from the market clearing condition above, thus outsider's information coincides with the knowledge of market clearing condition. Since the hedging function h and distributions of S and L are common knowledge, and values of conditional normal variances are independent from realizations[10], outsider can induce the following information from market clearing: $\frac{\mathrm{E}[X|s]}{a_i \mathrm{var}(X|S)} - l$. Multiplying this argument by a known constant (namely $a_i \mathrm{var}(X|S)$) would not matter for the informational content, therefore outsider's information is equivalent to the knowledge of the realization $\sigma = \mathrm{E}[X|s] - a_i \,\mathrm{var}(X|S)\, l$. Recall that S and L are jointly normally distributed. So Σ (the random distribution σ belongs to) is also normally distributed, and outsider's demand as given in (2) holds. Rewriting market clearing condition we have

$$p + A^* h(p) = \frac{A^*}{a_i^*}\mathrm{E}[X|s] + \frac{A^*}{a_o^*}\mathrm{E}[X|\sigma] - A^* l,$$

[10] See (A1) in Appendix A.

where A^*, a_i^*, and a_o^* are as defined above. Writing $\frac{A^*}{a_o^*} = 1 - \frac{A^*}{a_i^*}$, and using definition of f; the result follows. □

Note that the equilibrium price of risky security given by Proposition 1 is a function of insider's private signal s and liquidity supply l. In the Gaussian framework, $\mathrm{E}[X|s]$ is a linear increasing function of s, and given s the assessment of conditional expectation does not put a burden on the agents from the informational perspective since all the parameters necessary to extract its functional form are common knowledge. Therefore the comparative statics results in this paper do not change qualitatively if the equilibrium price is taken as a function of the vector $(\mathrm{E}[X|s], l)$ rather than (s, l). For this purpose we introduce the following notation: let N stand for the random variable $\mathrm{E}[X|S]$, and let ν be the realized value, i.e. $\mathrm{E}[X|s]$. Then the equilibrium price function takes the form[11]

$$P(\nu, l) = f^{-1}\Big(Q(\nu, l)\Big), \quad \text{where} \tag{3a}$$

$$\begin{aligned} Q(\nu, l) &= -\frac{A^*}{a_o^*}\Big\{1 - \frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma}\Big\}\mathrm{E}X + \Big\{\frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma} + \frac{A^*}{a_i^*}\Big(1 - \frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma}\Big)\Big\}\nu \\ &\quad -\Big\{a_i^*\frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma} + A^*\Big(1 - \frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma}\Big)\Big\}l. \end{aligned} \tag{3b}$$

2.3 Asymmetric price deviations and non-linear prices

As it follows from (3a), equilibrium price[12] is a function of insider's expectation of the risky payoff ν, and the liquidity supply l. Here, we would like to discuss how the asymmetry between crashes and frenzies emerges in our setup. If $P(\nu, l)$ were linear in (ν, l), negative and positive shocks of the same magnitudes would create price deviations of the same size. Then we could only attribute the asymmetry in favor of crashes to more frequent and significant negative shocks. As there is no evidence of more frequent negative news or liquidity shocks in the history of S&P 500, we are interested in asymmetric price deviations triggered by symmetric shocks. Formally, we have the following: given (ν_0, l_0), we say that there is an *asymmetry in*

[11] See (B2) in Appendix B for the derivation.

[12] From now on, the term "price" stands for the risky security price unless otherwise stated.

deviations at the equilibrium price $P(\nu_0, l_0)$ if for some $(\Delta\nu, \Delta l) > 0$[13]

$$\begin{aligned} P(\nu_0, l_0) - P(\nu_0 - \Delta\nu, l_0) &\neq P(\nu_0 + \Delta\nu, l_0) - P(\nu_0, l_0), \quad \textit{or} \\ P(\nu_0, l_0) - P(\nu_0, l_0 - \Delta l) &\neq P(\nu_0, l_0 + \Delta l) - P(\nu_0, l_0). \end{aligned}$$

Clearly, *non-linearity* of the equilibrium price function in ν or l is necessary and sufficient for asymmetry in price deviations. Recall that $f \equiv I + A^* h$. When there is no hedging supply, $f = I$ and $P(\nu, l)$ is linear in (ν, l) by (3a)-(3b). So asymmetric information by itself can not create asymmetric deviations in price. With non-zero A^*, non-linearity of hedging supply h becomes a necessary and sufficient condition for a non-linear equilibrium price function, and consequently for asymmetric deviations in price. Given (ν_0, l_0) we say information and liquidity shocks cause a *bias towards negative price deviations* within the set $U_{v_0} \times U_{l_0}$ if for all $(\Delta\nu, \Delta l) > 0$ such that $v_0 - \Delta v \in U_{v_0}$, $v_0 + \Delta v \in U_{v_0}$, $l_0 - \Delta l \in U_{l_0}$, $l_0 + \Delta l \in U_{l_0}$, the following holds:

$$\begin{aligned} P(\nu_0, l_0) - P(\nu_0 - \Delta\nu, l_0) &> P(\nu_0 + \Delta\nu, l_0) - P(\nu_0, l_0), \\ P(\nu_0, l_0) - P(\nu_0, l_0 - \Delta l) &> P(\nu_0, l_0 + \Delta l) - P(\nu_0, l_0). \end{aligned}$$

Suppose equilibrium price function P is continuously differentiable. Then there exists a bias towards negative price deviations within $U_{v_0} \times U_{l_0}$ if and only if $P(\nu, l)$ is *strictly concave* in ν and l within $U_{v_0} \times U_{l_0}$. This is due to the fact that for a strictly concave and continuously differentiable function g

$$g(x_1) < g(x_0) + g'(x_0)(x_1 - x_0),$$

and letting x_1 equal to first $x_0 - \Delta x$ and then $x_0 + \Delta x$ one gets

$$g(x_0) - g(x_0 - \Delta x) > g(x_0 + \Delta x) - g(x_0).$$

Note the following obvious that whenever $P(\nu, l)$ is globally concave in ν and l, all shocks will cause a bias towards negative price deviations in the economy. One can also interpret the *strict concavity* of equilibrium price P as *overreaction to negative shocks* compared to the price reaction to positive shocks.

[13] $(\Delta\nu, \Delta l) > 0$ if and only if both $\Delta\nu$ and Δl are strictly positive.

3 Amplification and asymmetry

In this section we present comparative statics of the equilibrium price $P(\nu, l)$. The first-order partial derivatives of $P(\nu, l)$ with respect to ν and l determine the sensitivity of price to changes in the information parameter and liquidity supply, respectively. The second-order partial derivatives determine the concavity of price function, hence it reveals the nature of bias within the asymmetric price deviations. Our purpose is to see how hedging activity affects the first and second-order partial derivatives of equilibrium price. If, in the presence of hedging, there is higher price sensitivity to changes in information and liquidity, we will be able to conclude that hedging amplifies price reactions. Also, if, with hedging, the equilibrium price becomes a concave function of the underlying parameters, this will imply that hedging creates a price asymmetry biased towards negative deviations.

First we examine the sensitivity of price. Taking partial derivatives of $P(\nu, l)$ with respect to ν and l yield

$$\frac{\partial P(\nu,l)}{\partial \nu} = (f^{-1})'\big(Q(\nu,l)\big)\left[\tfrac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma} + \tfrac{A^*}{a_i^*}\Big(1 - \tfrac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma}\Big)\right], \tag{4a}$$

$$\frac{\partial P(\nu,l)}{\partial l} = -(f^{-1})'\big(Q(\nu,l)\big)\left[a_i^* \tfrac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma} + A^*\Big(1 - \tfrac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma}\Big)\right]. \tag{4b}$$

Lemma B in Appendix C shows that $\frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma} \leq 1$. Therefore

$$\begin{aligned}
\mathrm{sign}\Big(\frac{\partial P(\nu,l)}{\partial \nu}\Big) &= \mathrm{sign}\Big((f^{-1})'\Big(Q(\nu,l)\Big),\\
\mathrm{sign}\Big(\frac{\partial P(\nu,l)}{\partial l}\Big) &= -\mathrm{sign}\Big((f^{-1})'\Big(Q(\nu,l)\Big)\Big).
\end{aligned}$$

Consider the case with no hedgers in the market, i.e. the case with $h \equiv 0$. Then $(f^{-1})' \equiv 1$ and

$$\mathrm{sign}\Big(\frac{\partial P(\nu,l)}{\partial l}\Big|_{h\equiv 0}\Big) < 0 < \mathrm{sign}\Big(\frac{\partial P(\nu,l)}{\partial \nu}\Big|_{h\equiv 0}\Big).$$

This means when there are no hedgers in the market the equilibrium price is a strictly increasing function of ν and a strictly decreasing function of l. This is naturally plausible since security prices tend to increase in the presence of good news and they tend to fall when liquidity of the security increases. Theoretically, presence of hedgers may pervert this observed characteristic of security prices, that is, prices may fall with good news and increase with liquidity supply. The following lemma provides the necessary and sufficient condition for hedging to lead to price reactions in accord with reality.

Lemma 1 *Let f^{-1} be differentiable. Then $P(\nu, l)$ is strictly increasing in ν and strictly decreasing in l if and only if*

(S1$'$) $I + A^*h$ *is strictly increasing.*

The proof simply follows from (4a)-(4b) and the fact that $(f^{-1})'(y) = \frac{1}{1+A^*h'(x)}$, given $y = f(x)$. Also, note that S1 necessarily holds when S1$'$ holds.

To further our analysis, we incorporate the size of hedging activity as a parameter into the hedging supply function by letting

$$h(p) = \alpha\Pi(p), \quad \forall p.$$

In the expression above, Π is a decreasing function of p, and α denotes the fraction of assets protected by hedging (portfolio insurance). We now have the following proposition:

Proposition 2 (Amplification) *Assume that S1$'$ holds and f^{-1} is differentiable. As the fraction α of assets protected by hedging increases, the equilibrium price function becomes more sensitive to changes in the information parameter ν and the liquidity parameter l. That is, $\left|\frac{\partial P(\nu,l)}{\partial \nu}\right|$ and $\left|\frac{\partial P(\nu,l)}{\partial l}\right|$ are increasing functions of α.*

Proposition 2 reveals the *amplifying effect* of hedging activity on price movements. The intuition is easy to see: once the price begins to fall (due to bad news or increasing liquidity supply), there will be more hedging supply of the security which will further push the prices to much lower levels. So, in the presence of hedging, one will see amplified price reactions to the triggering events (such as bad news or higher liquidity). Naturally, the bigger the size of hedging activity is, the larger the price reactions will be. Of course, the same argument works for the price hikes.

Next we analyze the second characteristic of large price movements: the *asymmetry* in favor of crashes. To that end, we need to check the concavity of equilibrium price with respect to the parameters ν and l (see §2.3). In the case of twice-differentiable price functions, concavity is determined by the second-order partial derivatives:

$$\frac{\partial^2 P(\nu,l)}{\partial \nu^2} = (f^{-1})''\big(Q(\nu,l)\big)\Big\{\frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma} + \frac{A^*}{a_i^*}\big(1 - \frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma}\big)\Big\}^2, \tag{5a}$$

$$\frac{\partial^2 P(\nu,l)}{\partial l^2} = (f^{-1})''\big(Q(\nu,l)\big)\Big\{a_i^*\frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma} + A^*\big(1 - \frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma}\big)\Big\}^2. \tag{5b}$$

They lead us to the following result:

Proposition 3 (Asymmetry) *Assume that* $S1'$ *holds and* f^{-1} *is twice-differentiable. If hedging supply* h *is a strictly convex function within the set*

$$P\big(U_{\nu_0}, U_{l_0}\big) = \{p : p = P(\nu, l) \ \textit{ s.t. } \ (\nu, l) \in U_{\nu_0} \times U_{l_0}\},$$

then:

(a) information and liquidity shocks cause a bias towards negative price deviations within $U_{\nu_0} \times U_{l_0}$*, i.e.* $P(\nu, l)$ *is strictly concave in* ν *and strictly concave in* l *for* $(\nu, l) \in U_{\nu_0} \times U_{l_0}$*,*

(b) the bias becomes more significant within $U_{\nu_0} \times U_{l_0}$ *as the fraction* α *of assets protected by hedging increases, i.e.* $\frac{\partial^2 P(\nu,l)}{\partial \nu^2}$ *and* $\frac{\partial^2 P(\nu,l)}{\partial l^2}$ *are decreasing functions of* α *within* $U_{\nu_0} \times U_{l_0}$*.*

It is easy to check that even if S1$'$ does not hold, the results above (on amplification and asymmetry) will hold within the domain

$$\big\{(\nu, l) : (I + A^* h)'\big(P(\nu, l)\big) > 0\big\}.$$

To sum up, under plausible conditions, whenever a shock (either of informational nature or liquidity based) occurs in the economy, the deviation in price is amplified due to hedging. Hence with hedging, the deviations are more likely to be significant, that is they are more likely to be a crash or a frenzy. Moreover if the hedging function is (globally) strictly convex, then a bias towards negative deviations is observed. We can summarize these results as follows:

Corollary 1 (Main Result) *Assume that* $S1'$ *holds and* f^{-1} *is twice-differentiable. If hedging supply* h *is a strictly convex function, then there exists a bias towards crashes in the economy.*

One criticism towards the results of this section might be the extent of their dependence on hedging. After all, having lots of irrational agents, programmed to behave in ways to create amplification and asymmetry, would not be much of an explanation for the characteristics we are examining. Therefore we would like to show that our results do not stem from an imposed environment with a lot of irrational hedgers accompanied by just enough rational traders to equate supply and demand. The main difference between rational traders (insider, outsider) and hedgers is that their demands react differently to price deviations. That is, the demand of rational traders is a decreasing function of price whereas the demand of hedgers is increasing in price. So we can determine the dominance of a group (namely rational traders or hedgers) in the market by checking the sensitivity of their aggregate demand with respect to price.

Proposition 4 (Market Demand) *Let f^{-1} be differentiable. $S1'$ holds if and only if the aggregate demand Z of the risky security is strictly decreasing in p, where*

$$Z(p) = D_i(p|\nu) + D_o(p|\sigma) - h(p).$$

This proposition shows that demand of rational traders prevail over that of hedgers if and only if S1$'$ holds. Since we get the results of this section with practically one assumption, namely S1$'$, we can say that our results hold within an environment where rationality prevails.

4 Put-option replication

In this section, we examine a specific hedging (portfolio insurance) strategy: the put-option replication. Put-option replication was the most popular hedging strategy during 1980's, in particular, during the October '87 crash. The formula for the put-option replication is taken from Gennotte and Leland (1990):[14] The hedging strategy is assumed to be applied to a fraction α of risky securities. The incremental hedging supply when new price is p, relative to the supply at the hypothetical initial price ($p^* = 1$), is given by

$$\hat{h}(p) = \alpha\Big(\Phi(d(1)) - \Phi(d(p))\Big),$$

where $\Phi(.)$ is the standard cumulative normal distribution function, and $d(.)$ is derived from the Black-Scholes formula

$$d(p) = \frac{\ln\left(\frac{p}{K}\right) + \frac{1}{2}\mathrm{var}(X|\Sigma)}{\sqrt{\mathrm{var}(X|\Sigma)}}.$$

with K as the striking price for the option (or the protection level in the replication case).[15]

Possibility of negative security prices is a caveat of the CARA-Gaussian framework. Naturally we focus on strictly positive prices for the analysis of put-option replication. Note that

$$\hat{h}'(p) = -\frac{\alpha\phi(d(p))}{p\sqrt{\mathrm{var}(X|\Sigma)}},$$

[14]Gennotte and Leland (1990) point out the differences in their formula compared to Black and Scholes (1973). They assume that interest rate has been normalized to zero, and assume a one-year time horizon. Moreover in theirs payoff is normally distributed (as in our model), whereas in Black and Scholes (1973) payoff follows a log-normal process.

[15]In the actual Black-Scholes formula, we would have $d(p) = \Big(\ln\left(\frac{p}{K}\right) + \frac{1}{2}\mathrm{var}(X|P)\Big)\left(\sqrt{\mathrm{var}(X|P)}\right)^{-1}$. However as we elaborated before, observing P is equivalent to observing Σ (see Proposition 1).

where $\phi(.)$ is the standard normal density function. Clearly, $\hat{h}$ is decreasing in the domain of strictly positive prices. Extracting $\hat{h}'(p)$, we get

$$\hat{h}'(p) = -\frac{\alpha}{p}\frac{\exp\Big(-\frac{1}{2}\big(\frac{\ln\frac{p}{K}+\frac{1}{2}\mathrm{var}(X|\Sigma)}{\sqrt{\mathrm{var}(X|\Sigma)}}\big)^2\Big)}{\sqrt{2\pi\mathrm{var}(X|\Sigma)}}.$$

Now it is easy to see the following:

Lemma 2 *Given $p_0 > 0$, if $\alpha \leq \frac{p_0\sqrt{2\pi\mathrm{var}(X|\Sigma)}}{A^*}$, then $\hat{h}'(p) > -\frac{1}{A^*}$ for all $p \in [p_0, \infty)$. Moreover, as α tends to 0, the set $\{p : \hat{h}'(p) > -\frac{1}{A^*}\}$ will converge to the domain of strictly positive prices $(0, \infty)$.*

So if α is sufficiently small, S1$'$ holds for $\hat{h}$ over a strict subset of positive prices. It is easy to check that all our proofs will work over this strict subset. To be more precise, our results on amplification and asymmetry still hold over the domain

$$\{(\nu, l) : \hat{h}'\big(P(\nu, l)\big) > -\frac{1}{A^*}\},$$

and this domain converges to $\{(\nu, l) : P(\nu, l) > 0\}$ as α tends to 0.

For the convexity of $\hat{h}$, we need to check the second-order partial derivative:

$$\begin{aligned}\hat{h}''(p) &= -\frac{\alpha}{\sqrt{\mathrm{var}(X|\Sigma)}}\frac{\phi'(d(p))d'(p)p - \phi(d(p))}{p^2} \\ &= \frac{\alpha\phi(d(p))}{p^2\sqrt{\mathrm{var}(X|\Sigma)}}\Big(\frac{d(p)}{\sqrt{\mathrm{var}(X|\Sigma)}} + 1\Big) \\ &= \frac{\alpha\exp(-d(p)^2)}{p^2\sqrt{2\pi\mathrm{var}(X|\Sigma)}}\Big(\frac{\ln\big(\frac{p}{K}\big)}{\mathrm{var}(X|\Sigma)} + 2\Big).\end{aligned}$$

The following result can be easily proved using this equation:

Lemma 3 *$\hat{h}$ is strictly convex over the domain $\{p : p > \frac{K}{e^{2\mathrm{var}(X|\Sigma)}}\}$. As $\mathrm{var}(X|\Sigma)$ tends to ∞, the domain where $\hat{h}$ is strictly convex will converge to the set of strictly positive prices.*

Now using Lemma 2, Lemma 3, and our results from Section 3, we obtain the following:

Proposition 5 (Put-option Replication) *If the vector of information and liquidity parameters are in the domain $\{(\nu, l) : P(\nu, l) > \frac{K}{e^{2\mathrm{var}(X|\Sigma)}}\}$, α is less than $\frac{K\sqrt{2\pi\mathrm{var}(X|\Sigma)}}{A^* e^{2\mathrm{var}(X|\Sigma)}}$, and hedgers employ put-option replication $\hat{h}$ as the hedging strategy, then there exists a bias towards crashes in the economy, i.e.*

(a) $\left|\frac{\partial P(\nu,l)}{\partial \nu}\right|$ *and* $\left|\frac{\partial P(\nu,l)}{\partial l}\right|$ *are increasing functions of* α*,*

(b) $P(\nu, l)$ *is strictly concave in* ν *and strictly concave in* l*,*

(c) $\frac{\partial^2 P(\nu,l)}{\partial \nu^2}$ *and* $\frac{\partial^2 P(\nu,l)}{\partial l^2}$ *are decreasing functions of* α*.*

Moreover as $\mathrm{var}(X|\Sigma)$ *tends to* ∞*, the domain where the bias towards crashes is observed will converge to* $\big\{(\nu, l) : P(\nu, l) > 0\big\}$.

Proposition 5 reveals that for a large domain of positive security prices there would be a bias towards crashes when market is highly volatile and put-option replication is the hedging strategy.

Though put-option replication and other portfolio insurance strategies played an important role in modern times, it is hard to use the same argument for the first half of the century. The sophisticated portfolio insurance strategies did not even exist then. However there is a hedging strategy which has been in use arguably as long as stock markets existed: stop-loss. In its most primitive form, hedgers sell their risky securities when the price falls below a predetermined level, say K. Use of this primitive hedging form clearly creates the asymmetry we want: there is an additional downward pressure on sales once price falls below K whereas there is no pressure when market goes up. Hence we get an asymmetry biased towards crashes with stop-loss as well.

5 A numerical example: back to 80's

The levels of risk aversion, hedging and market volatility necessary for substantial sizes of amplification and asymmetry are, of course, matters of concern. In other words, we do not want to generate amplification and asymmetry using implausible values for the parameters of our model. So we examine the following numerical example:

Let us take put-option replication, the most popular portfolio insurance strategy of 80's, as the hedging function. We assume α to be 0.05, which is not far from the hedging size in the '87 crash. The protection level K is assumed to be 85 percent of initial price. Let us fix the initial equilibrium price to be 1 so that K becomes 0.85. Assuming an expected 6 percent return on the risky security compared to a risk-free asset is reasonable for U.S. markets, thus we let $\mathrm{E}[X] = 1.06$. Outsider is assumed to be more risk averse than insider by letting $a_i = 0.70$

and $a_o = 1.40$. Take $\mathrm{var}(X|S)$, $\mathrm{var}(X|\Sigma)$ and $\frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma}$ to be 200, 400 and 0.5, respectively.[16] Note that these values illustrate the informational advantage of insider through $\frac{\mathrm{var}(X|\Sigma)}{\mathrm{var}(X|S)} = 2$. We assume l to be 0 as liquidity supply is not biased.

Then to create a 20 percent price deviation in the negative direction it takes a 2.9 percent fall in the insider's expectation on risky payoff (ν) whereas a positive price deviation of the same magnitude requires a 8.5 percent increase in ν. This example clearly depicts the asymmetry.

Moreover if there were no hedging in the market, a 20 percent price movement in any direction would require a 18.1 percent change in the information parameter ν. Clearly in the case with put-option replication, price is more sensitive to the parameter changes, which illustrates the amplification brought by hedging.

6 Roles of risk aversion and asymmetric information

Lee (1998) makes the following conjecture in the conclusion of his paper: "Under risk aversion it is more difficult to trigger a frenzy than a crash because a surprise of the same degree in the direction of the good state induces a smaller response than the one in the direction of the bad state." Granted Lee's model exploits a totally different mechanism, his conjecture actually pinpoints the role of risk aversion in our analysis. The following proposition demonstrates this:

Proposition 6 *Assume $S1'$. As insider or outsider tends to be risk neutral, asymmetry vanishes in the equilibrium price deviations.*

Proposition 6 simply follows from Proposition 1 and equations (3a)-(3b): f^{-1} converges to the identity function as either of the risk aversion parameters a_i or a_o converges to 0, which then implies $P(\nu, l)$ converging to a linear function. If the equilibrium price converges to a linear function, it simply means that asymmetry in price deviations vanishes.

This proposition is in accordance with Chalkley and Lee (1998) and Veronesi (1999). Both papers emphasize convexifying effect of risk aversion on price reactions to changes in underlying parameters. In our model, risk aversion allows hedging to be incorporated to the price function. If traders are risk neutral, then hedging does not affect price function at all, which consequently means there are no asymmetric deviations.

[16] $\frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma}$ always takes values between 0 and 1. See Lemma B (C1) in Appendix C.

Having elaborated on the role played by risk aversion in our analysis, next we discuss the role of asymmetric information. For convenience, we first define a measure for the level of asymmetry regarding information. Notice that the ratio $\frac{\mathrm{var}(X|\Sigma)}{\mathrm{var}(X|S)}$ gives the imprecision of the information of outsider relative to that of the insider, i.e. given the gaussian nature of our framework this ratio delivers insider's informational advantage over outsider. So we let

$$\mu := \frac{\mathrm{var}(X|\Sigma)}{\mathrm{var}(X|S)},$$

and call the ratio μ, $\mu > 1$, the *measure of asymmetric information*.[17] The bigger the measure μ gets, the larger the asymmetry between insider and outsider is. Now we can easily see how asymmetric information affects our analysis:

Proposition 7 (Asymmetric Information) *Assume $S1'$ and that $h'(.) < -\frac{1}{a_i^*}$. Also suppose that f^{-1} is continuously twice-differentiable and hedging supply h is strictly convex. There exits $\bar{\mu} > 1$ such that within the domain $(\bar{\mu}, \infty)$ of the asymmetric information measure μ*
(a) the equilibrium price function becomes more sensitive to changes in the information parameter ν and the liquidity parameter l as μ increases; i.e. $\left|\frac{\partial P(\nu,l)}{\partial \nu}\right|$ and $\left|\frac{\partial P(\nu,l)}{\partial l}\right|$ are increasing functions of μ,
(b) the bias towards negative deviations becomes more significant as μ increases; that is, $\frac{\partial^2 P(\nu,l)}{\partial \nu^2}$ and $\frac{\partial^2 P(\nu,l)}{\partial l^2}$ are decreasing functions of μ.

The only new assumption in this proposition, which has not been employed before, is

$$h'(.) < -\frac{1}{a_i^*} \equiv -\frac{1}{a_i \mathrm{var}(X|S)},$$

and this may be justified if the information of insider is sufficiently imprecise (i.e. if $\mathrm{var}(X|S)$ is sufficiently large). The proposition states that, with sufficiently large asymmetry between insider and outsider in terms of information owned, both amplification and asymmetry of price deviations will be more significant as the measure of asymmetric information μ increases. Hence asymmetric information certainly helps our cause.

Of course, one can still question the necessity of asymmetric information in our analysis. After all, risk aversion and hedging strategies are sufficient ingredients to create asymmetry in price deviations. That is, our analysis will go through without making use of asymmetric

[17] Since outsider's information is more imprecise compared to that of insider's, the measure of asymmetric information $\mu \equiv \frac{\mathrm{var}(X|\Sigma)}{\mathrm{var}(X|S)}$ is always strictly greater than 1.

information at all. However, such analysis will be hard to justify when it comes to numerical computations. For instance, in Section 5 we are able to generate significant amplification and asymmetry with risk aversion coefficients $a_i = 0.7$ and $a_o = 1.4$. Without asymmetric information, the same effect would require implausibly high risk aversion coefficients for CARA utility traders.

7 Trading behavior in the presence of hedgers

All previous sections have dealt with the effects of hedging on equilibrium price. Now we would like to analyze the effect of hedging on rational agents' trading behavior. Recall that equilibrium demand function of a rational trader is of the form

$$D_j(P(\nu,l)|I_j) = \frac{\mathrm{E}[X|I_j] - P(\nu,l)}{a_j \mathrm{var}(X|I_j)}, \tag{6}$$

where I_j stands for the Gaussian information of agent $j = i, o$. We can partition the rational demand into the *information effect* $\frac{\mathrm{E}[X|I_j]}{a_j \mathrm{var}(X|I_j)}$, and the *substitution effect* $-\frac{P(\nu,l)}{a_j \mathrm{var}(X|I_j)}$. The overcoming effect among these two determines the direction of the rational demand reaction whenever price deviates.

Clearly, portfolio allocation of a rational trader would be different depending on whether there are hedgers in the market or not, because the price is affected by the presence of hedgers. However we would like to analyze a more significant impact of hedging on the trading behavior. In particular, we want to see whether a rational trader ever changes the direction of his reaction to the information and liquidity shocks. The following proposition shows that he does indeed.

Proposition 8 (Trading Behavior) *Assume S*$1'$ *and that* f^{-1} *is differentiable. Then the following hold:*

(a) $D_o(P(\nu,l)|\sigma)$ *is decreasing in* ν *and increasing in* l.

(b) $D_i(P(\nu,l)|\nu)$ *is increasing in* l.

(c) If the fraction α *of assets protected by hedging is sufficiently small, then* $D_i(P(\nu,l)|\nu)$ *is increasing in* ν*. If* α *is sufficiently large, then* $D_i(P(\nu,l)|\nu)$ *is decreasing in* ν.

Part (c) of Proposition 8 depicts the significant impact of hedging that we were after. It is easy to see from the proof that substitution and information effects move in different directions with respect to the changes in the information parameter. To be more specific, information effect is an increasing function of ν, and substitution effect is decreasing in ν. Without hedging activity insider demands more of the risky security when good news arrive; that is, information effect overcomes the substitution effect. In the presence of hedgers, this is not necessarily the case. If the size of hedging is large enough, the price (hence the substitution effect) is amplified excessively by hedgers, virtually cancelling the information effect. Then insider's demand decreases when good news arrive. This is certainly a significant change for insider's trading behavior since he changes the direction of his demand reaction to information shocks.

On the other hand, we do not see hedging affecting outsider's trading behavior to the same extent it affects insider's. In particular, the direction of outsider's demand reaction to information and liquidity shocks does not differ with or without hedgers in the market. However the way outsider reacts to information shocks is interesting. Part (a) of Proposition 8 shows that whenever good news come (i.e. when insider's expectation about the risky security increases) outsider decreases her demand of the risky security regardless of the size of hedging activity. This might seem puzzling at first, because conventionally we would expect increasing demand following good news. The reason is actually the noise created by liquidity traders. When good news come the price increases, but outsider is not sure whether it is the good news or liquidity demand that increases the price. Therefore although her expectation on the risky security return increases, the price increase overcomes this effect due to the risk premium associated with the liquidity trading. This translates into substitution effect overcoming information effect in outsider's demand.

Finally, from part (b) of Proposition 8 we see that hedging does not change the direction of insider's demand reaction to liquidity shocks.

8 Discussion on the emergence of hedging

In this section we investigate the emergence of hedging strategies in the stock market. First let us verify that hedging strategy is not optimal for an outsider to employ. We know that hedging demand (i.e, negative hedging supply) is an increasing function of price. If the trigger for the price hike is an increase in the information parameter ν, following Proposition 6.1, outsider's demand decreases. If the trigger for the hike is a decrease in the liquidity supply l, outsider's demand again decreases. So whatever the origin of shock is, we always see hedging and rational demand dictating opposite directions in portfolio allocation. Therefore, hedging strategy is clearly sub-optimal for the outsider, that is it will create a significant ex-ante utility cost (given the observation of price) compared to employing the rational demand schedule. So why do people employ hedging strategies after all?

Ex-post, hedgers might be better off compared to outsiders in the case of information shocks. The reason is that both hedging demand and insider's rational demand are in the same direction (and both are opposite to outsider's direction of demand) when the size of hedging is sufficiently small. Since the insider has the privileged information, it is quite likely that insider is better off compared to outsider (however we cannot say this with certainty as insider's signal is noisy). Hence the hedger is also quite likely to be better off compared to outsider after an information shock. In the case of liquidity shocks, hedger's demand is opposite in direction to both insider and outsider. So if overwhelmingly information shocks trigger price deviations, employing hedging strategies might prove to be ex-post profitable due to the argument above. Of course, this explanation is far from a rigorous treatment of the matter.

Another interesting point is that whenever the size of hedging is sufficiently large, the hedger's demand is in the opposite direction of both insider and outsider. So a possibly winning strategy for one person will be an almost certainly losing strategy when many people employ it.

Appendix A: mathematical preliminaries

A1 Projection theorem. For jointly normally distributed random variables X and Θ, the following hold:

$$\begin{aligned} \mathrm{E}[X|\Theta=\theta] &= \mathrm{E}[X] + \frac{\mathrm{cov}(X,\Theta)}{\mathrm{var}\Theta}(\theta - \mathrm{E}\Theta), \\ \mathrm{var}(X|\Theta) &= \mathrm{var}(X) - \frac{(\mathrm{cov}(X,\Theta))^2}{\mathrm{var}\Theta}. \end{aligned}$$

A2 Rao's formula. For a normal random variable X, the following holds:

$$\mathrm{E}[\mathrm{e}^X] = \mathrm{e}^{\left(\mathrm{E}X + \frac{\mathrm{var}X}{2}\right)}.$$

Appendix B: derivations

B1 Derivation of excess demand functions. Since X is normal, W_j is also normal for $j = i, o$. By Rao's formula (A2) we have

$$\mathrm{E}[u^j(W_j)|I_j] = -\exp\left(-a_j D_j \mathrm{E}[X|I_j] - a_j(e_j - pD_j) + a_j^2 D_j^2 \frac{\mathrm{var}(X|I_j)}{2}\right).$$

Agent $j \in \{i, o\}$ solves the maximization problem, given in (1). The solution to this problem is

$$D_j(p) = \frac{\mathrm{E}[X|I_j] - p}{a_j \mathrm{var}(X|I_j)}.$$

B2 Derivation of (3b). Recall that $\sigma = \nu - a_i^* l$. Projection theorem (A1) implies

$$\mathrm{E}[X|\sigma] = \mathrm{E}X + \frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma}(\nu - a_i^* l - \mathrm{E}X).$$

Therefore

$$\begin{aligned} \mathrm{E}[X|\sigma] + \frac{A^*}{a_i^*}(\sigma - \mathrm{E}[X|\sigma]) &= \left\{\frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma} + \frac{A^*}{a_i^*}(1 - \frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma})\right\}\nu \\ &\quad -\left\{\frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma} + A^*(1 - \frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma})\right\}l \\ &\quad -\frac{A^*}{a_o^*}\left\{1 - \frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma}\right\}\mathrm{E}X. \end{aligned}$$

B3 Derivatives of the function $\mathbf{f^{-1}}$. Let $y = f(x)$. Then as $f \equiv I + A^* h$, we have

$$(f^{-1})'(y) = \frac{1}{1 + A^* h'(x)}, \quad (f^{-1})''(y) = -\frac{A^* h''(x)}{\left(1 + A^* h'(x)\right)^3}.$$

Appendix C: proofs

C1 Lemma B. $\frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma} \le 1$.

Proof: Note that

$$\begin{aligned} \mathrm{cov}(X,\Sigma) &= \mathrm{cov}\Big(X, \mathrm{E}[X|S] - a_i \mathrm{var}(X|S)L\Big) = \mathrm{cov}(X, \mathrm{E}[X,S]) \\ &= \mathrm{cov}\Big(X, \mathrm{E}X + \frac{\mathrm{cov}(X,S)}{\mathrm{var}S}(S - \mathrm{E}X)\Big) = \frac{(\mathrm{cov}(X,S))^2}{\mathrm{var}S}, \end{aligned}$$

$$\begin{aligned} \mathrm{var}\Sigma &= \mathrm{var}(\mathrm{E}[X|S]) + {a_i}^2(\mathrm{var}(X|S))^2 \mathrm{var}L \\ &= \mathrm{var}\Big(\mathrm{E}X + \frac{\mathrm{cov}(X,S)}{\mathrm{var}S}(S - \mathrm{E}X)\Big) + {a_i}^2(\mathrm{var}(X|S))^2 \mathrm{var}L \\ &= \frac{(\mathrm{cov}(X,S))^2}{\mathrm{var}S} + {a_i}^2(\mathrm{var}(X|S))^2 \mathrm{var}L. \end{aligned}$$

Hence the result follows. $\square$

C2 Proof of Proposition 2. Note that $h' = \alpha\Pi' > -\frac{1}{A^*}$ due to S1$'$. We also know from (C1) that $\frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma} \le 1$. Thus from (4a)-(4b) and (B3), given $p = P(\nu, l)$, one has

$$\begin{aligned} \Big|\frac{\partial P(\nu,l)}{\partial \nu}\Big| &= \tfrac{1}{1+\alpha A^*\Pi'(p)}\Big\{\tfrac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma} + \tfrac{A^*}{a_i^*}\Big(1 - \tfrac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma}\Big)\Big\}, \\ \Big|\frac{\partial P(\nu,l)}{\partial l}\Big| &= \tfrac{1}{1+\alpha A^*\Pi'(p)}\Big\{a_i^*\tfrac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma} + A^*\Big(1 - \tfrac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma}\Big)\Big\}. \end{aligned}$$

Since Π is a decreasing function, it is straightforward to see that $\big|\frac{\partial P(\nu,l)}{\partial\nu}\big|$ and $\big|\frac{\partial P(\nu,l)}{\partial l}\big|$ are increasing functions of α. $\square$

C3 Proof of Proposition 3. Due to S1$'$, $h' > -\frac{1}{A^*}$. Hence from (B3) it follows that if h is strictly convex within the set $P(U_{\nu_0}, U_{l_0})$, f^{-1} is strictly concave within $P(U_{\nu_0}, U_{l_0})$, consequently $P(\nu, l)$ is strictly concave in ν and strictly concave in l within $U_{\nu_0} \times U_{l_0}$. This proves (a).

From (5a)-(5b) and (B3), given $p = P(\nu, l)$ we have

$$\begin{aligned} \frac{\partial^2 P(\nu,l)}{\partial \nu^2} &= -\frac{\alpha A^*\Pi''(p)}{\big(1+\alpha A^*\Pi'(p)\big)^3}\Big\{\frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma} + \frac{A^*}{a_i^*}\Big(1 - \frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma}\Big)\Big\}^2, \\ \frac{\partial^2 P(\nu,l)}{\partial l^2} &= -\frac{\alpha A^*\Pi''(p)}{\big(1+\alpha A^*\Pi'(p)\big)^3}\Big\{a_i^*\frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma} + A^*\Big(1 - \frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma}\Big)\Big\}^2. \end{aligned}$$

Recall that Π is a strictly decreasing function. If h (and hence Π) is strictly convex in $P\big(U_{\nu_0}, U_{l_0}\big)$, one has $\Pi''(p) > 0$ for $p \in P\big(U_{\nu_0}, U_{l_0}\big)$, thus $\frac{\partial^2 P(\nu,l)}{\partial \nu^2}$ and $\frac{\partial^2 P(\nu,l)}{\partial l^2}$ are decreasing functions of α for $(\nu, l) \in U_{\nu_0} \times U_{l_0}$. Hence (b) is proved. $\Box$

C4 Proof of Proposition 4. We have $Z(p) = -\frac{p}{A^*} - h(p) + \frac{\mathrm{E}[X|s]}{a_i \mathrm{var}(X|S)} + \frac{\mathrm{E}[X|\sigma]}{a_o \mathrm{var}(X|\Sigma)}$ and $Z'(p) = -\frac{1}{A^*} - h'(p)$. Therefore S1$'$ holds if and only if Z is strictly decreasing in p. $\Box$

C5 Proof of Proposition 7. First, note that the assumptions employed in the statement of the proposition impose $-\frac{1}{A^*} < h'(.) < -\frac{1}{a_i^*}$.Recall from (C2) and (C3) that we have the following for $p = P(\nu, l)$:

$$
\begin{aligned}
\Big|\frac{\partial P(\nu,l)}{\partial \nu}\Big| &= \frac{1}{1 + A^* h'(p)} \Big\{ \frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma} + \frac{A^*}{a_i^*}\Big(1 - \frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma}\Big) \Big\}, \\
\Big|\frac{\partial P(\nu,l)}{\partial l}\Big| &= \frac{1}{1 + A^* h'(p)} \Big\{ a_i^* \frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma} + A^*\Big(1 - \frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma}\Big) \Big\}; \\
\frac{\partial^2 P(\nu,l)}{\partial \nu^2} &= -\frac{A^* h''(p)}{\big(1 + A^* h'(p)\big)^3} \Big\{ \frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma} + \frac{A^*}{a_i^*}\Big(1 - \frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma}\Big) \Big\}^2, \\
\frac{\partial^2 P(\nu,l)}{\partial l^2} &= -\frac{A^* h''(p)}{\big(1 + A^* h'(p)\big)^3} \Big\{ a_i^* \frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma} + A^*\Big(1 - \frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma}\Big) \Big\}^2.
\end{aligned}
$$

Observe that as $\mu \equiv \frac{\mathrm{var}(X|\Sigma)}{\mathrm{var}(X|S)} \to -\frac{1}{a_o \mathrm{var}(X|S)\left(\frac{1}{a_i^*} + h'(p)\right)}$ we have $1 + A^* h'(p)$ tending to 0, and consequently

$$
\Big|\frac{\partial P(\nu,l)}{\partial \nu}\Big|, \ \Big|\frac{\partial P(\nu,l)}{\partial l}\Big| \to \infty, \quad \frac{\partial^2 P(\nu,l)}{\partial \nu^2}, \ \frac{\partial^2 P(\nu,l)}{\partial l^2} \to -\infty.
$$

However, we need to check that $-\frac{1}{a_o \mathrm{var}(X|S)\left(\frac{1}{a_i^*} + h'(p)\right)} > 1$ as $\mathrm{var}(X|\Sigma) > \mathrm{var}(X|S)$. Suppose not: using the assumption that $h'(.) < -\frac{1}{a_i^*}$, we get

$$
-a_o \mathrm{var}(X|S)\left(\frac{1}{a_i^*} + h'(p)\right) > 1 \implies h'(p) < -\frac{1}{a_o \mathrm{var}(X|S)} - \frac{1}{a_i^*} < -\frac{1}{a_o \mathrm{var}(X|\Sigma)} - \frac{1}{a_i^*} = -\frac{1}{A^*},
$$

which violates another assumption, namely $h'(.) > -\frac{1}{A}$. Thus $-\frac{1}{a_o \mathrm{var}(X|S)\left(\frac{1}{a_i^*} + h'(p)\right)} > 1$.

Following the limit results derived above and the fact that $P(\nu, l)$ is continuously twice-differentiable, there exits $\bar{\mu} > 1$ such that within the domain $(\bar{\mu}, \infty)$: $\big|\frac{\partial P(\nu,l)}{\partial \nu}\big|$, $\big|\frac{\partial P(\nu,l)}{\partial l}\big|$ are increasing in μ and $\frac{\partial^2 P(\nu,l)}{\partial \nu^2}$, $\frac{\partial^2 P(\nu,l)}{\partial l^2}$ are decreasing in μ. $\Box$

C6 Proof of Proposition 8.

(a) From the extraction of $\mathrm{E}[X|\sigma]$ it follows that $\frac{\partial \mathrm{E}[X|\sigma]}{\partial \nu} = \frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma}$ and $\frac{\partial \mathrm{E}[X|\sigma]}{\partial l} = -{a_i}^* \frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma}$. Recall from (4a)-(4b) that

$$\begin{aligned}\frac{\partial P(\nu,l)}{\partial \nu} &= (f^{-1})'\big(Q(\nu,l)\big)\Big\{\frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma} + \frac{A^*}{a_i^*}\Big(1-\frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma}\Big)\Big\},\\ \frac{\partial P(\nu,l)}{\partial l} &= -(f^{-1})'\big(Q(\nu,l)\big)\Big\{a_i^*\frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma} + A^*\Big(1-\frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma}\Big)\Big\}.\end{aligned}$$

Following Lemma B (C1), $\frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma} + \frac{A^*}{a_i^*}\Big(1-\frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma}\Big) \geq \frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma}$. Moreover h is a strictly decreasing function, hence $h'(.) < 1$. Then it follows from (B3) that

$$(f^{-1})'\big(Q(\nu,l)\big) = \frac{1}{1 + A^* h'\big(P(\nu,l)\big)} \geq 1$$

under S1$'$. So $\frac{\partial P(\nu,l)}{\partial \nu} \geq \frac{\partial \mathrm{E}[X|\sigma]}{\partial \nu}$ and $\frac{\partial P(\nu,l)}{\partial l} \leq \frac{\partial \mathrm{E}[X|\sigma]}{\partial l}$. Therefore $D_o(P(\nu,l)|\sigma)$ is decreasing in ν and increasing in l from (6).

(b) We have $\frac{\partial \mathrm{E}[X|s]}{\partial l} = \frac{\partial \nu}{\partial l} = 0$. On the other hand, $\frac{\partial P(\nu,l)}{\partial l} < 0$ due to S1$'$ (see Lemma 1). Thus from (6) one observes that $D_i(P(\nu,l)|\nu)$ is increasing in l.

(c) We have $\frac{\partial \mathrm{E}[X|s]}{\partial \nu} = \frac{\partial \nu}{\partial \nu} = 1$. Recall from (4a) that

$$\frac{\partial P}{\partial \nu} = (f^{-1})'\big(Q(\nu,l)\big)\Big\{\frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma} + \frac{A^*}{a_i^*}\Big(1-\frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma}\Big)\Big\}.$$

First of all; $\frac{A^*}{a_i^*} = \frac{a_o^*}{a_i^*+a_o^*} \leq 1$, and $\frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma} \leq 1$ from Lemma B (C1). So

$$\frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma} + \frac{A^*}{a_i^*}\Big(1-\frac{\mathrm{cov}(X,\Sigma)}{\mathrm{var}\Sigma}\Big) \leq 1.$$

On the other hand, we have shown in part (a) that $(f^{-1})'\big(Q(\nu,l)\big) \geq 1$. Therefore $\frac{\partial P(\nu,l)}{\partial \nu}$ can be greater or less than 1 depending on the exact value of $(f^{-1})'\big(Q(\nu,l)\big)$. In particular, following from (B3),

$$(f^{-1})'\big(Q(\nu,l)\big) \rightarrow \begin{cases} 1 & \text{as} \quad \alpha \rightarrow 0 \\ \infty & \text{as} \quad \alpha \rightarrow -\frac{1}{A^*\Pi'\big(f^{-1}\big(Q(\nu,l)\big)\big)}. \end{cases}$$

Note that under S1$'$, α cannot take values larger than $-\frac{1}{A^*\Pi'\big(f^{-1}(.)\big)}$, and also note that $-\frac{1}{A^*\Pi'\big(f^{-1}(.)\big)} \geq 0$ as Π is a decreasing function. Therefore $D_i(P(\nu,l)|\nu)$ is increasing in ν for sufficiently small α, and it is decreasing in ν for sufficiently large α. $\square$

References

[1] Admati, A.R.: A noisy rational expectations equilibrium for multi-asset securities markets, Econometrica **53**, 629-657 (1985)

[2] Boldrin, M., Levine, D.K.: Growth cycles and market crashes, Journal of Economic Theory **96**, 13-39 (2001)

[3] Brady, N. et al.: Report of the presidential task force on market mechanisms. Washington: U.S. Government Printing Office 1988

[4] Bronshtein, I.N., Semendyayev, K.A.: Handbook of mathematics. English translation edited by K.A. Hirsch, reprint of 3rd edition. Berlin Heidelberg: Springer-Verlag 1998

[5] Brunnermeier, M.K.: Asset pricing under asymmetric information: bubbles, crashes, technical analysis and herding. Oxford New York: University Press 2001

[6] Bulow, J., Klemperer P.: Rational frenzies and crashes, The Journal of Political Economy **102**, 1-23 (1994)

[7] Chalkley, M., Lee, I.H.: Learning and asymmetric business cycles, Review of Economic Dynamics **1**, 623-645 (1998).

[8] Cutler, D.M., Poterba, J.M., Summers, L.H.: What moves stock prices?, Journal of Portfolio Management **15**, 4-12 (1989)

[9] Demange, G., Laroque, G.: Private information and design of securities, Journal of Economic Theory **65**, 233-257 (1995)

[10] Gennotte, G., Leland, H.: Market liquidity, hedging, and crashes, American Economic Review **80**, 999-1021 (1990)

[11] Glosten, L.R., Milgrom, P.R.: Bid, ask and transaction prices in a specialist market with heterogeneously informed traders, Journal of Financial Economics **14**, 71-100 (1985)

[12] Grossman, S.J.: An analysis of the implications for stock and futures price volatility of program trading and dynamic hedging strategies, Journal of Business **61**, 275-298 (1988)

[13] Grossman, S.J., Stiglitz, J.E.: On the impossibility of informationally efficient markets, American Economic Review **70**, 393-408 (1980)

[14] Hellwig, M.: On the aggregation of information in competitive markets, Journal of Economic Theory **22**, 477-498 (1980)

[15] Hong, H., Stein, J.C.: Differences of opinion, short-sales constraints and market crashes, Review of Financial Studies **16**, 487-525 (2003)

[16] Jacklin, C.J., Kleidon, A.W., Pfleiderer, P.: Underestimation of portfolio insurance and the crash of October 1987, Review of Financial Studies **5**, 35-63 (1992)

[17] Lee, I.H.: Market crashes and informational avalanches, Review of Economic Studies **65**, 741-760 (1998)

[18] Ozsoylev, H.N.: Asymmetric information and financial markets. Ph.D. Thesis, University of Minnesota 2004

[19] Rubinstein, M., Leland, H.: Replicating options with positions in stock and cash, Financial Analysts Journal, July-August, 63-72 (1981)

[20] Shiller, R.J.: Market volatility. Cambridge: MIT Press 1989

[21] Shiller, R.J.: Irrational exuberance. Princeton: Princeton University Press 2000

[22] Veldkamp, L.L.: Slow boom, sudden crash, Journal of Economic Theory (in press)

[23] Veronesi, P.: Stock market overreaction to bad news in good times: a rational expectations equilibrium model, Review of Financial Studies **12**, 975-1007 (1999)

[24] Zeira, J.: Informational overshooting, booms and crashes, Journal of Monetary Economics **43**, 237-258 (1999)