A Necessary and Sufficient Condition for Convergence of Statistical to Strategic Equilibria of Market Games

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Abstract

In our model, we treat a market game where traders are heterogeneous not only with respect to their rationality level but also with the formation of their subjective beliefs for the strategy of their opponents. Under these conditions, the market mechanism results a statistical equilibrium, where traders randomise among their available actions due to their limited rationality. Here, we provide a necessary and sufficient condition for convergence of statistical to strategic equilibria of market games, when traders become more informed and increasingly rational.

Keywords: market games, bounded rationality, rational learning.

1 Introduction

This paper extends market games by incorporating traders with differential levels of rationality, as in Voliotis [8]. In particular, we derive a necessary and sufficient condition for the convergence of statistical equilibria of Voliotis [8] to the rational expectations equilibria of market games. The treatment of bounded rationality rests upon to McKelvey and Palfrey [5] and Chen, Friedman and Thisse [2]. According to this specification, traders choose their random strategy as a best response to their individual beliefs about their opponents' strategies. Therefore, their subjective optimal strategy is initially suboptimal since it depends upon beliefs that are inaccurate. As information

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is gradually revealed, their suboptimal strategy converges to the strategy associated with full information. The merging of beliefs to the full information strategy is modeled as in Blackwell and Dubins [1] and Kalai and Lehrer [3, 4], whereas the condition we provide is equivalent to the concept of relative entropy as introduced by Lehrer and Smorodinski [6].

The paper is organised as follows. In section 2, the model is presented. Section 3 formulates the model in a dynamic context, whereas Section 4 identifies a necessary and sufficient condition for convergence to strategic equilibria. Finally, section 5 offers some concluding remarks.

2 The model

Consider an exchange economy, consisting of a finite set of perishable commodities, $L = \{1, \dots, L\}$ and a finite set of traders $H = \{1, \dots, H\}$, with H sufficiently large. For each trader $h \in H$, let $X^h \subset \mathbb{R}^L$ be his consumption set. Also let trader h have an initial endowment ω^h, for all $l \in L$ such that $\omega_l^h >> 0$ and a utility function u^h which is assumed to be continuous, concave and strongly monotonic C^2. The economy is described by $e = (u^h, \omega^h)_{h \in H}$.

The game form. The strategic market games in contrast to other general equilibrium models require the complete specification of the way which trade is conducted. Several market mechanisms have been investigated, proposing a different way of bidding for the formation of market prices and consumption allocations. In most of the cases bidding is in terms of money, goods or prices, always depending on the suggested market mechanism. Here, without loss of generality we assume that the available bidding is given by the budget set of traders, B^h. The generic market mechanism will be described by a strategic outcome function that maps for any strategy profile $b \in \Pi_{h \in H} B^h$ the final consumption allocations. Formally, the game is defined by a list $(B^h, \phi^h)_{h \in H}$, where for all $h \in H$, B^h denotes the budget set of trader h, while $\phi^h : B \to X^h$ denotes the strategic outcome function of trader h.The payoff function will simply be the indirect utility function $u^h = u^h(x^h(b))$ whereas $x^h(b) \in X^h$ is given by the strategic outcome function.

Further, in this model we assume that traders are boundedly rational in the sense depicted in Chen *et.al.* [2] and Voliotis [8], i.e. they are characterised by a stochastic choice, which is ruled by their degree of rationality. In particular, the stochastic strategy of a trader h will be given by a probability measure σ^h, defined over the Borel field of B^h. In this specification, the traders randomise among their available bids in an interacting environment, hence their random strategy, according to the Luce rule, takes the form,

$$\sigma^h(b^h|\sigma^{-h}) = \frac{\upsilon^h(b^h, \sigma^{-h})}{\sum_{b^h \in B^h} \upsilon^h(b^h, \sigma^{-h})} \tag{1}$$

where $-h$ denotes the set $H\backslash\{h\}$, and v^h is the expected payoff, $v_h(b^h) = \sum \sigma^{-h}(b^{-h}) \cdot u^h(b^h, b^{-h})$. The above rule states that the choice probabilities of alternative strategies will be deduced as the relative size of the derived expected payoff. In addition, if we permit an idiosyncratic level of rationality for each trader, designated by a parameter μ, we define the *idiosyncratic expected utility* $f = f^h(v^h, \mu^h)$, as a monotonic transformation of v, strictly increasing to μ^h [1]

With respect to solution concepts

DEFINITION 2.1 *A strategy profile $b^* = (b^h)_{h \in H}$ is the strategic equilibrium of a Sell-All model (SAM), if and only if for every trader $h \in H$, there is no other strategy b^h such that,*

$$u^h(b^h, b^{-h*}) > u^h(b^*).$$

The existence of this equilibrium has been proved by a theorem in Shapley and Shubik [7]. Let us now define the optimality condition for the stochastic case, and then provide the associated equilibrium concept.

DEFINITION 2.2 *We say that the random strategy of trader h, $\sigma^h(b^h|\sigma_{-h}, \mu^h)$ is a best response to $\sigma^h{-}h$ if for any pair of pure actions $b^{h(\alpha)}, b^{h(\beta)} \in B^h$,*

$$f^h(b^{h(k)}) > f^h(b^{h(z)}) \Rightarrow \sigma_h(b^{h(k)}|\sigma^{-h}, \mu^h) > \sigma^h(b^{h(z)}|\sigma^{-h}, \mu^h) \qquad (2)$$

DEFINITION 2.3 *We say that the profile $\sigma_l = (\sigma_{1l}, \sigma_{2l}, \ldots, \sigma_{Hl})$ is a Strategic Market Game Quantal Response Equilibrium (SMGQRE) for the market of commodity l only if $\forall h \in H$ and $b_l^h \in B_l^h$,*

$$\sigma^{h*}(b_l^h|\sigma^h) = \frac{f^h(v^h(b_l^h, \sigma^{-h}), \mu^h)}{\sum_{b_l^h \in B_l^h} f^h(v^h(b_l^h, \sigma^{-h}), \mu^h)} \quad \text{is a best response.}$$

When this is true for all markets, then we have the SMGQRE of the game. [2]

3 Information learning in a repeated game

Let us consider the case that the game is played infinitely many times, with boundedly rational as well as myopic traders. This is not a standard repeated game of sequential decision making since our objective is to define an environment of subjective strategic behavior, in a dynamic adjustment process. By

[1]As μ increases, so does traders' degree of rationality. At the limit, i.e. $\mu = +\infty$, traders are perfectly rational, i.e. $\sigma^h(b^h|\sigma^{-h}) = 1$ for some $b^h \in B^h$. A transformation that works properly and increases non linearly to μ is the $f = v^\mu$.

[2]See Voliotis [8] for the existence proof.

subjective strategic behavior is understood that traders do not possess actual knowledge of others' random strategies but form subjective beliefs about them. In the dynamic context of the game, we assume that traders are able to observe the game realisations at each time and update their beliefs accordingly. It turns out that under certain conditions, the game will be cumulative in information for all traders. Put differently, the *beliefs* of a trader h will point-wise converge to the actual opponents' random strategy, i.e. $\lim_{t\to\infty} \tilde{\sigma}^{-h}_{h(t)} = \sigma^{-h}$. Since the last observation of random strategy is up-to-date, then the random strategy takes the form of a first order dependence,

$$\sigma^h(b^h|\sigma^{-h}) = \frac{f^h(v^h(b^h, \tilde{\sigma}^{-h}_{h,(t-1)}), \mu^h)}{\sum_{b^h\in B^h} f^h((v^h(b^h, \tilde{\sigma}^{-h}_{h,(t-1)}), \mu^h))} \tag{3}$$

Note that the information learning process is not explicitly defined, i.e. how beliefs $\tilde{\sigma}^{-h}_{h,(t-1)}, \forall t$ converge to σ^{-h}, and is merely assumed throughout.

4 A necessary and sufficient condition for convergence

Now we are ready to state a condition that guarantees the convergence to the strategic equilibria. In particular, we consider the logarithm of the ratio so as when the beliefs coincide with the true random strategy it becomes equal to zero.

DEFINITION 4.1 *Let $D : \Delta(B^h) \to \mathbb{R}_+$ defined to be the likelihood ratio* [3] *function,*

$$D^h = -\sum_{b^h\in B^h} \ln \frac{\sigma^h_{(t)}(b^h|\tilde{\sigma}^{-h}_{h,(t-1)})}{\sigma^h_{(t)}(b^h|\sigma^{-h})}, \quad \forall h \in H, \tag{4}$$

where $\Delta(B^h)$ is the unit simplex of the space of strategies of trader h, with the convention that when the denominator is zero, the function is defined to be zero.

THEOREM 4.1 *The SMGQRE will converge to the strategic equilibrium, if and only if,*

$$\lim_{\substack{\mu_t\to\infty \\ \tilde{\sigma}^{-h}_{h,(t)}\to\sigma^{-h}}} D^h = 0 \;\; \forall\, h \in H. \tag{5}$$

[3]We use the actual differential of the likelihood ratio of beliefs from the true random strategy

In order to prove theorem 4.1, we first provide some basic definitions and prove a necessary lemma. Let us first provide the notions of ϵ-closedness and of weak ϵ-closedness. Let $\mathbb{B}$ be the Borel field generated by B^h.

DEFINITION 4.2 *(Blackwell and Dubins [1]) Let $\tilde{\sigma}^{-h}$ and σ^{-h} be the beliefs of trader h and the actual random strategy for $-h$, respectively. For $\epsilon > 0$, we say that $\tilde{\sigma}^{-h}$ is ϵ-close to σ^{-h} if*

$$|\tilde{\sigma}^{-h}(A) - \sigma^{-h}(A)| < \epsilon \quad \forall A \in \mathbb{B}. \tag{6}$$

DEFINITION 4.3 *(Kalai and Lehrer [3]) Let $\tilde{\sigma}^{-h}$, $\sigma^{-h} \geq 1 - \epsilon$. For $\epsilon > 0$, we say that $\tilde{\sigma}^{-h}$ is weakly ϵ-close to σ^{-h} if*

$$(1-\epsilon)\tilde{\sigma}^{-h}(A) \leq \sigma^{-h}(A) \leq (1+\epsilon)\tilde{\sigma}^{-h}(A) \;\; \forall A \in \mathbb{B}. \tag{7}$$

DEFINITION 4.4 *A sequence $\{\tilde{\sigma}^{-h}_{h,(t)}, t \in N\}$, (weakly) pointwise converge to σ^{-h} if there is a period T such that for every $t > T$ the $\tilde{\sigma}^{-h}_{h,(t)}$ is (weakly) ϵ-close to σ^{-h}.*

We are now ready to prove the following lemma.

LEMMA 4.1 *Suppose that trader's beliefs weakly pointwise converge to the actual reaction function. Then, for every $\epsilon > 0$ there is a $t(\epsilon)$, given μ^h, such that,*

$$1 - \epsilon < \frac{\sigma^h_{(t(\epsilon))}(b^h|\tilde{\sigma}^{-h}_{(h,t(\epsilon)-1)}, \mu^h)}{\sigma^h(b^h|\sigma^{-h}, \mu^h)} < 1 + \epsilon, \;\; \forall h \in H. \tag{8}$$

Proof

Each trader in the game has a closed set of strategies B^h and hence the set of all strategy profiles B is closed as well. In addition, the set of strategy profiles B is bounded since for each element $b \in B$ there is M sufficiently large such that $0 \leq b < M$. We set out M to be equal to $\sum e^h_l + 1$. It follows that B is a compact set.

By assumption, the payoff u is continuous everywhere in B. Also, by compactness, it follows that it is also uniformly continuous. Since it also bounded in B, the expected utility $v^h_l(\sigma^h)$ is well defined.

When $\{\tilde{\sigma}^{-h}_{h,(t)}, t \in N\}$ weakly pointwise converge a.e. in σ^{-h} and u^h is uniformly continuous, i.e. $v^h_l(\tilde{\sigma}^{-h}_{h,t}) \to v^h_l(\sigma^{-h})$ for all h, as t tends to infinity.

Thus, for a fixed μ^h and for every $h \in H$,

$$\lim_{t\to\infty} f(v^h_l(\sigma^h_{(t)}), \mu^h) = f(v^h_l(\sigma^h), \mu^h).$$

Define now a random variable X_t be the normalised ratio,

$$X_t = \frac{f^h(v^h(\sigma^h, \tilde{\sigma}^{-h}_{h,(t-1)}), \mu^h)}{\sum_{\sigma^h \in B^h} f^h((v^h(\sigma^h, \tilde{\sigma}^{-h}_{h,(t-1)}), \mu^h))} \text{ and } X^* = \frac{f(v^h_l(\sigma), \mu^h)}{\sum_{b^h \in B^h} f(v^h_l(\sigma), \mu^h)}.$$

Then, X_t weakly converges almost everywhere to the random variable X^*. Equivalently,

$$\frac{X_t}{X^*} \rightarrow 1 \text{ as } t \text{ tends to infinity.}$$

However, by equation (3) this quotient is the ratio of beliefs to the actual random strategy.

$$\frac{X_t}{X^*} = \frac{\sigma^h(b^h|\tilde{\sigma}^{-h}_{h,(t-1)}, \mu^h)}{\sigma^h(b^h|\sigma^{-h}, \mu^h)} \rightarrow 1 \text{ as } t \text{ tends to infinity.}$$

Hence, for every $\epsilon > 0$ there is a $t(\epsilon)$ such that for all $t \geq t(\epsilon)$,

$$1 - \epsilon < \frac{\sigma^h_{t(\epsilon)}(b^h|\tilde{\sigma}^{-h}_{h,(t-1)}, \mu^h)}{\sigma^h(b^h|\sigma^{-h}, \mu^h)} < 1 + \epsilon.$$

$\square$.

Now we proceed proving theorem 4.1,

Proof (Proposition 4.1)

($\Rightarrow$)

Without loss of generality, let us depict the strategic equilibrium of rational expectations as a random strategy profile σ^*, assigning probability one to each pure strategy component b^{h*} of the equilibrium strategy profile b^*, and zero elsewhere. By theorem in Voliotis [8], for increasing values of μ^h_t, we attain the strategic equilibrium in the limit, i.e. $\lim_{t\rightarrow\infty} \sigma^h(b^h|\sigma^{-h}, \mu^h_t) = \sigma^{h*}$. Therefore, substituting into equation 5, we simply have to prove the following,

$$\lim_{t\rightarrow\infty} D(\sigma^h(b^h|\tilde{\sigma}^{-h}_{h,(t-1)}), \sigma^{h*}(b^h|\sigma^{-h})) = 0. \tag{9}$$

By lemma 4.1 there is a $t(\epsilon)$ for every $\epsilon > 0$ such that,

$$1 - \epsilon < \frac{\sigma^h_{t(\epsilon)}(b^h|\tilde{\sigma}^{-h}_{h,(t(\epsilon)-1)})}{\sigma^{h*}(b^h|\sigma^{-h})} < 1 + \epsilon$$

As t goes to infinity, ϵ will tend to zero, and therefore

$$\frac{\sigma^h_{t(\epsilon)}(b^h|\tilde{\sigma}^{h}_{h,(t(\epsilon)-1)})}{\sigma^{h*}(b^h|\sigma^{-h})} = 1 \tag{10}$$

ISubstituting equation 10 into equation 9 we obtain,

$$\begin{aligned}\lim_{t\to\infty} D &= \lim_{t\to\infty} - \sum_{b^h \in B^h} \ln \frac{\sigma_t^h(b^h|\tilde{\sigma}_{h,(t-1)}^{-h})}{\sigma^{h*}(b^h|\sigma^{-h})} \\ &= \lim_{t\to\infty} - \sum_{b^h \in B^h} \ln 1 \\ &= 0.\end{aligned}$$

($\Leftarrow$)

We need to show that if $\lim_{t\to\infty} D = 0$ then $\lim_{\mu_t^h\to\infty} \sigma_t^h(b^h|\tilde{\sigma}_{h,(t-1)}^{-h}) = \sigma^{h*}$. Suppose by contradiction that $\lim D = 0$ but the sequence $\tilde{\sigma}_t^h$ weakly point-wise converges to a different point from σ^{h*}, say to the point $\sigma^{h'}$, where $\sigma^{h'}$ is a non degenerate probability belonging to neighborhood of σ^{h*} . Since $\sigma_t^h(b^h|\tilde{\sigma}_{h,(t-1)}^{-h}) \to \sigma^{h'}$ by lemma 4.1 as t tends to infinity, we have $\frac{\sigma_t^h(b^h|\tilde{\sigma}_{h,(t-1)}^{-h})}{\sigma^{h'}} \to$ 1.

Respectively, for the σ^{h*} there will be a nonnegative k [4], different from unity, such that $\frac{\sigma_t^h(b^h|\tilde{\sigma}^{-h}(t-1))}{\sigma^{h*}} \to k$. Recalling that σ^{h*} is a degenerate probability distribution that ascribes to a pure strategy, say to $\overline{b}^h \in B^h$, probability one, if the $\tilde{\sigma}_t^h$ does not converge to σ^{h*}, apparently in the limit $\frac{\sigma^{h'}(\overline{b}^h)}{\sigma^{h*}(\overline{b}^h)} \neq 1$

Substituting this factor in equation 5, as t tends to infinity, we get,

$$\lim_{t\to\infty} D = \lim_{t\to\infty} \sum_{b^h \in B^h} \ln \frac{\sigma_t^h(b^h|\tilde{\sigma}_{h,(t-1)}^{-h})}{\sigma^{h*}(b^h|\sigma^{-h})} \neq 0$$

which contradicts to our hypothesis that $\lim_{t\to\infty} D = 0$ $\square$.

5 Concluding Remarks

1) We introduce with bounded rationality the general class of strategic market games . Bounded rationality does not conform with symmetric information, since traders act randomly and they do not have complete knowledge of their own preferences! Therefore, it is not obvious that a strategic equilibrium can always be reached by a SMGQR equilibrium, when traders learn to play rationally. This paper offers a necessary and sufficient condition for the traders to learn how to play the rational expectations strategic equilibrium.

2)The condition can be violated either by the existence of upper bounds to the rationality level of one or more traders or to a problematic information dissemination. In either case, the strategic equilibrium point will never be reached,

[4]In fact, we expect k to be asymptotically zero.

however, it is possible to reach an equilibrium point in an ϵ-neighborhood of the SAM. For such equilibria, a weaker convergence condition should be defined.

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