<u>The Valuation of a Firm Advertising Optimally.</u>

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Abstract

In this paper we model the value of a firm based on its current earnings and cash balances. The value is modelled on the assumption that earnings follow a mean-reverting process. The effect of advertising on earnings is modelled, and the condition for optimal advertising derived. The value of the firm is derived as the solution to a partial differential equation. The way in which this value depends on the legal structure and banking arrangements of the firm is discussed.

1. Introduction

The use of stochastic differential equations in modelling derivative financial products is well established. One of the most famous results in the financial literature, the Black-Scholes formula for the value of a European call, is an example of this. In evaluating the value of a European call, the model derives the Black-Scholes equation which the value of the contract must satisfy under certain assumptions about hedging and arbitrage, and solves it under boundary conditions which reflect the structure of the contract.

This approach is useful as it enables a relatively simple modelling methodology. Once the form of the underlying processes on which the value of a given contract depends have been specified, results in stochastic calculus can be used to derive a differential equation for the value of the contract with boundary conditions which reflect the structure of the contract. For example, for a European vanilla call, the payoff of the larger of underlying asset value minus the strike price or nothing, is used as a terminal condition. For more complicated contracts, more complex boundary conditions are imposed. This approach has been used in many models for valuing derivative contracts (see eg. Hull (1993), Hull and White (1990), Wilmott, Dewynne and Howison(1993)).

In their paper, Black and Scholes point out that a firm's liabilities can be priced as options. They regard the stock of a firm as being a call option on the value released when the firm shuts down and a "final dividend" is paid out. The strike price is the value of the firm's bonds outstanding, the level which the value of the firm must clear if the stockholders are to receive anything at termination. This analogy has also been cited by

Brealey and Myers (1996) in the valuation of a firm's debt. Holding a firm's debt is regarded as identical to lending money with a zero default risk, together with writing a put to the stockholders. This put represents the limited liability which the stockholders benefit from: they have the option to sell the firm to the bondholders if the terminal value does not exceed the value of the firm's outstanding bonds. Option pricing analysis can, it is argued, be transferred to the problem of valuing a firm's financial liabilities. Models pricing a firm's liabilities as contingent claims on its underlying value include Benoussan, Crouhy and Galai (1994), Black and Cox (1976), Cooper and Mello (1994), Longstaff and Schwartz (1994) and Merton (1974,1990).

More recently, this method has been applied to the control and valuation of a firm's decision. Research and development programs, market entry and exit, and the value of reserves of resources, have all been analyzed in this way. Pindyck (1988) considers the implications for the value of the firm of the fact that many investment decisions are irreversible, so that not investing in sunk capacity now offers an option to do so later, which has value in the face of future uncertainty. He finds that a large proportion of firm value, probably as much as 50%, should be derived from the "option value" of the opportunities it has for the future, rather than simply the asset value of its current capital. This result is confirmed by Abel, Dixit, Eberly and Pindyck (1996). Brennan and Schwartz (1985) apply the option pricing method to evaluating a copper mine which offers the option to continue or close in the face of uncertain future copper prices. McDonald and Siegel (1985) consider a similar problem of valuing a firm given that it has an option to shut down.

A key reference in this area is Dixit and Pindyck (1994).They use option pricing methods in the most general sense, applying stochastic calculus to analysis of firms' decisions. R&D programs, entry and exit to markets and the value of holding natural resources are all covered. Copeland, Koller and Murrin (1990) also have examples of this approach, using option pricing to develop useful capital budgeting rules. Newton, and Newton and Pearson (1994) show how option pricing can be applied to R&D.

All these models of firm value rest on two crucial assumptions. Firstly, it is clear that the behavior of any such model of a derivative claim is inextricably linked to the choice of process for the quantity underlying the value of the claim. These models generally assume that some reasonably high level process follows a simple geometric random walk. Secondly, these models typically rely on assumptions about hedging opportunities which are invalid or unrealistic.

This was pointed out in a recent paper on firm valuation (Apabhai et al. (1996)). The model derived in the paper is built on the specification of more fundamental processes than are typically used: earnings is regarded as the key underlying variable. The free cash generated by earnings is paid into a bank account during the firm's lifetime. This cash is then allocated at the end of the firm's life. Current value, regarded as the present discounted expected value of this terminal value, is modelled as a derivative claim on the processes followed by earnings and cash. No assumptions are made regarding the possibility of hedging against fluctuations in a firm's value. An alternative approach was adopted by Milne and Robertson (1996), who modelled the cash flow, rather than the earnings process, as a random variable and investigated firm behaviour under the threat of liquidation.

This paper follows an approach similar to Apabhai et al.. The underlying process modelled is deeper than simply assuming some process for the evolution of firm value over time. Moreover, no use is made of perfect hedging possibilities and arbitrage pricing relationships in the model. A number of authors (for example Duffie and Huang (1985), Rabb and Schwager (1993)) demonstrate results in which it is theoretically possible to replicate the payoff of any financial or real asset, or "span" its payoff. However, not only is complete spanning a practical difficulty (eg. many firms' shares are not traded), but there are many situations in which the party interested in the value being modelled will not attempt to hedge, making any assumption to the contrary irrelevant. As a consequence of this, risk remains in the model, which might prompt a higher interest rate than the risk-free rate to be used in discounting future values.

The structure of the paper is as follows. Firstly, the basis of the model, the underlying earnings process, is considered. Then the firm's ability to control its environment via advertising is incorporated, and the partial differential equations for firm value with and without advertising derived. The first is relatively simple, the second requires the use of dynamic programming and is more complicated. These equations are solved under a variety of boundary conditions which reflect the firm's banking and ownership arrangements, and the outcomes discussed.

2. The choice of earnings process

The paper by Apabhai et al. (1996) assumes that the firm's annualized earnings in the single market in which it is assumed to operate can be modelled by a geometric random walk. They acknowledge that this is essentially an arbitrary assumption, but it is one which is widely made in many areas of modelling.

However, on sufficiently long series of data, the random walk has been rejected as a description of certain key markets. The oil market is a clear example, as studies of the oil price have shown. Over a 100 year time series, some form of mean-reverting pattern has been found (Wey (1993)). Moreover, a comparison of a 30 year and 117 year time series analyses demonstrates that while in the shorter sample the hypothesis of a random walk cannot be rejected, in the longer it is, in favour of a mean-reverting process. Therefore there are certain markets for which we should acknowledge that the random walk is not an adequate description of the processes involved.

In such a market the speed of mean-reversion is clearly very low. Even over a 30 year period, the hypothesis of a random walk as the actual process cannot be rejected. Our modelling horizon may be substantially shorter than 30 years. If we are interested in only a small set of data, or short-run modelling, the assumption of a random walk seems relatively harmless and could be justified by the wealth of results applicable to the process. Nevertheless, it has been shown that to ignore the mean-reverting properties of oil prices in evaluating the asset value of undeveloped reserves can lead to errors of the order of 10%. Many models therefore use a mean-reverting process as their basis.

Regardless of whether or not such data we might have available rejects the random walk hypothesis, we might also prefer to use a different stochastic process in modelling a particular market to reflect theoretical considerations. Some markets may be regarded as possessing mean-reverting characteristics. Commodity markets are generally felt to fall into this category. Whatever the behavior of, say, the price of copper in a given period of time, it is to be expected that in the long-run the price of copper will be drawn back to its long-run marginal cost of production. In other words, while short-term volatility will have big effects on the price of copper, in the long-run the "price equals marginal cost" rule for a perfectly competitive product applies to this commodity. As a result, models of commodity prices generally specify a mean-reverting process as their basis.

We argue here that a mean-reverting process is a useful process for the description of firm's earnings in other markets. Admittedly, a huge literature in economics dispenses with the hypothesis of perfect competition in all but the smallest subset of real world markets. The commodity price argument does not then apply, since price can diverge from long-run marginal cost in the long-run for any of a number of reasons. The rise of a straightforward monopoly is one possibility, or the market may be heavily subsidized or taxed, or the strategies of oligopolistic incumbents may change.

Nevertheless, there would appear to be a large set of markets in which firms' earnings would be expected to be mean-reverting: monopolistically competitive markets. In reality, many goods markets can be classified as monopolistically competitive. In such a market, products are sufficiently differentiated that firms face a downwards sloping demand curve and have some degree of market power, but there are no barriers to entry; no supernormal profits are made. Such a market is relatively stable, as long as demand and cost conditions are relatively stable. If demand and costs do not change, the theory of monopolistic competition predicts that a firm will continue to produce the same output at the same price throughout its life. We can expect costs and demand to be stable in the long run, with stochastic shocks causing movements away from a mean level. There will therefore be strong mean-reverting characteristics to a firm's earnings.

In this paper, we assume that a firm's earnings follow the mean-reverting process

$$dE = q(\overline{E} - E)Edt + sEdX$$

where E is earnings, q a parameter for the speed of mean-reversion, and dX a Wiener process. This process has a characteristic worth noting: the level $E=0$ is non-attainable. This supplies a useful boundary condition.

3. The effect of advertising

Advertising can be included in this framework. An increase in advertising spending is likely to have a positive effect on earnings. In the framework of monopolistic competition, it might be expected that greater spending on advertising will cause an outward shift in the firm's demand curve. This then corresponds to an increase in mean earnings. The factor by which mean earnings will increase is not immediately clear, since both costs and revenue will have changed, and there may be a second-round effect if entry becomes attractive. However, while earnings will continue to fluctuate, the level around which they do so will certainly have increased.

The relationship of advertising to earnings needs to be specified quite carefully. There are three characteristics which should be reflected in the model. The first is that advertising will increase earnings. This characteristic is built in with a wide variety of markets in mind. In analyzing the effect of advertising, economists generally distinguish between "search" and "experience" goods. Search goods are those which offer a known level of satisfaction to the consumer, but points of sale are not immediately apparent. Experience goods are those which may be highly visible, but which offer the consumer an unknown level of satisfaction. For search goods, the increased earnings after advertising can be seen as new consumers entering the market as greater numbers are informed of the existence of the product. Markets for experience goods would see more consumers switching products perhaps because more advertising is more persuasive; or perhaps because advertising acts as a signal of product quality. Firms with poor quality experience goods would not invest much in advertising because they are unlikely to attract sufficient repeat sales to repay the outlay. Both these channels for the effects of advertising have been covered in the literature. Either way, greater advertising should yield greater earnings.

The second characteristic is that there will be diminishing marginal returns to advertising. In other words, a firm cannot increase its earnings without bound simply by advertising more. The argument for diminishing marginal returns to advertising is again equally applicable to either type of market. A wider or more intensive advertising campaign for a search good will reach fewer and fewer consumers who are searching for the good who are not already consumers. Meanwhile, greater exposure for experience goods will find fewer and fewer consumers sufficiently close in terms of their preferences that they can be persuaded to buy the product.

The third characteristic is that consumers are forgetful. The stock of customers built up through advertising will decay if the advertising does not continue.

A negative exponential relationship fits these requirements of the. To introduce a general framework into the model, the earnings process is assumed to be

$$dE = q\left(\overline{E} - E - l\left(1 - e^{-ga}\right)\right)Edt + sEdX$$

where a is annualised advertising expenditure and l and g are parameters whose value is to be specified.

The relationship between advertising expenditure and mean earnings under this assumption is simple. If a (annualized advertising expenditure) is zero, the term multiplying l is zero. Earnings will tend to return to the basic mean $\overline{E}$, at a rate determined by the level of E and the parameter q. If a increases away from zero, the coefficient on l increases, increasing the mean to which earnings will revert. This is costly to perform; such considerations are included in the expression for the firm's cash balances.

4. The value of the firm in the absence of advertising

The derivation of the differential equation for firm value without advertising follows that of Apabhai et al. (1996). The earnings process ignoring advertising is:

(1) $$dE = q(\overline{E} - E)Edt + sEdX\ .$$

The cash in the bank, which earns interest continuously at the rate r, at any given moment in time is given by:

(2) $$C_t = \int_0^t ((1-k)E(t) - E^*)e^{r(t-t)}dt\ ,$$

where k is the proportion of fixed costs, E^* is the annualised fixed costs, r is the bank deposit rate, and C is cash. It follows that:

(3) $$dC = ((1-k)E - E^* + rC)dt\ .$$

As explained above, the current value of the firm is defined simply as the expected present discounted value of the cash balances the firm has when it is wound up, i.e.:

(4) $$V(E, C, t) = e^{r(t-T)}E(C_T)\ .$$

From this expression, the Kolmogorov equation for the moments of a random variable can be derived to show that V follows the Kolmogorov-type equation

$$(5)\quad \frac{\partial V}{\partial t}+\frac{\partial V}{\partial E}\theta E(\overline{E}-E)+\frac{\partial V}{\partial C}((1-k)E-E^{*}+rC)+\frac{1}{2}\sigma^{2}E^{2}\frac{\partial^{2}V}{\partial E^{2}}-rV=0.$$

<u>The choice of boundary conditions.</u>

The boundary conditions under which this is solved are to be chosen to reflect the legal and corporate structure of the firm. Some possibilities are:

Partnership:

This affects the final payoff to the owners' or investors in the firm. If the firm is wound up and owes money, partners in an unlimited liability firm are jointly liable for its losses; they are responsible for any debts when the firm is wound up. Therefore the terminal payoff to interested parties is

$$V(T) = C(T)$$

The banking arangements of the firm are also important, since they affect the range of C over which the equation for V is to be solved. If the firm can run its current account in any state, the equation is to be solved in $C \in (-\infty, \infty)$. Conversely, if the bank will not allow the firm an overdraft, the equation is to be solved in $C \in [0, \infty)$.

In addition to a final condition in t, we need conditions in E and C to solve the equation. The conditions used are:

$V_{CC} \rightarrow 0$ *as* $C \rightarrow \pm\infty$

$V_{EE} \rightarrow 0$ *as* $E \rightarrow \infty$

$V(E,C,t) = \frac{E^*}{r}\left(e^{r(t-T} - 1\right) + C$ as $E \to 0$ (this is the solution to the ordinary differential equation obtained if E were ever to reach zero and "switch off" for the rest of the firm's life).

Fig. 1 shows a numerical solution for a partnership with any cash balances allowed. We have obtained all numerical solutions with parameter values $\overline{E} = 10000, E^* = 500, (T-t) = 5, q = 0.5, l = 1, g = 0.5, r = r = 6\%, k = 10\%$ and $s = 0.2$.

Limited liability:

If a firm is of limited liability, its shareholders are not liable for any debts it may have accrued during its lifetime. They attract any positive cash remaining at termination, but are not liable for any losses. The terminal payoff is therefore:

$$V(T) = max(C(T),0)$$

Fig. 2 shows a solution for a limited liability firm with any cash balances permissible, and the same parameter values as above.

Optimal trading strategies

If it is ever the case that *V<C*, then the firm is expected to lose money later in its lifetime: expected terminal cash balances are lower than current value. It would therefore be preferable for the firm to be wound up now. It is possible to introduce a “free boundary” into the problem if the firm could be closed down early in this way. The decision structure is then in fact identical to that of an American option, where early exercise is permitted. If the value of the option ever falls below its intrinsic value (the

value received when it is struck) it is optimal to exercise the option early. Introducing the constraint that $V \geq C$ mirrors this exactly.

Again, this problem does not permit an explicit solution. However, numerical solutions using an explicit finite-difference scheme have been found. One such solution is shown below in *fig. 3*. It is clear from the results that with a mean-reverting process for earnings, it is only optimal to close down at very low levels of earnings.

There have been no assumptions made about hedging possibilities or actuality. The amount of cash in the bank at any given time is random, but we use the expectation of terminal cash in our definition of firm value. Given that risk remains in the model, the rate r used in discounting may be chosen to include some kind of risk premium to acknowledge this. We can use a higher rate $r = r + p$ to discount future values, where p represents a risk-premium. Apabhai et al. (1996) discuss practical methods for choosing p.

5. The value of the firm including advertising

As before, the model is of a business which will be run for a set period of time, and then wound up. Its current value is again defined as the present discounted value of the cash balances earned by the firm by the end of its life, given that the firm is assumed to be advertising optimally. The firms earnings are described by a mean-reverting process, with the effect of advertising built in.

The earnings process is:

(6) $$dE = q(\overline{E} - E + l(1 - e^{-ga}))Edt + sEdX \ .$$

The expected percentage change is $q(\bar{E} - E + l(1-e^{-ga}))$ and linear in E; the expected absolute change in E is $q\bar{E}E + l(1-e^{-ga})E - qE^2$ and quadratic in E.

The amount of cash the firm has in the bank at any given point is

$$(7) \qquad C_t = \int_0^t ((1-k)E(t) - a(t) - E^*)\exp(r(t-t))dt \ .$$

As before, the firm's value is regarded as the expected value of terminal cash balances. In addition, the firm is assumed to be continuously maximizing this definition of value, i.e. if V' is defined as V was previously, and V is now firm value under an optimal advertising policy

$$(8) \qquad V'(E,C,t) = E(C_T)exp(r(t\text{-}T)).$$

Therefore its current value on the assumption of continuous maximization is:

$$(9) \qquad V_t = \max_{a\geq 0} V'(E,C,t) = \max_{a\geq 0} E(C_T)\exp(r(T-t)) \ .$$

The equation for V is more difficult to obtain than before, as we are dealing with a dynamic optimisation problem. The most useful tool to apply to this problem is dynamic programming. This breaks a sequence of decisions into two components: the immediate decision, and a "value function" which represents the consequences of all subsequent decisions. We can then use Bellman's Principle of Optimality (see eg. Dreyfus (1965)) to split up the problem. This principle states that an optimal policy has the property that whatever the initial action, all subsequent actions constitute an optimal policy with respect to the optimisation problem beginning from that initial action: we are continually optimising. This breaking up of the dynamic optimisation problem and assumption of continuous maximisation can be represented mathematically. If V_t is the value of the firm now, a_t is our action, E_t is the state of the world when we act and $p(a_t, E_t)$ is the immediate payoff to our action, then incorporating discounting,

(10) $$V_t = \max_{a_t}\left[p(a_t, E_t) + (1 + rdt)^{-1} E(V_{t+1}(a_{t+1}))\right].$$

This is the Bellman equation, or fundamental equation of optimality, for such a dynamic optimisation problem. All this says is that next period, we inherit V_{t+1} , which is similarly recursively defined, and in the meantime our action, contingent on the state of the world, yields payoff $p(a_t, E_t)$.

In this case, we have no immediate payoff because all value is derived from the expectation of terminal cash. Therefore for a firm choosing its advertising optimally, with the obvious constraint that $a \geq 0$, its current value across some time interval dt (we implicitly let dt go to zero to get into continuous time) must satisfy

(11) $$V_t = \max_{a\geq 0}\left[(1 + rdt)^{-1} E(V_{t+dt})\right].$$

It is obvious that $V_{t+dt} = V_t + dV$, so that we can write (suppressing the time index for clarity)

(12) $$V_t = \max_{a\geq 0}\left[(1 + rdt)^{-1} E(V + dV)\right].$$

Given that $E(V)$ is V and $max(V)=V$ (by definition, since V is always the outcome of the optimal policy), it then follows that

(13) $$V_t = (1 + rdt)^{-1}\left[V + \max_{a\geq 0} E(dV)\right].$$

Implying straightforwardly that

(14) $$rVdt = \max_{a\geq 0} E(dV).$$

We now need to evaluate the expectation. Using Ito's Lemma, and given that *V* is some function of *E*, *C* and *t*,

$$(15) \qquad E[dV] = \frac{\partial V}{\partial t}dt + \mu_E \frac{\partial V}{\partial E}dt + \mu_C \frac{\partial V}{\partial C}dt + \frac{1}{2}\sigma_E^2 \frac{\partial^2 V}{\partial E^2}dt \; .$$

Substituting this into equation (14) and dividing by *dt*, it follows that *V* satisfies

(16)

$$rV = \max_{a \geq 0}\left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial E}(\overline{E} - E + \lambda(1-e^{-\gamma a}))\theta E + \frac{\partial V}{\partial C}((1-\kappa)E - a - E^* + rC) + \frac{1}{2}\sigma^2 E^2 \frac{\partial^2 V}{\partial E^2}\right)$$

.

We cannot have negative advertising so *a* is restricted to the range $a \geq 0$. We therefore have a form of constrained optimisation problem (see eg. Chiang (1984)). In order to impose the conditions for optimality subject to $a \geq 0$ we have two complementarily slack conditions for optimal advertising (written a^*). These are

(17a)

$$\Lambda = \frac{\partial}{\partial a}\left(\frac{\partial v}{\partial t} + \frac{\partial V}{\partial E}\theta\left(\overline{E} - E + \lambda\left(1-e^{-\gamma a}\right)\right)E + \frac{\partial V}{\partial C}((1-\kappa)E - a - E^* + rC) + \frac{1}{2}\sigma^2 E^2 \frac{\partial^2 V}{\partial E^2}\right) \leq 0$$

(17b) $a \geq 0$,

with $\Lambda . a = 0$ as the complementary slackness condition.

There are therefore two key types of solution. Firstly, the interior solutions where a^* is strictly greater than zero. At these points, by simple differentiation of the first condition above, the condition

$$(18) \qquad \frac{\partial V}{\partial E}\lambda\gamma\theta E e^{-\gamma a^*} = \frac{\partial V}{\partial C}$$

emerges, from which it follows that

(19) $a^{*} = -\frac{1}{g}\ln\left(\frac{\P V/\P C}{qE\,l\,g\left(\P V/\P E\right)}\right).$

Secondly, we have the possibility of boundary solutions, where the first condition (17a) for *a** is satisfied as an inequality, and *a*=0*. There is a third possibility, a boundary solution where both constraints are satisfied with equality, but this does not affect the discussion. From these two possible types of solution, it follows that

(20) $a^{*} = \max\left(-\frac{1}{g}\ln\left(\frac{\P V/\P C}{qE\,l\,g\left(\P V/\P E\right)}\right),0\right).$

As a check that this is a maximum, differentiate (17a) with respect to *a* again to check that

(21) $\frac{\P}{\P a}\left(\frac{\P V}{\P E}\,lgqEe^{-ga^{*}} - \frac{\P V}{\P C}\right) < 0$

is satisfied. For this to hold, it is sufficient that

(22) $-\frac{\P V}{\P E}\,l g^{2} qEe^{-ga^{*}} < 0$

which is satisfied for all admissible values (it is asserted, but can be shown, that $\frac{\P V}{\P E} \geq 0$).

We can therefore be sure that we have found a maximum. Substituting this expression for *a** into equation (17), it then follows that *V* satisfies the partial differential equation

(23)
$$\frac{\P V}{\P t} + \frac{\P V}{\P E}J\left(\overline{E} - E + l\left(1 - \exp\left(g\,\max\left(-\frac{1}{g}\ln\left(\frac{\P V/\P C}{qE\,l\,g\left(\P V/\P E\right)}\right),0\right)\right)\right)\right)E +$$
$$\frac{\P V}{\P C}\left((1-K)E - \max\left(-\frac{1}{g}\ln\left(\frac{\P V/\P C}{qE\,l\,g\left(\P V/\P E\right)}\right),0\right) + rC\right) + \frac{1}{2}s^{2}E^{2}\frac{\P^{2}V}{\P E^{2}} - rV = 0$$

<u>Boundary conditions:</u>

The boundary conditions under which this equation is solved again depend upon the organizational structure and decision rules of the firm. Different boundary conditions will clearly lead to different evaluations of the firm. Boundary conditions will capture whether the firm is of limited or unlimited liability and whether the firm is allowed an overdraft. The boundary conditions used in each case are identical to those used before. Solutions are shown for parameter values of

$\overline{E} = 10000, E^* = 500, (T-t) = 5, q = 0.5, l = 1, g = 0.5, r = r = 6\%, k = 10\%$ and $s = 0.2$

below. *Fig. 4* shows the value of an optimally advertising partnership with any bank balance allowed; *fig. 5* shows the value of an optimally advertising limited liability firm with any cash balances.

The effect of advertising on firm value is not immediately apparent from the above diagrams. *Fig. 6* shows the value added by optimal advertising for a partnership (the "difference" between the solutions shown in *figs. 1* and *4*), and *fig. 7* the value added by optimal advertising for a limited liability company (the difference between *figs. 2* and *5*).

The level of *C* seems largely irrelevant for the choice of advertising, and for the effect of an optimal advertising policy on firm value, in *fig. 6*. Earnings drive the behaviour of the model. At low earnings, the effect of advertising on firm value is high as we have a high level of $\frac{\partial V}{\partial E}$; this effect becomes less and less pronounced as *E* increases, as the dropping off of $\frac{\partial V}{\partial E}$ dominates the increased effect arising from higher *E*.

In *fig. 7*, the same relationship to earnings is apparent, while cash affects the effect of advertising on firm value once more via the probability that there will be some terminal value in the firm, given that its value for all *C's* not greater than zero, is zero. At low levels of *C* and *E*, it becomes harder and harder to affect firm value.

Finally, *Fig. 8*shows a numerical solution for an optimally advertising firm with an option to close. Again, it is apparent that it is optimal to close down only at very low levels of earnings.

6. Conclusion

The use of stochastic differential equations in finance is common practice. These equations are used to value derivative securities, and have also been used to analyze and value various opportunities and decisions faced by a firm.

Moreover, the technique is easily applicable to the valuation of entire firms. Once the model of the stochastic environment in which the firm operates has been specified, the equation satisfied by firm value can easily be derived. This can be done simply by assuming some arbitrary process for firm value; it is however better done, as here, by modelling more fundamental processes on which the value of the firm depends.

The inclusion here of the effects of advertising illustrates a further possible extension: to value the entire firm while acknowledging that the firm has a substantial degree of control over its environment. Advertising is not the be and end all; other control variables could easily be introduced into the framework.

A model of advertising has therefore been incorporated into a model designed to represent some sort of monopolistically competitive market. Numerical solutions have been derived, as shown above.

Extensions to this model could obviously include further decision variables; they also include different specifications of the effect of advertising. Moreover, the basic mean-reverting process describing the evolution of earnings, while justified here, could be altered for applications elsewhere.

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