On Trading American Options

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Abstract. This paper proves that the optimal exercise time for the holder of an American option depends upon the physical drift of the underlying asset and the utility of the option holder. We illustrate our results by applying them to several families of utility functions, namely the CARA, the HARA, and the expected return. While the option holder maximises his utility, the issuer gains from the difference between the price maximising exercise boundary and the exercise boundary performed by the option holder. We provide the numerical results which describe the effect of the physical drift and the risk aversion on the issuer's expected profit.

1 Introduction

The American option is an option contract that allows the option holder to exercise before the maturity if he is better off doing so. Because of the flexibility of choosing the exercise time, the price of the option is calculated as the value of the option in the worst case for the issuer among all feasible exercise strategies that the option holder may perform. Typically the price maximising exercise time, and hence the least favourable exercise time for the issuer, is described as an optimal stopping time, and the resulting pricing equation becomes a free-boundary partial differential equation (PDE). Although it may sound that it requires a rather sophisticated mathematics to price an American option, the fundamental concept of the absence of arbitrage is still the integral part of determining the price. The issuer can construct a hedging portfolio involving the trading of the underlying assets in such a way that the value of the replicating portfolio (i.e., the upfront premium for the option plus the result of the trading) is not less than his liability even when his customer exercises the option at the least favourable time.

The option holder may hedge his position as well, constructing his portfolio exactly opposite to that of the the issuer's and exercising his option at the price maximising exercise time. In this case, both the issuer and the option holder have their balance equal to zero. If this was what they intended, they could have chosen not to trade the option at the beginning and saved their effort in maintaining their hedge positions. Thus it is reasonable to assume that the option holder engages in some other strategies. For example, he may adopt a stop-loss strategy: buy-and-hold the option until he decides to exercise it. Unlike the previous case, the option holder may gain or lose depending upon the behavior of the underlying asset price while the potential loss is not more than the premium he paid. The issuer gains unless the option holder exercises at the first time that the asset price reaches the price maximising exercise boundary. One of the questions we address in this paper is "Should the option holder exercise at the price maximising exercise time?"

Trading an option is not a two-person-zero-sum game because both the issuer and the holder can trade the underlying asset with other investors. Hence, the worst case for the issuer is not necessarily the best case for the holder. It is not true that the option holder is better off exercising the option at the price maximising exercise time. First, we consider the physical drift of the underlying asset. The price of an option depends upon the risk-neutral drift, not the physical drift of the underlying asset. The reason is that the presence of an option immediately allows one to construct locally riskless portfolios, and hence the risk-free rate is the only one that governs its price. As a result, the price maximising exercise boundary is also independent of the physical drift. Can we assert that the optimal exercise boundary for the option holder is not affected by the market direction? Almost certainly not. For example, it is well known that the price maximising exercise time for the American call is the maturity of the contract, provided that the underlying stock pays no dividend. If there is any evidence that the price of the underlying asset is expected to fall, however, a wise investor would exercise his call earlier before it expires worthless. Second, each investor has his own risk preference. Two different rational decision makers may exercise differently, even when they agree with the probability distribution of outcomes. It is nonsense to argue that a single exercise strategy is the optimal strategy for every investor.

This paper establishes the optimal exercise boundary provided that the option holder is a utility maximising investor. The optimal exercise boundary, or the utility maximising exercise boundary, depends upon the risk aversion and the physical drift. In Theorem 3.1 we confirm the followings: (i) if the option holder is sufficiently risk averse, early exercise is optimal even for a call; (ii) the optimal exercise time is a non-decreasing function of the physical drift, if the option is a call; (iii) if the option is a put, the optimal exercise time is a non-increasing function of the physical drift. We illustrate these results with several families of utility functions: the constant absolute risk averse (CARA), the hyperbolic absolute risk averse (HARA), and the linear utility (i.e., the expected return). Some of the highlights are:

(a) If the option holder's utility is of the CARA type, early exercise prevails for both call and put regardless of the absolute risk aversion parameter.

(b) Certain HARA utilities may yield two separate exercise boundaries.

(c) Upon the expected return criterion, a call option is exercised early only when the physical drift is surpassed by the risk-free rate.

Another accomplishment in this paper is the equation for the expected profit selling American options. As we stated earlier, the issuer gains from the difference between the price maximising exercise time and the exercise time performed by his customer. The profit grows as the occupation time of the asset price in the region between the exercise boundaries of the price maximisation and the utility maximisation. The difference between the value of the option and the exercise value is the final piece of the profit. We provide numerical results on how the physical drift and the risk aversion affect the issuer's profit.

The paper is structured as follows. In the next section we review the classical results of pricing and hedging American options. In Section 3, we find the optimal exercise time for the utility maximising investor. In Section 4, we analyse the effect of option holder's optimal exercise strategy on the issuer's profit. Section 5 contains concluding remarks.

2 Preliminary : Pricing and Hedging

The early exercise feature makes the valuation of the American option more intriguing than that of the European counterpart. The main concepts are the optimal stopping and the corresponding parabolic variational inequalities. Myneni (1992) surveyed literature on the subject and summarized key results. Here we state the standing assumptions for the rest of the paper and review the variational inequalities.

The classical theory of option pricing is predicated on many assumptions for market completeness. We assume that the market is frictionless, that short-selling is allowed without restriction, that one can trade assets as frequently as one wishes, that all risk-free assets grow at the common rate r which is known *a priori*, and that there is a unique risk-neutral equivalent martingale measure. The last assumption becomes less abstract when we assume that the price of the underlying asset follows a geometric Brownian motion and that market participants are not capable of foreseeing the future. Thus, in what follows the price of the underlying asset evolves as:

$$dS_t = \sigma S_t dW_t + \mu S_t dt \tag{1}$$

where W is a standard Brownian motion. In addition the filtration is natural, meaning that the stream of information consists of the observations of the asset price only.

As shown in Harrison and Pliska (1981), the complete market assumption allows a trader to replicate the payoff of an arbitrary contingent claim by trading the underlying assets. We start by assuming that the issuer of the option maintains Δ shares of the underlying asset to hedge his position. In other words, the value of the issuer's portfolio is given by $\Delta S - v$ where v is the value of the option. This portfolio must grow at least at the risk-free rate r:

$$\Delta dS_t - dv \geq r(\Delta S_t - v)dt\,. \tag{2}$$

Because of the Markovian nature of the underlying asset price (1), the value of the option v at time t is a function of t and the asset price S_t. For the time being, we assume that v is continuously differentiable with respect to t and twice continuously differentiable with respect to s. Then we have

$$dv(t, S_t) = v_t(t, S_t)dt + v_s(t, S_t)dS_t + \frac{1}{2}\sigma^2 S_t^2 v_{ss}(t, S_t)dt\,. \tag{3}$$

which follows from Itô's formula. It is required for the issuer to pick $\Delta = v_s$ in order to fulfill (2) because the random growth dS_t is of the order $\sqrt{dt}$ and is much bigger than terms with dt. Rearranging (2) after replacing Δ by v_s yields:

$$\mathcal{L}v = v_t + \frac{1}{2}\sigma^2 s^2 v_{ss} + r(s v_s - v) \leq 0 \tag{4}$$

for each s. The value of the option will never fall below an immediate exercise value. Otherwise the issuer loses. This yields the second condition for v:

$$v \geq \phi \tag{5}$$

where ϕ is the payoff of the option: $\phi(s) = \max(s - K, 0)$ for a call with strike K and $\phi(s) = \max(K - s, 0)$ for a put. Each time t, the option holder may or may not exercise his option. If $v > \phi$ at the moment, then exercising the option is not the least favourable to the issuer because he can claim a non-zero profit $v - \phi$ instantly. In this case, $\mathcal{L}v = 0$ because the issuer has an arbitrage opportunity if $\mathcal{L}v$ were strictly less than zero. Therefore we obtain the third condition:

$$(\mathcal{L}v) \cdot (v - \phi) = 0\,. \tag{6}$$

The inequalities (4), (5), and (6) subject to $v(T, s) = \phi(s)$ form a parabolic obstacle problem. We refer to Friedman (1988) for the existence and the uniqueness of the solution to such problems. Jaillet, Lamberton, and Lapeyre (1990) showed that the solution of the parabolic variational inequalities (4), (5), and (6) has a continuous gradient at the free boundary (i.e., a smooth fit), and Van Moerbeke (1976) showed that the optimal stopping boundary is continuously differentiable. Thus Itô's formula (3) is valid at least in a weak sense: see San Martin and Protter (1993) for detail.

In the theory of optimal stopping, the space-time domain defined by $v > \phi$ is called the continuation region as the stopping is premature in this region and the graph of its boundary is called the optimal stopping boundary. In this paper we call this the price maximising exercise boundary, distinguishing it from the optimal stopping boundary from the utility maximisation problem in the next section.

3 Utility Maximising Exercise Time

We assume that the option holder possesses a utility function $U : \mathbb{R} \to \mathbb{R}$ that is strictly increasing and twice continuously differentiable. The investor, who purchases an American option at time 0, will select his exercise time by maximising the expected utility of the discounted wealth. The class of feasible exercise times consists of stopping times that are less than or equal to T, the maturity of the option. This includes exercise times that are strategically selected based upon the price of the asset up to date as well as pre-scheduled times (i.e., non-random). A feasible exercise time will be denoted by τ. If the option holder never exercises the option we set $\tau = T$. As before ϕ is designated for the payoff. Then, at time t, the option holder faces the following optimal stopping problem:

$$u(t,s) = \operatorname*{ess\,sup}_{t\le\tau\le T} E^{t,s}\Big[U(e^{-r\tau}\phi(S_\tau))\Big] \tag{7}$$

where $E^{t,s}$ is the conditional expectation given that $S_t = s$, τ is the option holder's exercise time, and ϕ is the payoff. The essential supremum is taken over all the feasible exercise times. Finally the expectation is governed by the physical measure not the risk-neutral equivalent martingale measure. We consider only when (7) is well defined. A sufficient condition is that $U \circ \phi$ is bounded by a polynomial.

We could have defined u as the expected utility of $e^{-r\tau}\phi(S_\tau) - v(0, S_0)$, the discounted payoff minus the option price. In our definition, the option price is a part of the utility function U, as we treat the option price as a constant.

As in the case of the price maximisation, the optimal stopping problem (7) is equivalent to a parabolic obstacle problem. Thus u satisfies a set of variational inequalities. We will describe the variational inequalities financially, omitting technical details. For notational convenience, we define $g(t,s) = U(e^{-rt}\phi(s))$. First we check that

$$u \ge g\,. \tag{8}$$

This is because the maximum expected utility is not smaller than the utility of the immediate exercise which is a special case of feasible stopping times. Next we will explain the following inequality for $t < T$:

$$u(t,s) \ge E\Big[u(t+\delta, S_{t+\delta})\Big] \tag{9}$$

for each δ that makes $t+\delta$ a feasible exercise time. Note that the right side of (9) coincides with the expected utility when the option holder pursues the optimal stopping only after δ elapses. In other words, the option holder is dormant until time $t+\delta$ and he tries to find an optimal exercise time from then on. Thus the value of this expected utility cannot exceed the maximum expected utility which is on the left side of (9). The implication of (9) is the following inequality:

$$\mathcal{L}_\mu u = u_t + \mu s u_s + \frac{1}{2}\sigma^2 s^2 u_{ss} \leq 0 \tag{10}$$

which is obtained by applying Itô's formula on u. If $\mathcal{L}_\mu u < 0$, then the maximum expected utility is expected to fall in an infinitesmal time, and hence the optimal strategy is to exercise the option immediately. That is, $u = g$. Therefore u must satisfy

$$(\mathcal{L}_\mu u) \cdot (u - g) = 0\,. \tag{11}$$

The set of variational inequalities (8), (10), and (11) with terminal data $g(T, s)$ characterises the maximum expected utility u. The optimal exercise time is the first time that the asset price S_t hits the free boundary of the inequalities.

Next we consider $h = e^{rt}U^{-1} \circ u$, the maximum expected certainty equivalence. U^{-1}, the inverse of U, is well defined as U is an increasing function of wealth. The merit of using this change of variable is that it facilitates us comparing the utility maximisation to the price maximisation. We confirm that h must satisfy the following variational inequalities:

$$\begin{gathered} h \geq \phi \\ \mathcal{D}h = h_t + \frac{1}{2}\sigma^2 s^2 \Big(h_{ss} + \frac{U''}{U'}(e^{-rt}h)e^{-rt}(h_s)^2 \Big) + \mu s h_s - rh \leq 0 \\ (\mathcal{D}h) \cdot (h - \phi) = 0 \end{gathered} \tag{12}$$

subject to $h(T, s) = \phi(s)$. Therefore the utility maximising exercise boundary depends upon the physical drift and Pratt's measure of absolute risk aversion $-U''/U'$, and is different from the price maximising exercise boundary. The distortion in discount is caused by the non-linearity of the utility funciton.

Theorem 3.1. *The utility maximising exercise time for an American option has the following properties:*

(i) If the absolute risk aversion is sufficiently large, then there is a positive probability of early exercise for both call and put.

(ii) The exercise time is a non-decreasing in μ, when the option is a call.

(iii) The exercise time is a non-increasing in μ, when the option is a put.

Proof. Note that the exercise region coincides with the space-time domain of $\mathcal{D}h < 0$. If the absolute risk aversion $-U''/U'$ tends to infinity uniformly in its argument, then $\{(t,s) : \mathcal{D}h < 0,\ 0 \le t < T,\ s > 0\}$ is a set of a positive measure. Since the support of a non-degenerate geometric Brownian motion (i.e., $\sigma^2 > 0$) occupies the entire positive plane, the utility exercise maximising time can be less than the maturity with a positive probability. This proves (i). When the option is a call, h_s is positive. Thus $\mathcal{D}h$ becomes more negative when μ becomes smaller. If the option is a put, h_s is negative, and hence $\mathcal{D}h$ becomes more negative when μ becomes larger. Therefore we have (ii) and (iii).

□

Our next task is to locate the the boundary when the time to maturity is arbitrarily close to zero. Note that the certainty equivalence h tends to ϕ as $t \to T$ and the utility maximising exercise boundary (as a function of time) is continuously differentiable. Thus when t is near T, the utility maximising exercise boundary is close to the boundary of $\mathcal{D}\phi < 0$. This is *the boundary at maturity.* If $\phi(s) = \max(s-K,0)$ (i.e., a call option), then the boundary is above the strike K for each $t \in [0,T)$ and hence the boundary at maturity is

$$\partial\Big[\ s > K\ :\ \frac{1}{2}\sigma^2 s^2 \frac{U''}{U'}\Big(e^{-rT}(s-K)\Big)e^{-rT} + (\mu - r)s + rK < 0\ \Big]. \tag{13}$$

Here, the symbol ∂ is used for indicating the boundary a set. Similarly, if $\phi(s) = \max(K - s, 0)$ (i.e., a put option), the boundary at maturity is

$$\partial\Big[\ s < K\ :\ \frac{1}{2}\sigma^2 s^2 \frac{U''}{U'}\Big(e^{-rT}(K-s)\Big)e^{-rT} - (\mu - r)s - rK < 0\ \Big]. \tag{14}$$

Sometimes (13) and (14) may contain more than one element. In such a case, we have more than one free boundary. In the remaining of this section, we provide an explicit expression for the boundary at maturity when the option holder's utility belongs to one of the following categories: the CARA, the HARA, and the expected return (i.e., the linear utility).

3.1 Constant Absolute Risk Aversion

This is the case when the absolute risk aversion is a constant regardless of the wealth of the investor. That is, $-U''/U' \equiv \lambda$ for a positive constant λ. Up to a constant, the utility is of the form $U(\omega) = -\alpha e^{-\lambda\omega}$ for a positive constant α.

First we consider a call option. We confirm that the boundary at maturity (13) is reduced to

$$\max\Big\{K, \frac{1}{\lambda\sigma^2}\left(\mu - r + \sqrt{(\mu-r)^2 + 2\lambda\sigma^2 K r e^{-rT}}\right)e^{rT}\Big\}. \tag{15}$$

Note that (15) tends to infinity as λ tends to zero. Hence as the risk aversion of the option holder vanishes, the utility maximising exercise time tends to the maturity which coincides with the price maximising exercise time. Next we consider a put option. The inequality in (14) is

$$-\frac{1}{2}\sigma^2\lambda e^{-rT}s^2 - (\mu - r)s - rK < 0\,. \tag{16}$$

If the physical drift is at least the risk-free rate ($\mu \geq r$), (16) is true for all positive s. Thus the boundary at maturity is K. Suppose that $\mu < r$. The quadratic inequality (16) is always satisfied if

$$d = (m - r)^2 - 2\lambda\sigma^2 K r e^{-rT} < 0.$$

in this case the boundary at maturity is also K. Now suppose that $d \geq 0$ as well as $\mu < r$. Solving the quadratic inequality (16), we obtain the boundary at maturity:

$$\min\Big\{K, \frac{1}{\lambda\sigma^2}\Big(r - \mu + \sqrt{(r-\mu)^2 - 2\lambda\sigma^2 K r e^{-rT}}\Big)e^{rT}\Big\}.$$

3.2 Hyperbolic Absolute Risk Aversion

Merton (1990) provides a complete description to this family of utility functions. The hyperbolic absolute risk aversion means $-U''/U'(\omega) = \lambda/(\omega+\alpha)$ for a positive constant λ. This utility applies to the case when the wealth of the investor is bounded below: $\omega+\alpha > 0$. Thus the richer the investor is, the less he is risk averse. Up to a constant shift,

$$U(\omega) = \begin{cases} \dfrac{1}{\beta^\lambda}\dfrac{(\omega+\alpha)^{1-\lambda}}{1-\lambda}, & \text{if } \lambda \neq 1 \\ \dfrac{1}{\beta}\log(\omega+\alpha), & \text{otherwise} \end{cases}$$

where $\beta > 0$. The parameter α is assumed positive as the option payoff could be zero.

A simple algebra reduces the inequalities in (13) and (14) to quadratic inequalities. For example (13) is equivalent to

$$\partial\Big[s > K : As^2 + Bs + C < 0\Big] \tag{17}$$

where $A = (\mu - r - \frac{1}{2}\sigma^2\lambda)e^{-rT}$, $B = (\mu - r)(\alpha - e^{-rT}K) + re^{-rT}K$, and $C = rK(\alpha - e^{-rT}K)$. The continuation region and the exercise boundary depends upon the choice of parameters. An unusual case is when parameters satisfy the followings:

$$r + \frac{1}{2}\sigma^2\lambda < \mu < \frac{1}{2}\sigma^2\lambda\frac{e^{-rT}K}{\alpha}.$$

In this case, the continuation region near the maturity is separated by the exercise region:

$$\Big[s > K : As^2 + Bs + C < 0\Big] = [s : K < s < \frac{-B + \sqrt{B^2 - 4AC}}{2A}].$$

If the physical drift is sufficiently large, the option is very valuable to the holder when the option is very in-the-money. If not, the curvature reduces the holder's utility. Also note that there is no exercise boundary if

$$\mu > r + \frac{1}{2}\sigma^2\lambda \qquad \text{and} \qquad \alpha > e^{-rT}K.$$

This is the case when the physical drift is large while the risk aversion is not.

3.3 The Expected Return

This is a special case of $U(\omega) = \alpha\omega + \beta$ for a positive constant α. As U'' vanishes in this case, our analysis on the boundary at maturity becomes straightforward.

When the option is a call, the inequality in (13) becomes $(\mu - r)s + rK < 0$. This is never satisfied if $\mu \geq r$. Thus the utility maximising exercise time is the maturity when the physical drift is at least the risk-free rate. If $\mu < r$, on the other hand, the boundary at maturity is

$$\max\Big\{K, \frac{r}{r-\mu}K\Big\}.$$

Next we consider a put option. If $\mu \geq r$, then the inequality in (14) is always satisfied. Thus the boundary at maturity is K in this case. If $\mu < r$, the boundary at maturity becomes

$$\min\Big\{K, \frac{r}{r-\mu}K\Big\}.$$

4 Profit by Selling American Options

In the previous section, we observed that the option holder's exercise time could differ from the price maximising exercise time, when he optimises his utility. When this happens, the issuer gains from the difference. In this section we examine the profit by selling American options to utility maximising investors.

The issuer charges $v(0, S_0)$ at time 0 as he sells an American option. He will hedge his short position in option as described in Section 2 until his customer exercises the option or the option expires. The discounted potential liability of the issuer is $e^{-r\tau}\phi(S_\tau)$ where τ is the actual time that his customer exercises. When the option holder never exercises, $\tau = T$ by convention. Thus the present value of the issuer's profit becomes:

$$P = v(0, S_0) + \int_0^\tau e^{-rt}\Delta\Big(dS_t - rS_t dt\Big) - e^{-r\tau}\phi(S_\tau) \tag{18}$$

The second term in the right side of (18) is the result of delta hedging with the cost of carry. First we add and subtract $e^{-r\tau}v(\tau, S_\tau)$ from the profit P. Applying Itô's formula on v yields:

$$v(0, S_0) + \int_0^\tau e^{-rt}\Delta\Big(dS_t - rS_t dt\Big) - e^{-r\tau}v(\tau, S_\tau) = -\int_0^\tau dt e^{-rt}\mathcal{L}v$$

where $\mathcal{L}$ is the Black-Scholes differential operator defined in (4). Thus we may rewrite the profit (18) as

$$P = -\int_0^\tau dt e^{-rt}\mathcal{L}v + e^{-rt}\Big(v(\tau, S_\tau) - \phi(S_\tau)\Big). \tag{19}$$

We define the expected profit at time t as

$$\psi(t, s) = E^{t,s}\Big[-\int_0^\tau dt e^{-rt}\mathcal{L}v + e^{-rt}\Big(v(\tau, S_\tau) - \phi(S_\tau)\Big)\Big] \tag{20}$$

We will show that ψ satisfies a diffusion equation with a moving boundary which is known *a priori*. Recall that h is the maximum expected certainty equivalence of the option holder and its free boundary gives the optimal exercise time τ. Let $\mathcal{H}$ and $\mathcal{V}$ be the domains defined by $h > \phi$ and $v > \phi$, respectively. These are the regions of continuation for the utility maximisation and the price maximisation. We also define $\mathcal{G} = \mathcal{H} \backslash \mathcal{V}$, see Figure 1. Since $\mathcal{L}v$ vanishes on $\mathcal{V}$, the expected profit ψ satisfies

$$\psi_t + \mu s \psi_s + \frac{1}{2}\sigma^2 s^2 \psi_{ss} - e^{-rt}\mathcal{L}v \mathbb{I}_{\mathcal{G}} = 0 \tag{21}$$

subject to $\psi(T, s) = 0$ and $\psi = e^{-rt}(v - \phi)$ on $\partial\mathcal{H}$, the utility maximising exercise boundary. The indicator $\mathbb{I}_{\mathcal{G}}$ is one if (t, s) belongs to $\mathcal{G}$ and zero otherwise. If the option is a call, the left side of (21) vanishes because $\mathcal{G}$ is empty. If the option is a put, then $v = \phi$ on the complement of $\mathcal{V}$, and therefore

$$e^{-rt}\mathcal{L}v\mathbb{I}_{\mathcal{G}} = e^{-rt}\mathcal{L}\phi\mathbb{I}_{\mathcal{G}} = -re^{-rt}K\mathbb{I}_{\mathcal{G}}$$

where K is the strike price. Here we used the fact that the price maximising exercise boundary is not above the strike when the option is a put.

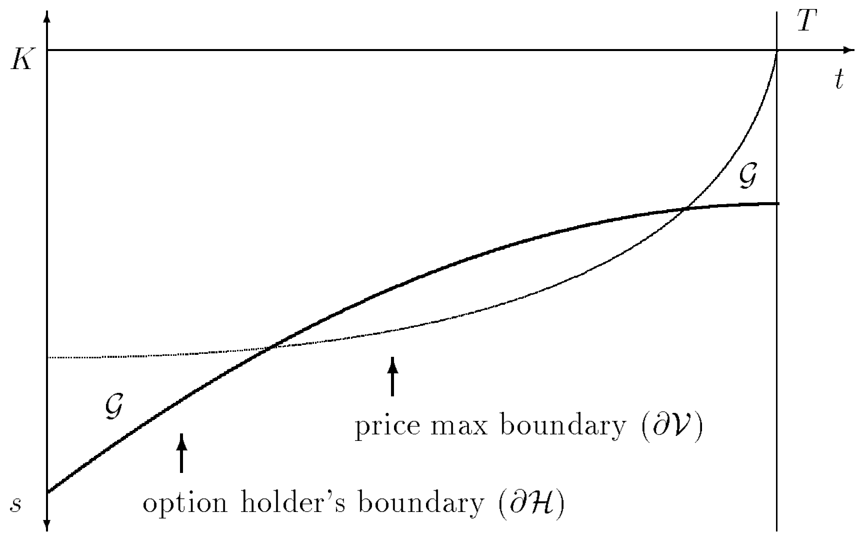

Figure 1: Overlapping exercise boundaries

Figure 2 shows the expected profit by selling an at-the-money American put to an investor who maximises the expected return. In this case, the option holder's criterion in choosing the exercise time is free from the risk aversion, and hence the outcome is considered as the marginal effect of the physical drift to the issuer's expected gain. The initial asset price is 50, the asset volatility is 20% annum, the maturity of the option is 6 months, and the risk-free rate is 8% annum. When the physical drift coincides with the risk free rate, the exercise boundary that maximises the expected return coincides with the price maximising exercise boundary, and hence there is no profit for the issuer. When the physical drift surpasses the risk-free rate, the holder's exercise boundary is inside the price maximising exercise boundary. In this case, $\mathcal{G}$ is empty and the only source of the issuer's profit is the difference between the value of the option and the exercise value (i.e., the value of ψ on the moving boundary $\partial\mathcal{H}$). If the physical drift is less than the risk-free rate, then the holder's exercise boundary is outside of the price maximising exercise boundary, and hence the issuer's profit grows with the occupation time of the asset price in between two boundaries. This explains the asymmetry in the picture.

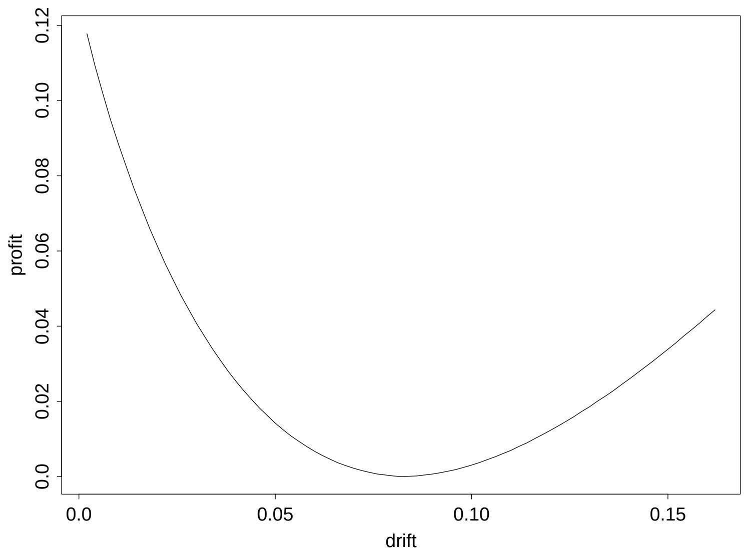

Figure 2: The effect of the pysical drift

Figure 3 is the issuer's expected profit as a function of the absolute risk aversion. The option holder's exercise time maximises the expected CARA utility, while the physical drift coincides with the risk-free rate 8% annum. Thus, the outcome is the marginal effect of the absolute risk aversion to the issuer's expected profit. Again, the option is an at-the-money American put, the initial asset price is 50, the asset volatility is 20% annum, the maturity is 6 months. If the absolute risk aversion vanishes and the physical drift and the risk-free rate coincide, then the utility maximising exercise boundary coincides with the price maximising one, and hence the expected profit vanishes.

5 Concluding Remarks

The theory of optimal stopping has been applied to the valuation of the American option. People are prone to use the terminologies of the theory of optimal stopping even when they talk about American options. For example, the price maximising exercise boundary has been referred to the optimal exercise boundary, while it is optimal to neither the issuer

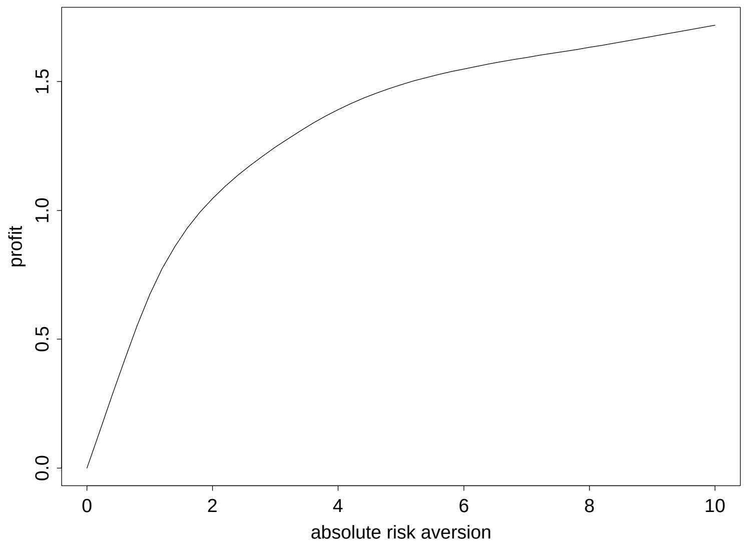

Figure 3: The effect of the absolute risk aversion

nor the option holder. This causes confusion to students, practitioners, and even academic researchers in the field. There are two obvious sources of confusion. First, financial software packages almost invariably value contracts, and find exercise strategies, from the option writer-hedger point of view. Rarely does it have anything to say about what is optimal for the contract holder. Some research papers even suggest that we price an American option by estimating the price maximising exercise boundary from the empirical data of exercise boundaries. Throughout this paper, we have explained why these are wrong. The holder of an American option should pursue his own profit maximisation and choose the right exercise time for himself apart from the price maximising exercise boundary, unless the last thing he wants to see is when the issuer gains. As the exercise times of the market participants are affected by the market direction, their risk aversion factors, and their financial structures, the estimate of the price maximising exercise boundary from the empirical data is not valid.

In this paper, we have assumed that the option holder maximises his expected utility while his strategy is rather academic: buy-and-hold the option until he decides to exercise. A more realistic set-up is to allow the option holder to sell his option to another investor and to construct a dynamic portfolio including the underlying asset. Nevertheless the results in this paper are not violating common senses.

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