A Model for the Value of a Business, Some Optimisation Problems in its Operating Procedures and the Valuation of its Debt

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Abstract

In this paper we present a model for the value of a firm based on observable variables and parameters: the annual turnover, the expenses, interest rates. This value is the solution of a parabolic partial differential equation. We show how the value of the company depends on its legal status such as its liability (i.e. whether it is a Limited Company or a sole trader/partnership). We give examples of how the operating procedures can be optimised (e.g. whether the firm should close down, relocate etc.). Finally, we show how the model can be used to value the debt issued by the firm.

1. Introduction

Many papers have modelled the value of a firm, and the classic reference texts on this subject are by Dixit & Pindyck (1994) and Copeland, Koller & Murrin (1990). Many others have modelled contingent claims on the value of the firm, including its debt. The latter often take as the starting point a stochastic model for the value of the firm. Examples of these are the work by Bensoussan, Crouhy & Galai (1994), Black & Cox (1976), Cooper & Mello (1991), Longstaff & Schwartz (1994) and Merton (1974, 1990). The last of these may be described as the most important in this field. All of these models assume that the firm's value V is given by a stochastic process such as

$$dV = mVdt + sVdX \qquad (1)$$

where μ and σ are usually constant and represent the drift and volatility of the value respectively. This assumption is unsatisfactory because of the difficulty in measuring V at any time, and in measuring the two parameters. In practice, the variable V is unobservable. Nevertheless, these models have proved popular for two reasons: first, their similarity to the acclaimed Black-Scholes theory for options gives them a certain credibility, and, second, there are often meaningful explicit solutions of the resulting parabolic partial differential equations. However, we believe that the problems of firm valuation, business strategy and valuation of debt need to be addressed at a more fundamental level. For this reason in this paper we present a model that has easily observed parameters and variables and is applicable to a number of problems.

Another common assumption in the literature is that there are no arbitrage opportunities and thus that the ideas of hedging and risk-neutral valuation may be applied. This is certainly not the case for the majority of businesses since they are not publicly owned and do not have traded shares. The assumption of risk-neutrality is often made by authors to escape the problem of having to value risk. In this paper we shall not assume no arbitrage, we will carry a real drift term through our analysis, and make several comments about the value of risk.

In this paper we model the earnings of the firm as a stochastic variable. The value of the firm then depends on these earnings, and on its expenses. To be precise, we assume that all income after expenses are deposited in a bank account earning a fixed rate of interest (we make a simple assumption about the effect of taxes). We then say that the value of the firm is equal to the present value of the expected cash in the bank on a certain date in the future. The need for a final date is a weakness shared by many models and is due to the difficulty in assigning a finite present value to an infinite stream of cashflows that increase faster than the rate of interest.

The value of the firm satisfies a partial differential equation in three variables, instantaneous earnings E, instantaneous cash in bank C and time t. The legal status of the firm and its operating procedures appear in the problem via boundary conditions. There are three types of conditions (i) at the end date, representing the extent of the liability of the firm to debt, (ii) on the plane $C = 0$, representing the possibility that the company could decide to close down (or be closed down by a bank, say) if it ever goes in the red and (iii) a constraint that the value of the firm must be above a certain level, representing optimal operating procedures of the firm, for example whether to relocate.

We then model the decision of the company to relocate its premises. Finally, we show how to value, in this framework, debt issued by the firm.

In Section 2 we motivate the modelling by giving a simple real-life example. In Section 3 we derive the governing partial differential equation. We describe the boundary and final conditions for this equation in Section 4. In Section 5 we describe one simple optimisation problem. In Section 6 we model the decision of whether to relocate the firm, and this also becomes an optimisation problem. The problem of valuing debt issued by the firm is discussed in Section 7.

2. A motivating case study

In any mathematical modelling problem it is always good practice to build up a model from the simplest foundations. It is rarely successful to try and combine all realistic features into a model from the start. For this reason, we motivate our work by describing the small publishing business Oxford Financial Press (OFP), its origins and its operating procedures.

OFP publishes the derivatives text *Option Pricing: Mathematical Models and Computation*. It was founded in mid-1993 and published *Option Pricing* in September that year. Books are sold mainly by mail order, but increasingly also by a few selected bookshops. Books are advertised in a variety of ways including direct mail. The costs associated with each sale have both fixed and variable elements. The former includes office rental and advertising. There are many sources of variable costs including postage, packaging, labour (paid on a commission basis), fees to credit card companies, bank charges for currency conversion to pounds sterling.

The sales of *Option Pricing* vary a great deal. There are good weeks with a large number of sales, and bad weeks with few. There is a large correlation between a new advertisement appearing and a surge in sales.

Although there are plans to expand the business, it currently has only one product. This product, while initially unique, will increasingly suffer from competition and the lifespan of the business will probably not extend beyond a few more years.

Since the business is only a small part of the activities of the partners, the profit has so far all been retained in the firm's bank accounts; it has not been used for expansion for example.

The above observations all play a role in the following model of a firm (see Jamie, 1994, for more details). This model will have applications in many situations, some of which we shall see here.

The first part of the modelling is to treat the annualised earnings of the firm, E, as a random quantity. We choose the following lognormal random walk as the traditional starting point, although, of course, this is highly arbitrary and artificial:

$$dE = mEdt + sEdX \ . \qquad (2)$$

The earnings/turnover is not the profit; the profit comes after expenses have been accounted for. Here we shall assume that the expenses are both variable, proportional to E, and fixed. The annualised expenses are thus

$$kE + E^* . \qquad (3)$$

The earnings less the expenses are invested in a bank, earning interest at a rate of r. If C is the balance in the current account at time t, then

$$C = \int_0^t \left((1-k)E(t) - E^*\right)e^{r(t-t)}dt \ . \qquad (4)$$

This assumes that the business began at time t=0.

Differentiating (4) we find that

$$dC = \left((1-k)E - E^* + rC\right)dt \ . \qquad (5)$$

Hence if $(1-k)E < E^*$ the balance in the current account will decrease.

The modelling of taxes is quite complicated but here we make a very simple assumption: the firm pays, in taxes, an amount proportional to their total profit if that profit is positive and nothing otherwise. Thus (5) becomes

$$dC = \left((1-k)E - E^{*} + rC - \alpha \max((1-k)E - E^{*} + rC, 0)\right)dt \,. \tag{6}$$

This is only an approximation to the effect of taxes, it does not allow taxes to be at discrete intervals (in the above, taxes are paid continuously) nor allow losses to be carried over.

Finally, we assume that the product ceases to sell at time $t=T$. (Alternatively, we could make the drift and volatility time dependent, tending to zero.)

3. The value of the firm as the solution of a partial differential equation

The simplest model for the worth of the firm at any time is the present value of the *expected* current account balance at time T:

$$V(E,C,t) = e^{r(t-T)}\mathrm{E}\left[C(T)\right]. \tag{7}$$

The discounting in this equation is at the risk-free interest rate r. (A case could easily be made for discounting at the risk-free rate adjusted for tax.)

From this definition of V, we find that it satisfies the Kolmogorov-type partial differential equation

$$\begin{aligned}\frac{\partial V}{\partial t} + \frac{\sigma^2 E^2}{2}\frac{\partial^2 V}{\partial E^2} + \mu E\frac{\partial V}{\partial E} + \\ \left((1-k)E - E^{*} + rC - \alpha \max((1-k)E - E^{*} + rC, 0)\right)\frac{\partial V}{\partial C} - rV = 0\end{aligned} \tag{8}$$

Observe that the drift term μ appears explicitly in this equation. This is in contrast with the Black-Scholes equation for the value of an option and is due, in our problem, to the absence of a traded instrument with which to hedge the risk (the randomness) in the firm's value.

At this point we comment on the rate of discounting and on the valuation of risk. The amount of cash in the bank at time T will be random. However, it is natural to examine the expected value of $C(T)$ and use this in valuing the firm. It is also natural to assign a lower value to a risky asset than to a riskless one with the same return. One way of adjusting the firm's value to allow for this is to discount the expected value at a rate higher than the risk-free rate. In other words, the final term in (8) could be replaced with $r+p$ where p is a risk premium. The question then is, what value to use for p? An obvious choice is to make it a constant, with a value chosen depending on the calculation to be made: different values may be chosen when calculating the value of the firm from when valuing debt. Alternatively, if the value of the firm or the debt has been calculated by some other means, the *implied* risk premium could be calculated (as is commonly done with traded bonds where the company/country might default). Finally, we could choose a p that depends on the amount of *instantaneous* risk. To find out what this is, we must examine the stochastic differential equation for V. This equation is

$$dV = rVdt + \sigma E\frac{\partial V}{\partial E}dX \,. \tag{9}$$

From this we see that the risk, as measured by the relative randomness in V, is

$$\frac{\sigma E \, \partial V / \partial E}{V}.$$

A possible, though complicated, choice for the risk premium p would be proportional to this risk.

(We note, in passing, that we will see, in all of the examples we consider, that

$$V(E,C,t) \sim f(t)E \qquad \text{as} \qquad E \to \infty$$

for some $f(t)$. So that when the firm's earnings are high (and provided C is not too small), i.e. when bankruptcy is unlikely, our model (9) approaches (1) but with a time-dependent random term. However, when there is a large likelihood of bankruptcy the models diverge, but it is at precisely such times that it is important to model the business accurately.)

To complete the specification of this first problem, we must impose boundary and final conditions.

Henceforth, we assume that there are no taxes, $\alpha = 0$.

4. Boundary and final conditions

One of the interesting aspects of the present modelling is the specification of boundary and final conditions for equation (8). These conditions depend crucially on the operating procedure of the firm's decision makers, and, where relevant, anyone from whom the firm has borrowed money.

The examples in this section are by no means the only possible cases.

4.1 Partnership, no restriction on borrowing

If the business is run as a partnership, meaning that the partners are liable for the firm's debts, and if there is no restriction on the business borrowing from the bank, then we can imagine a situation where the business continues running regardless of its success or otherwise. In other words, if there is a negative amount in the bank at any time then the business continues to run. The problem for V is specified for all C, both positive and negative.

The boundary conditions are

$$\frac{\partial^2 V}{\partial C^2} \to 0 \quad \text{as} \quad |C| \to \infty,$$

$$\frac{\partial^2 V}{\partial E^2} \to 0 \quad \text{as} \quad E \to \infty,$$

$$V(0,C,t) = \frac{E^*}{r}\left(e^{r(t-T)} - 1\right) + C$$

and $\quad V(E,C,T) = C.$

The problem is to be solved in

$$C \in (-\infty, \infty).$$

The solution of this is

$$V(E,C,t) = \frac{E^*}{r}\left(e^{r(t-T)} - 1\right) + C + \frac{(1-k)E}{r-m}\left(1 - e^{(r-m)(t-T)}\right)..$$

Note that this does not depend on the volatility of the earnings. This is obvious: take expectations inside the integral in Equation (4).

4.2 Partnership, current account always in the black

An alternative set of boundary conditions arises if the firm is still a partnership, but if the partners decide to call a halt to trading the moment that the current account balance falls to zero. The cash *C* is now constrained to be positive. The boundary and final conditions become

$$\frac{\partial^2 V}{\partial C^2} \to 0 \quad \text{as} \quad \mathrm{C} \to \infty,$$

$$\frac{\partial^2 V}{\partial E^2} \to 0 \quad \text{as} \quad E \to \infty,$$

$$V(0,C,t) = \begin{cases} 0 & \text{for } C \le -\dfrac{E^*}{r}\left(e^{r(t-T)} - 1\right) \\ \dfrac{E^*}{r}\left(e^{r(t-T)} - 1\right) + C & \text{for } C > -\dfrac{E^*}{r}\left(e^{r(t-T)} - 1\right). \end{cases}$$

$$V(E,0,t) = 0$$

and $\quad V(E,C,T) = C.$

The problem is to be solved in

$$C \in [0,\infty).$$

The solution of (8) with these conditions was found by a simple explicit finite-difference scheme. The solution for

$$E^* = 30{,}000 \qquad m = 10\ \% \qquad s = 25\% \qquad r = 5\% \qquad k = 7\% \qquad T - t = 10$$

is shown in Figure 1.

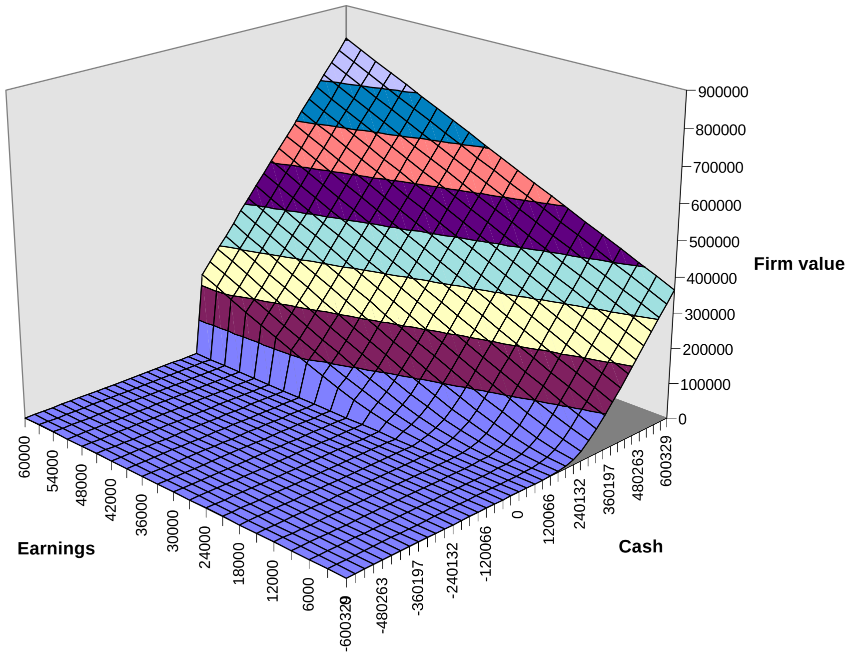

Figure 1

4.3 Limited liability

Another possibility is that the business is a limited liability company. In this case, the directors are not personally liable for any debt of the company (provided they have not acted negligently). We consider the case where the company is run regardless of the state of its current account. The conditions are

$$\frac{\partial^2 V}{\partial C^2} \to 0 \quad \text{as} \quad |C| \to \infty,$$

$$\frac{\partial^2 V}{\partial E^2} \to 0 \quad \text{as} \quad E \to \infty,$$

$$V(0,C,t) = \frac{E^*}{r}\left(e^{r(t-T)} - 1\right) + C$$

and $$V(E,C,T) = \max(C,0).$$

The problem is to be solved in

$$C \in (-\infty, \infty).$$

The only difference between these conditions and those in our first example is in the final condition, the value of the company cannot be negative. The solution of this problem with the same values for parameters and variables as above is shown in Figure 2.

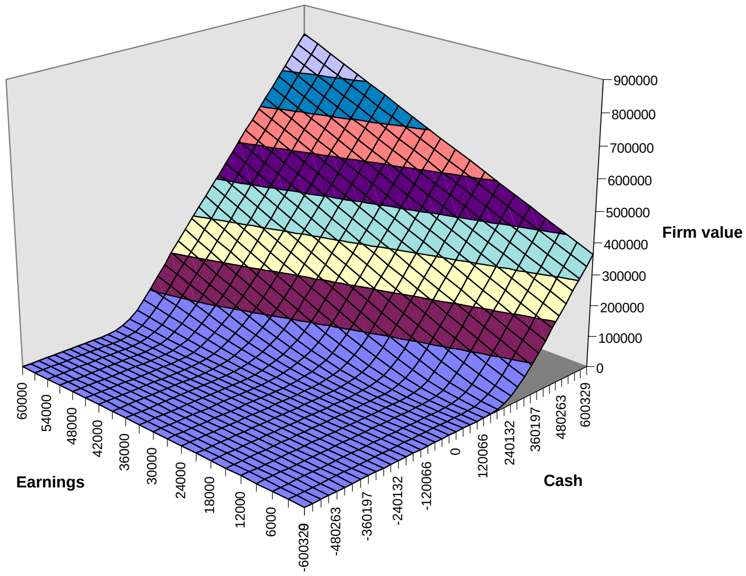

Figure 2

4.4 Grace period allowed by bank

Whatever the legal status of the business, the bank will not be too happy if it is overdrawn for any length of time. We can thus introduce a ‘grace period’. This is a period allowed by the bank, at the start of trading, during which the firm may be overdrawn. This short time gives the business breathing space in which to grow. If, however, the firm is still overdrawn at the end of this period then the bank can effectively close down the business.

The problem for *V* must be solved for all *C* while the grace period is in operation, but after this period it is only to be solved for positive *C*. The details of the rest of the boundary conditions will again depend on the legal liabilities of the business. We will not pursue this case further.

5. An optimal trading strategy: free boundary problem I

In the above we have restricted the control exerted by the firm’s decision makers to be based on an examination of the bank balance alone: either they trade regardless, or they close up shop when they go into the red. In this section we consider more sophisticated control strategies, based on simple optimisation procedures.

We have been presenting models for the value of the firm based on its random earnings. The value of the firm has been taken to be the present value of the expected cash held by the company at a future time *T*. It is possible in such a model (depending on boundary and final conditions and on parameter values) for this value to fall below the current cash balance *C*. In such a situation the model is telling us that we expect to lose money in the future. Following such a result, it would be natural for the firm to close down: why operate at all if you are not going to increase the value of the business?

We now introduce an optimisation problem into the running of the business. We wind up the firm the moment that its value falls to the amount in the bank. This is modelled by the constraint

$$V(E,C,t) \geq C. \qquad (10)$$

The value of the firm is maximised subject to this constraint by imposing continuity of the gradient of *V*.

This problem is mathematically almost identical to the early exercise free boundary problem for American options (see, for example, Wilmott, Dewynne & Howison, 1993). The free boundary in the present problem separates the region where the business continues to operate from the region where it is closed down. Applying this constraint to the problems in Section 4 will result in firm values that are no less than the value without the constraint, and may be significantly higher. The results below were all found by an explicit finite-difference method.

5.1 Partnership, value cannot be less than current cash balance

The addition of constraint (10), and smoothness, to the boundary and final conditions in Section 4.1 above results in a value for the firm which is greater than the value found in that section. A plot of the value is shown in Figure 3 for parameter and variable values used above. The effect on the value of the business can be seen in the portion of the graph where the earnings are low. For sufficiently small E we see that it is optimal to close down the business.

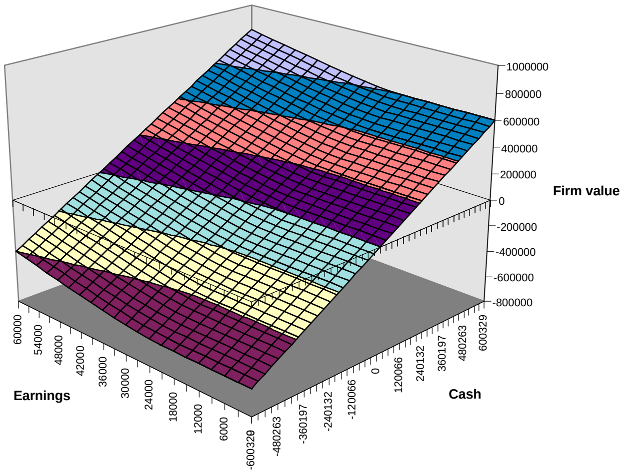

Figure 3

5.2 Limited liability, value cannot be less than current cash balance

We show, in Figure 4, results for the limited liability model of Section 4.3 with the added constraint (10).

6. Optimal relocation: free boundary problem II

In this section we describe an optimisation problem based on the above model for the value of a firm in which the firm has to decide whether to relocate its premises. There are many reasons why such a move might be favourable. There may be advantages in transport costs (of materials, product or labour), taxes (local), cost of materials (including fuel), rental of factory or hire of equipment, cost of labour, legal restrictions (age of workforce, employers pension contributions) etc. There will also be disadvantages to such a move, for example the building of a new factory, hiring and training a new workforce etc.

In order to put a value on the option to relocate in our framework we must introduce two functions, both representing the value of the company. These functions are $V_b(E,C,t)$ and $V_a(E,C,t)$ which are the value of the firm *before* relocating, having the option to relocate, and the value *after* relocating, respectively.

The difference between the firm's value before and after relocating will depend on the earnings and costs of business in the original location and after moving. We shall assume that the differences between the two locations an be represented by different parameter values: we use subscript '*b*' to denote the old values and '*a*' to denote the new. Furthermore, we shall assume that there is a fixed cost of relocation C^*, which must be paid on moving (this may be negative if, for example, there is a substantial amount made from the sale of the old premises).

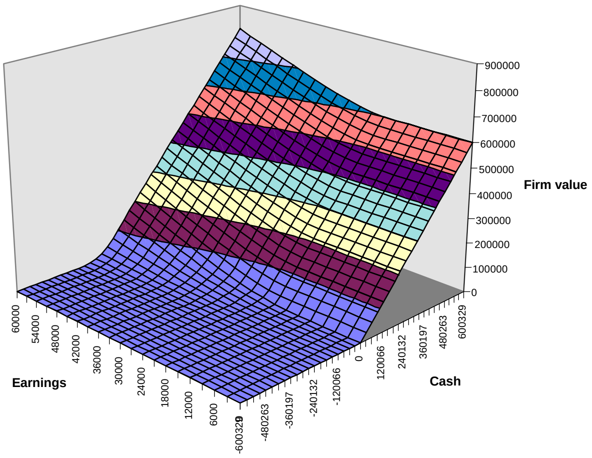

Figure 4

The first problem to solve is for $V_a(E,C,t)$. This problem is identical to the type presented in Sections 4 and 5 since this value contains no optionality. Thus we have

$$\frac{\partial V_a}{\partial t} + \frac{\sigma_a^2 E^2}{2}\frac{\partial^2 V_a}{\partial E^2} + \mu_a E \frac{\partial V_a}{\partial E} + \left((1-k_a)E - E_a^* + r_a C\right)\frac{\partial V_a}{\partial C} - r_a V_a = 0 . \qquad (11)$$

To this we must add final and boundary conditions and constraints depending on the operating procedures of the business and its legal status.

More interesting is the problem for $V_b(E,C,t)$, that is, the problem while the firm is in its original location having the option to relocate. The value of the firm in this case satisfies equation (9) but with all subscripts '*b*'. The ability to relocate the business is represented by the constraint

$$V_b(E,C,t) \geq V_a(E, C - C^*, t), \qquad (12)$$

together with smoothness. This inequality simply says that the value of the firm with the option to relocate is at least the value after relocating with allowance made for the cost of the relocation. The boundary between where there is equality in (12) and strict inequality marks the relocation boundary. The smoothness condition ensures that the relocation boundary is optimal, meaning that the value for the firm before relocation is maximised.

In Figure 5 we show the difference between the firm's value before and after relocation assuming it to be of limited liability (as in Section 4.3) and with the above parameter value for the 'before' case and for the 'after' case we have chosen $E^* = 10{,}000$. The cost of relocation $C^* = 100{,}000$. This example shows the play off between the lower expenses in the new location and the one-off cost of moving. There is a significant advantage in moving if the earnings are low but there is a substantial amount saved. If the time horizon is increased then the move becomes increasingly favourable since the cost of relocation becomes less significant than the lower expenses.

Any of the boundary and final conditions and constraints described in Sections 4 and 5 could be used together with constraint (12).

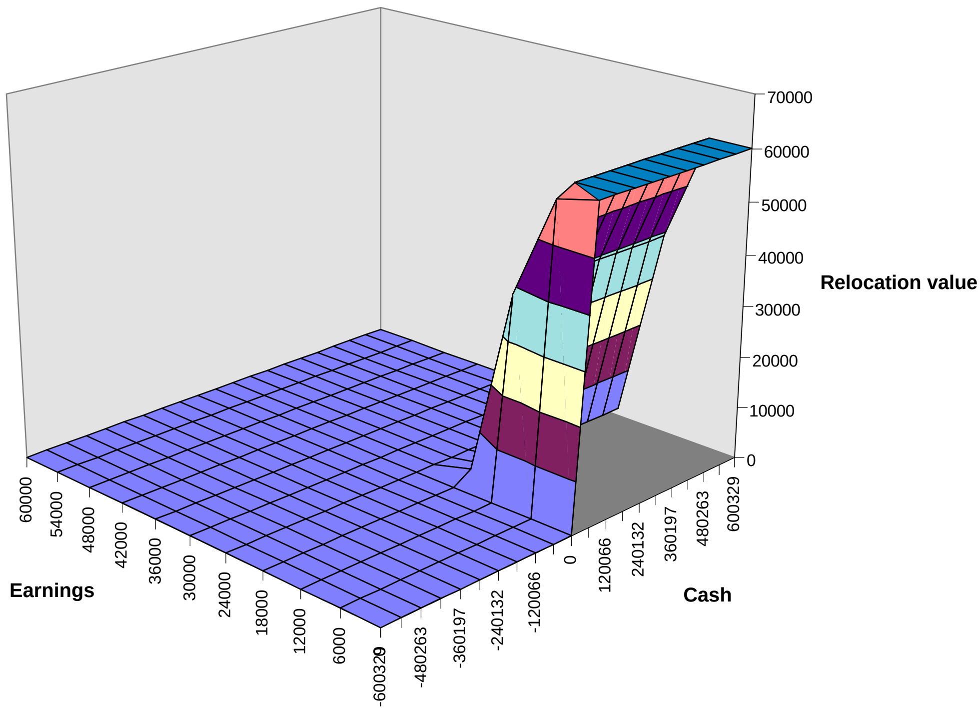

Figure 5

7. Valuing debt

In this section we use the present model to put a value on a loan to the business (see Georgikopoulos, 1995, for more details). We shall assume that the firm has limited liability as in Section 4.3, but any of the other models could be used.

Suppose that the business must repay an amount D at time T_D. We make the simple assumption that if the firm has D in the bank at this time then it is repaid. If it has less than this then it repays as much as it has. If there is a negative amount in the bank then it repays nothing. This gives a repayment at time T_D of

$$\max(\min(C,D),0)\,.$$

We shall assume that the value of the loan at time t is equal to the present value (discounting at rate r) of the expected repayment. We call this quantity $W(E,C,t)$ and it satisfies

$$\frac{\partial W}{\partial t}+\frac{\sigma^2E^2}{2}\frac{\partial^2 W}{\partial E^2}+\mu E\frac{\partial W}{\partial E}+\left((1-k)E-E^*+rC\right)\frac{\partial W}{\partial C}-rW=0\,, \tag{13}$$

with

$$W(E,C,T_D)=\max(\min(C,D),0)\,.$$

In Figure 6 we show results for a loan of 100,000 to be repaid in three years' time.

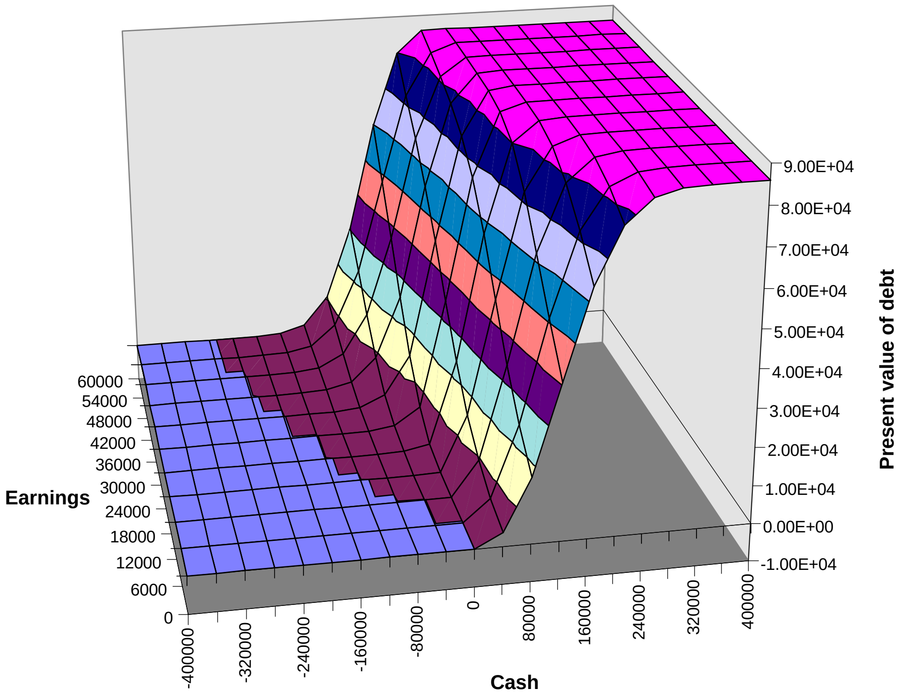

Figure 6

8. Conclusions

In this paper we have presented a simple model for the value of a firm using observable quantities as the variables and parameters. We have shown how the extent of liability can be represented as boundary conditions for a partial differential equation. We have shown how the firm can optimise its value by deciding when, if ever, to close down the business. The ideas were then applied to the question of whether or not the company should relocate. Finally, we used the model to value the debt of the firm.

The models can be extended in many ways which we hope to explore in future work. Possible directions for this work are the modelling of a merger between two companies (Kim, 1995), whether friendly or hostile, and the decision whether or not a small business should register for VAT.

9. Acknowledgement

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