Crash Modelling, Value at Risk and Optimal Hedging

by

Philip Hua (Bankers Trust and Imperial College, London)
Paul Wilmott (Oxford University and Imperial College, London)

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Addresses for communication: Paul Wilmott
Mathematical Institute
24-29 St Giles'
Oxford
OX1 3LB

pw@oxfordfinancial.co.uk

Tel/fax: 44 (0) 1483 32647
Mobile: 44 (0) 802 898092

Abstract

In this paper we present a new model for pricing and hedging a portfolio of derivatives that takes into account the effect of an extreme movement in the underlying. We make no assumptions about the timing of this 'crash' or the probability distribution of its size, except that we put an upper bound on the latter. The pricing and hedging follow from the assumption that the worst scenario actually happens i.e. the size and time of the crash are such as to give the option its worst value. The optimal static hedge follows from the desire to make the best of this worst value. There are many applications for this crash modelling, we shall focus on using the model to evaluate the Value at Risk for a portfolio of options.

Introduction

The true business of a financial institution is to manage risk. While the term ‘Value at risk’ (VAR) may have become familiar, its definition and its measurement differ significantly at different institutional levels. A book runner manages his market risk using the Black-Scholes framework where the value at risk from his derivatives transactions may be mitigated by having a zero exposure to various ‘greeks’. At an institutional level, risk management concerns creating a consistent framework to assess and communicate the maximum loss under some predefined ‘worst scenarios’ over a specific time horizon. Unfortunately it requires significant effort to compute the maximum losses for a derivatives portfolio using the traders’ risk data.

The rationale for the two different languages is obvious. The trader manages ‘normal event’ risk, where the world operates much closer to a Black-Scholes one of random walks and dynamic (delta) hedging. The institution, however, views its portfolio on a ‘big picture’ scale and focuses on ‘tail events’ where liquidity and large jumps are important (Figure 1).

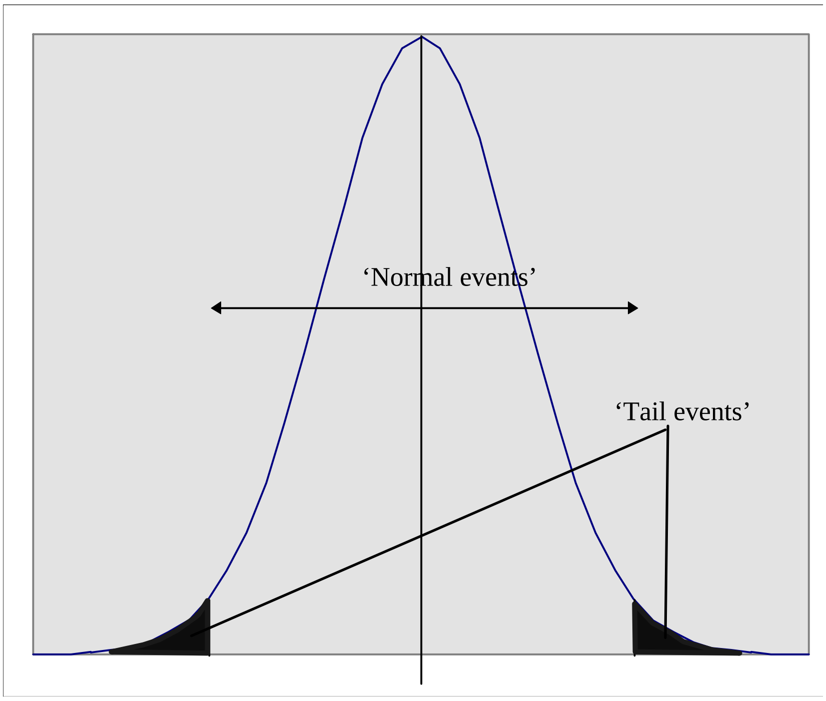

Figure 1: ‘Normal events’ and ‘tail events’

It is the latter meaning that the current generation of VAR models have attempted to model, although their implementations differ, depending mostly on the nature of the business, the regulatory requirements and the sophistication of the technology infrastructure. The criticisms of this type of VAR modelling have been well documented in *Risk* magazine. In general, the simplifying assumptions made by the existing VAR models do not adequately capture the nuances of exotic and complex structure trades over time, and can lead to substantial underestimation of the risks. On the other hand, because of cancellation effects and the inherent ‘nonlinearity’ of VAR measurement, the risks can be overestimated!

A marriage of the definitions will give a useful tool to both book runners and senior management. A true measure of the risk in a portfolio will answer the question ‘What is the value of a crash to my portfolio?’

The approach taken here in finding the value at risk for a portfolio is to model the cost to a portfolio of a crash in the underlying. The advantage of this bottom up VAR modelling is to allow us to accurately value the cost of a crash and also to find an optimal static hedge to minimise this cost and so reduce the value at risk.

Mathematical modelling

The main idea in the following model is simple. *We assume that the worst will happen*. We value all contracts assuming this, and then, unless we are very unlucky and the worst *does* happen, we will be pleasantly surprised. In this context, 'pleasantly surprised' means that we make more money than we expected. To start with we value any path-independent option portfolio in this framework and later we show how to make the 'worst' less bad. (The final optimisation problem is inspired by the work of Avellaneda & Paras.)

The binomial + crash model
We will model the underlying asset price behaviour as the classical binomial tree, but with the addition of a third state, corresponding to a large movement in the asset. So, really, we have a trinomial walk but with the lowest branch being to a significantly more distant asset value. The up and down diffusive branches are modelled in the usual binomial fashion (see Wilmott, Dewynne & Howison, 1993). For simplicity, we will assume that the crash, when it happens, is from S to $(1-k)S$ with k given; this assumption can easily be dropped to allow k to cover a range of values, or even to allow a dramatic *rise* in the value of the underlying. We introduce the subscript 'b' to denote values of the option/portfolio *before* the crash and 'a' to denote values *after*. Thus V_a is the value of the option position after the crash. This is a function of S and t and, since we are only permitting one crash, V_a must be exactly the Black-Scholes option value.

As shown in Figures 2 and 3, the underlying asset which starts at value S (point O) can go to one of three values: S^u, if the asset rises, S^d, if the asset falls or $(1-k)S$, if there is a crash. These three points are denoted by A, B and C respectively. The values for S^u and S^d are chosen in the usual manner for the traditional binomial model.

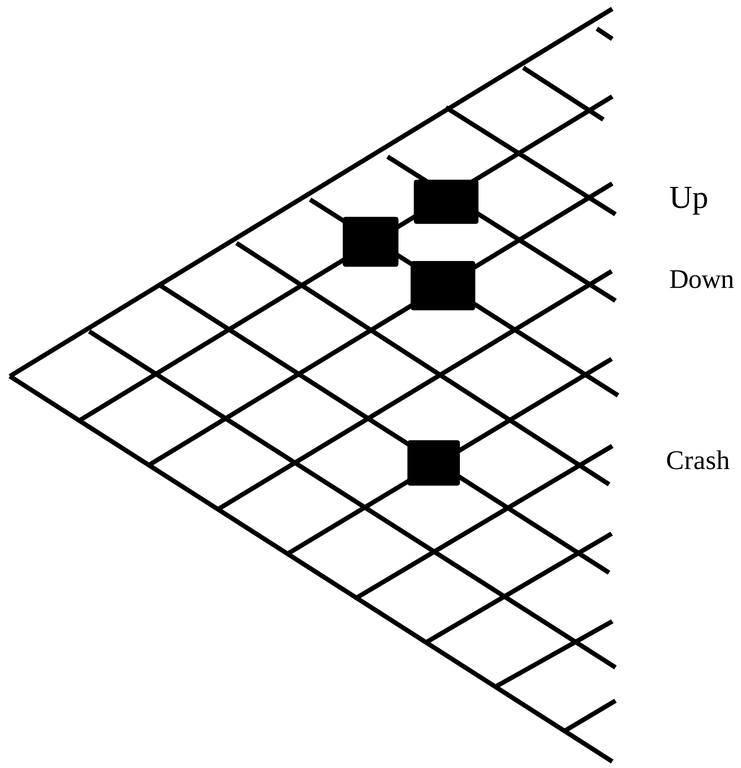

Figure 2: The tree structure

Before the asset price moves, we set up a 'hedged' portfolio, consisting of our option position and $-D$ of the underlying asset. At this time our portfolio has value V_b. We must find both an optimal D and then V_b.

A time dt later the asset value has moved to one of the three states, A, B or C and at the same time the option value becomes either V_b^u (for state A), V_b^d (for state B) or the Black-Scholes value V_a (for state C).

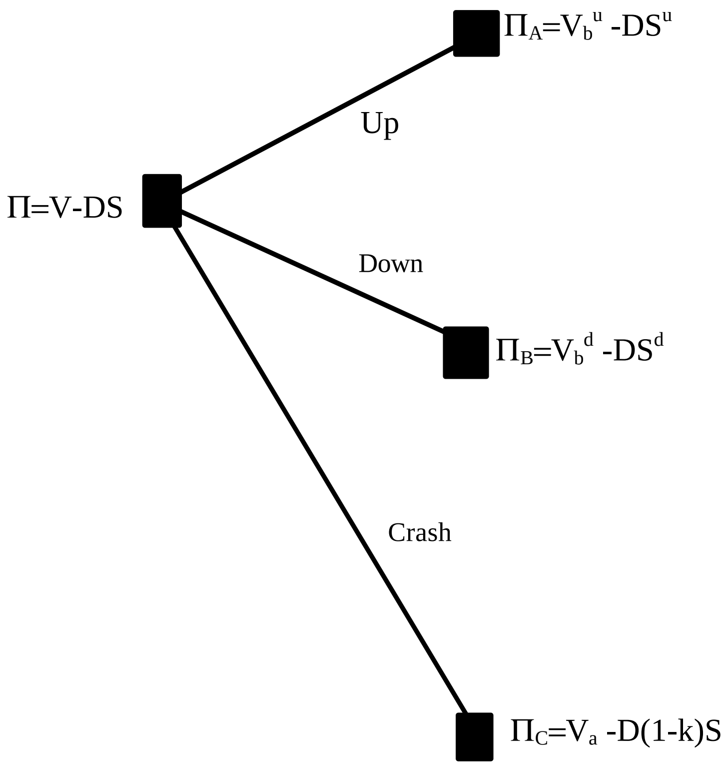

Figure 3: The tree and portfolio values

The change in the value of the portfolio, between times t and $t+dt$ (denoted by $d\Pi$) is given by the following expressions for the three possible states:

A (diffusive rise): $d\Pi_A = V_b^u - DS^u + DS - V_b$

B (diffusive fall): $d\Pi_B = V_b^d - DS^d + DS - V_b$

C (crash): $d\Pi_C = V_a + DkS - V_b$

These three functions are plotted against D in Figures 4 and 5. We are going to choose the hedge ratio D so as to maximise the pessimistic, worst outcome among the three possible.

There are two cases to consider, shown in Figures 4 and 5. The former, Case I, is when the worst case scenario is *not* the crash but is the simple diffusive movement of S. In this case V_a is sufficiently large for a crash to be *beneficial*:

$$V_a \geq V_b^u + (S - S^u - kS)\frac{\left(V_b^u - V_b^d\right)}{S^u - S^d}. \qquad (1)$$

If V_a is smaller than this, then the worst scenario is a crash; this is Case II.

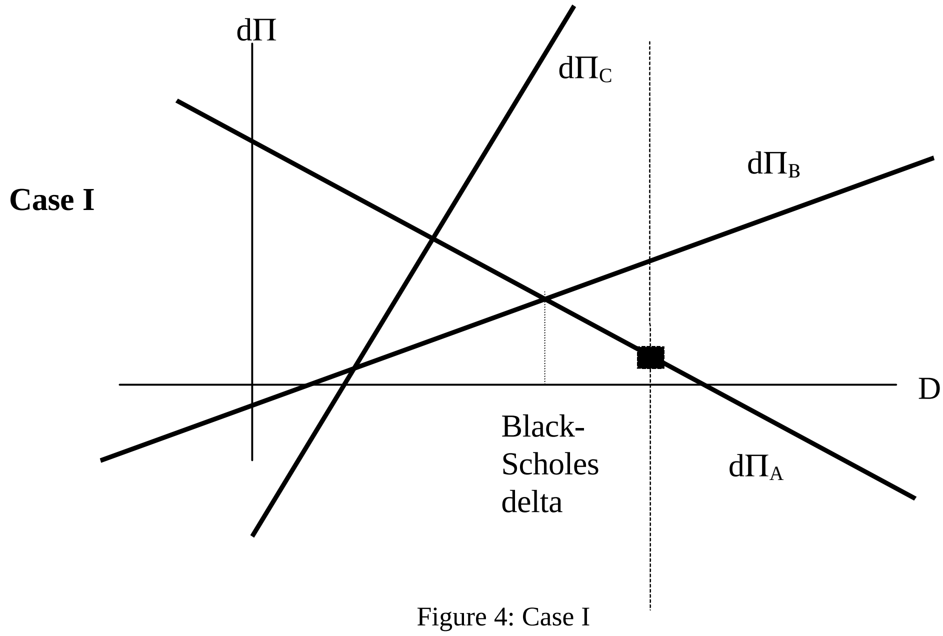

Figure 4: Case I

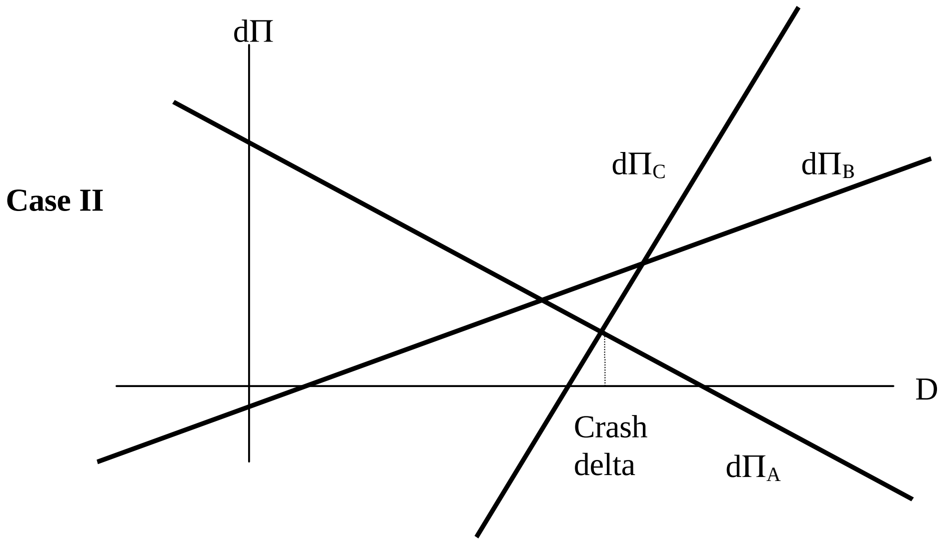

Figure 5: Case II

Case I: Black-Scholes hedging

Refer to Figure 4. In this figure we see the three lines representing $d\Pi$ for each of the moves to A, B and C. Pick a value for the hedge ratio D (for example, see the dashed vertical line in Figure 4), and determine on which of the three lines lies the worst possible value for $d\Pi$ (in the example in the figure, the point is the black square and lies on the A line). Change your value of D to maximise this worst value.

In this case the maximal-lowest value for $d\Pi$ occurs at the point where

$$d\Pi_A = d\Pi_B \quad ,$$

that is $$D = \frac{V_b^u - V_b^d}{S^u - S^d} .$$

(This will be recognised as the expression for the hedge ratio in a Black-Scholes world.)

Having chosen *D*, we now determine V_b by setting the return on the portfolio equal to the risk-free interest rate. Thus we set

$$d\Pi_A = r\Pi dt$$

to get

$$V_b = \frac{1}{1+rdt}\left(V_b^u + (S - S^u + rSdt)\frac{\left(V_b^u - V_b^d\right)}{S^u - S^d} \right) . \qquad (2)$$

This is the equation to solve if we are in Case I. Note that it corresponds exactly to the usual binomial version of the Black-Scholes equation, there is no mention of the value of the portfolio at the point C. As *dt* goes to zero, Equation (2) becomes the Black-Scholes partial differential equation.

Case II: Crash hedging

Refer to Figure 5. In this case the value for V_a is low enough for a crash to give the lowest value for the jump in the portfolio. We therefore choose *D* to maximise this worst case. Thus we choose

$$d\Pi_A = d\Pi_C \quad ,$$

that is, $$D = \frac{V_a - V_b^u}{S - S^u - kS} . \qquad (3)$$

Now set

$$d\Pi_A = r\Pi dt$$

to get

$$V_b = \frac{1}{1+rdt}\left(V_a + S(k + rdt)\frac{\left(V_a - V_b^u\right)}{S - S^u - kS} \right) . \qquad (4)$$

This is the equation to solve when we are in Case II. Note that this is different from the usual binomial equation, and does not give the Black-Scholes pde as *dt* goes to zero. Also (3) is not the Black-Scholes delta. To appreciate that delta hedging is not necessarily optimal, consider the simple example of the butterfly spread. If the butterfly spread is delta hedged on the right ‘wing’ of the butterfly, where the delta is negative, a large fall in the underlying will result in a large loss from the hedge, whereas the loss in the butterfly spread will be relatively small. This could result in a negative value for a contract, even though its payoff is everywhere positive!

Examples

All that remains to be done is to solve equations (2) and (4) (which one is valid at any asset value and at any point in time depends on whether or not (1) is satisfied). This is easily done by working backwards down the tree from expiry in the usual bi/trinomial fashion.

For our example, let us examine the cost of a 15% crash on a portfolio consisting of:

C/P	Strike	Expiry	Bid	Ask	Quantity
c	100	75 days			-3
c	80	75 days			2
c	90	75 days	11.2	12.0	0

Table 1: The available contracts and the initial position

(For the moment, the bid-ask prices will not concern us.) The volatility of the underlying is 17.5% and the risk-free interest rate is 6%.

We have arbitrarily chosen a crash of 15%, although, in practice, one would relate the crash size to the volatility of the underlying and a time horizon.

The solution to our problem is shown in Figure 6.

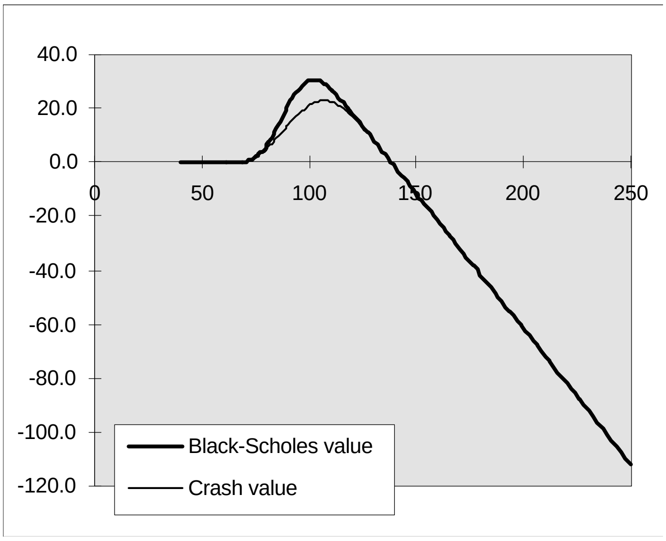

Figure 6: Example showing Crash value and Black-Scholes value

Observe how the value of the portfolio assuming the worst (21.2 when the spot is 100), is lower than the Black-Scholes value (30.5). This is especially clear where the portfolio's gamma is highly negative. This is because when the gamma is positive, a crash is beneficial to the portfolio's value. When the gamma is close to zero, the delta hedge is very accurate and we are insensitive to a crash. If the asset price is currently 100, the difference between the before and after portfolio values is 30.5-21.2 = 9.3. This is the value at risk under the worst-case scenario.

Optimal static hedging: VAR reduction

The 9.3 value at risk is due to the negative gamma around the asset price of 100. An obvious hedging strategy that will offset some of this risk is to buy some positive gamma as a 'static' hedge. In other words, we want to buy an option or options having a counterbalancing effect on the value at risk. We are willing to pay a premium for such an option i.e. we will pay more than the Black-Scholes fair value for such a static hedge because of the extra benefit that it gives us in reducing our exposure to a crash. Moreover, if we have a choice of contracts with which to statically hedge we should buy the most 'efficient' one. To see what this means consider the above example in more detail.

Recall that the value of our initial portfolio under the worst-case scenario is 21.2. How many of the 90 Calls should we buy (for 12) or sell (for 11.2) to make the best of this scenario? Suppose that we buy λ of these Calls. We will now find the optimal value for λ.

The cost of this hedge is

$$\lambda C(\lambda)$$

where $C(\lambda)$ is 12 if λ is positive and 11.2 otherwise. Now solve Equations (2) and (4) with the final total payoffs

$$V_a(S,T) = V_b(S,T) = 2\max(S-80,0) - 3\max(S-100,0) + \lambda\max(S-90,0) .$$

This is the payoff at time T for the statically hedged portfolio.

The net value of our original portfolio (that is, the portfolio of the 80 and 100 Calls) is therefore

$$V_b(100,0) - \lambda C(\lambda) \qquad (5)$$

i.e. the worst-case value for the new portfolio less the cost of the static hedge. The arguments of the before-crash option value are 100 and 0 because they are today's asset value and date. The optimality in this hedge arises when one chooses the quantity λ to maximise the value (5).[1]

With the bid-ask spread in the 90 Calls as given in Table 1, we find that buying 3.5 of the Calls maximises expression (5). The components of the optimally hedged portfolio are given in Table 2. The value of the new portfolio is 70.7 in a Black-Scholes world and 65.0 under our worst-case scenario. The value at risk has been reduced from 9.3 to 5.7. The optimal portfolio values before and after the crash are shown in Figure 7.

C/P	Strike	Expiry	Bid	Ask	Quantity
c	100	75 days			-3
c	80	75 days			2
c	90	75 days	11.2	12	3.5

Table 2: The optimally hedged portfolio

[1] This optimal hedging is identical in spirit to that of Avellaneda & Paras (1996). They were considering the hedging of portfolios assuming uncertain volatility.

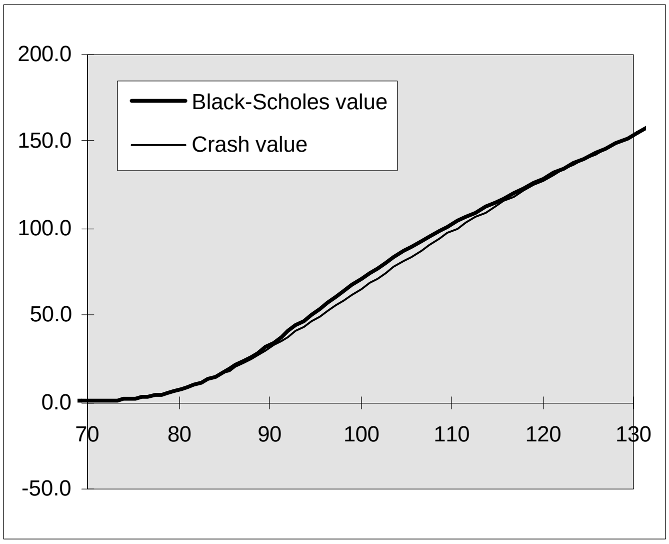

Figure 7: Optimally hedged portfolio, before and after crash

Conclusion

We have presented a model for the effect of an extreme market movement on the value of portfolios of derivative products. This can be related to Value at risk. In particular, we have shown how to employ static hedging to minimise this VAR. In conclusion, we note that the above is not a jump diffusion model since we have deliberately not specified any probability distribution for the size or the timing of the jump: we model the worst-case scenario.

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Philip Hua is Vice President for Global Risk Management at Bankers Trust, London, and a doctoral student at Imperial College, London.

Paul Wilmott is a Royal Society University Research Fellow in the Mathematical Institute, Oxford, and the Department of Mathematics, Imperial College, London. He is an author of *Option Pricing*, an Editor-in-chief and founder of *Applied Mathematical Finance*, and researcher in many branches of finance. For more information, see www.oxfordfinancial.co.uk.