The Pricing of Risky Bonds: Current Models and Future Directions

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1 Introduction

The modelling of credit risk, credit derivatives and non-hedgeable securities in general, is currently in a poor state. Ideas from equity option theory have been *adopted* for credit risk, but have not been *adapted* for the peculiarities of this more complex world.

This brief paper is a review and critique of current ideas and models, and includes suggestions for a more sophisticated, realistic and ultimately more sensible approach. The bibliography at the end should prove a useful source for the current state of the art.

2 Traditional approaches to credit-risk modelling

Before discussing credit risk issues, we will mention the importance of the Black–Scholes theory for derivatives since this theory is currently being applied to risky securities of all kinds.

The Black–Scholes model is used by everyone working in derivatives, whether they are salesmen, traders or quants. It is used confidently in situations for which it was not designed, often successfully. The ideas of delta hedging and risk-neutral pricing have taken a formidable grip on the minds of academics and practitioners alike. In many ways, especially with regards to commercial success, the Black–Scholes model is remarkably robust. But the assumptions underlying the theory are many and various. The one that will concern us here is that options can be continuously hedged with the underlying asset. This is almost never the case with risky bonds. Hedging amounts to balancing random price movements in one contract with random, but perfectly correlated, price movements in some other asset. But what if you own a contract which is not perfectly correlated with anything else?

The typical risky bond is issued by a company as a means of borrowing money. The bond is called 'risky' because there is often a reasonable chance that the issuing firm will go broke before it is due to pay back the borrowed principal. If this happens the bond holder will not be too pleased, and will have lost much or even all of his investment. This may not matter if fluctuations in the price of the bond can be hedged with fluctuations in some other contract. But what if the issuing company has issued no other bonds, and does not even have traded stock? It is perfectly possible, even common, for there to be nothing with which to hedge. How then can we justify the use of Black–Scholes-type theories and models? We would argue that we can't.

The currently popular models for credit risk fall roughly into the two categories of 'firm value models' and 'instantaneous risk of default models.'

3 Firm value models

Historically, the first attempt at modelling risk of default was via the modelling of the value of the firm. Typically, one starts with some stochastic differential equation for the value of the firm

$$dV = \mu V \; dt + \sigma V \; dX$$

(a lognormal random walk is perennially popular) and makes statements about the likelihood of the firm going bankrupt. Bankruptcy is usually represented by the firm value hitting some lower bound V_b, say. Should V reach this level, default is triggered. We will not write down the governing partial differential equation (it is a backward parabolic equation) since it is very easy to criticise this model. How do you measure V at any time? How do you estimate the parameters, μ, σ and V_b?

A better approach, and one that involves more easily measured variables and parameters, is to model the earnings of the firm:

$$dE = \mu E \; dt + \sigma E \; dX$$

(or whatever one believes to be realistic). From a model for these earnings together with estimates for the running costs of the business, one can estimate the income stream and thus the cash balance, C:

$$C = \int_0^t (E - E^*) e^{r(t-\tau)} d\tau.$$

Here E^* are the firm's expenses and r the interest rate for deposited cash. Bankruptcy of the firm is now represented by, for example, a negative bank balance, although many, many other models are possible depending on the legal structure of the firm (limited or unlimited liability) and its relationship with its bank. One can estimate the variables and parameters for this model from an analysis of the firm's accounts. This approach, although it makes a great deal of sense, has not proved popular. Practitioners are naturally unwilling to go to the required lengths of estimation.

4 Instantaneous risk of default models

The most popular models of the moment, probably because they require the least thought to implement and are borrowed directly from the Black–Scholes world, are the instantaneous risk of default models. In these models default is triggered by some completely exogenous event, usually governed by a Poisson process. There is thus no attempt made to relate the likelihood of default to the performance of the firm issuing the bond.

If default is triggered by a Poisson event with intensity p then the *expected* value of a risky zero-coupon bond, $V(t)$, when interest rates are constant, satisfies

$$\frac{dV}{dt} - (r + p)V = 0$$

with

$$V(T) = 1$$

if the bond receives \$1 at time T.

Does it make sense to use an expectation of the payoff as a value? Probably not for such an extreme, bimodal, payoff that you get with a risky bond i.e. all or nothing. If it were possible to hedge this bond with another that is *subject to exactly the same default event* then we enter the world of risk-neutral pricing. Is that more satisfactory? Again no, unless we can model the market price of default risk.

Although this approach is clearly naive and unrealistic, it is still commonly used. Moreover, the faults that are obvious in this simple model are obscured by the complexities that are often added. For example, sometimes the model is combined with a stochastic interest rate model. The 'pricing' of risky bonds is then via a backward parabolic partial differential equation, reassuringly similar to the Black–Scholes equation.

Fashionable at the moment is parameter fitting or calibration. This amounts to choosing model parameters in such a way that theoretical prices output from a model match exactly the prices of contracts quoted in the market. To facilitate this fitting it obviously helps to have enough parameters to play around with. To this end one sometimes sees stochastic risk of default models such as

$$dp = \alpha\, dt + \beta\, dX.$$

Again we are in the 'safe' world of the diffusion equation.

5 Utility-based pricing

If today's models are deficient, what does the future hold in store for credit risk modelling? The answer is apparent if one looks at personal experience.

We will offer the reader of this paper a bet on the outcome of a single coin toss. Heads we win 10 cents, tails you win 10 cents. Anyone with the slightest spirit of adventure will accept this bet. Now we offer the same bet but with \$1000 riding on the outcome. Will you still accept it?[1] Few people would accept this gamble even though the expected outcome is exactly the same as in the first bet.

One way of examining investors' attitudes towards risk is via utility functions. The utility function is a function of a person's wealth, will vary from investor to another, and represents the value one assigns to an amount of money. The simplest example is to consider the marginal personal value of £100 to a UK academic, and the marginal value of the same amount to Bill Gates.

Clearly utility functions should be nonlinear in wealth. Interestingly, a linear utility function, which sees the £100 as being equally important to the academic as to Bill Gates,

[1] At this point we must stress that our offer is purely hypothetical!

returns us to the world of expected values and the easily-criticised models described above. Now we know why they are unsatisfactory.

Denote our wealth by W and our utility of wealth by $U(W)$ and ask questions about our *expected utility*. The expected utility is far more meaningful that the simple expected payoff. In other words, given some random outcomes, of a bet for example, and the probablity of these outcomes we can examine the quantity

$$E[U(W)].$$

Here W will range over all the outcomes that are possible. In our first bet example W will be either $+0.1$ or -0.1.

How can we compare bets or investments? One way is to compare their utilities and invest in the one with the greatest expected utility. Another way is to determine for each investment the 'certainty equivalent.' This quantity, call it V, solves

$$U(V) = E[U(W)]$$

and is the certain amount we would exchange for the uncertain investment.

We will not go into the details in this brief article but simply quote the relevant risky bond pricing equation and make a few comments. When interest rates are constant the certainty equivalent price of the risky bond is given by the solution of

$$\frac{dV}{dt} - \left(rV + p\frac{U(V)}{\frac{dU}{dV}} \right) = 0, \tag{1}$$

where p is still the risk of default, with

$$V(T) = 1.$$

6 Consequences

The nonlinearity of the governing equation (1) is extremely important. Some of the consequences are summarised here.

- Different prices for contracts depending on what else is in the portfolio. If there are other instruments with the same default exposure we can statically hedge one with the other.
- Any static hedging can be optimised to give any contract the highest possible value.
- People can disagree on the price of a contract. There is no such thing as a fair value for any risky contract.
- There will generally be a range of values for the contract that make it appealing to either buy or sell.
- For a given market price of a contract there will be an optimal quantity that one would like to hold.

As will be clear from this article, the above thoughts represent 'work in progress.' For more up-to-date information see www.maths.ox.ac.uk/mfg.

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