THE VALUE OF MARKET RESEARCH WHEN A FIRM IS LEARNING: REAL OPTION PRICING AND OPTIMAL FILTERING.

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Abstract

In this paper we model the value to a firm of undertaking market research into a particular product opportunity. The way in which information about the potential of the project arrives and knowledge evolves during the life of the project is modelled using the theory of optimal filtering. The value of the project and optimal entrance decision rule is then derived as the solution to a partial differential equation, using boundary conditions which reflect the structure of the project.

1. Introduction

This paper is motivated by the well-established use of option pricing methods in evaluating firms' market opportunities. The key reference in this area is Dixit and Pindyck (1994). Among other valuation and control problems, they model the value to a firm of the option to invest in a product at a later date, given that the intrinsic value of the project is uncertain. Other authors have evaluated R&D projects and investment opportunities under conditions of uncertainty. Grossman and Shapiro (1986) derive the optimal time path of R&D outlays when the difficulty of a project is unknown and the benefits from it only accrue once it is completed. Brennan and Schwartz (1985) use option pricing to value a copper mine as a derivative claim on future revenues. In more general models, McDonald and Siegel (1986) value a "perpetual option" offered to a firm by the right to launch a product at some later date, and Newton and Pearson (1994) discuss the general application of option pricing theory to R&D valuation.

Such real option pricing models are shown by Quigg (1993) to have strong empirical support. She shows that a model of land values which includes the option value of waiting to develop a site has significant predictive power for transactions and prices over and above simple intrinsic value. Empirical work by Pakes (1986) has also looked at the option value of the right to a later investment offered by a patent, and demonstrated evidence for a large amount of inherent option value.

The crucial factor in the specification of a model of the value of any derivative claim is the specification of the process followed by the underlying variable on which the derivative claim rests. In exact analogy to financial option models, in all the real option models the underlying value of the investment opportunity is assumed to follow a constant variance process; either a geometric random walk or a simple mean-reverting process.

In many of these models the firm can only decide to invest at one later date (in analogy to European options). While these models capture the value of the increase in information until the exercise date they are unrealistic in assuming that there is only one possible and fixed time to invest which cannot be chosen by the management.

The time to invest was endogenised in later models in which the firm can observe how the potential value of a project evolves, and then lock into an especially favourable value, similar to the early exercise of American options. In these models the firm's management is always perfectly informed about the current value of the state variable, and the option to wait derives its value from the firm's ability to wait for an even more favourable value, and not from information acquisition by the management.

This is undesirable for many projects, as it implies that the firm does not learn about the actual level or process for intrinsic value with any greater certainty. In some cases admittedly the firm can actually observe the underlying variable on which the value of its project rests. In many real world situations however this is not true: a product may have a certain but unobservable value, or its prospects may depend on a variable whose process is not observable. During the time for which the firm is investigating such a product, it seems reasonable that a firm would be learning about the processes involved, so that the uncertainty inherent in its environment is not constant, while the choice of time to invest should be endogenous.

Models of firms' activities, in which the stochastic environment is not given but can be influenced, have been developed before. Cukierman (1980) demonstrates a model in which a firm decides which of a range of investment projects to undertake, and "buys" information by waiting, reducing the probability of losses from launching an unsuccessful product. Demers (1991) has a model of a firm which is uncertain about the demand in a particular market, and updates its beliefs according to a Bayesian rule after receiving informational signals. In both cases, the firm learns about its environment with greater accuracy as time goes by.

It would seem worthwhile to combine these two aspects - learning about the true state of the economy and endogenous timing of the investment - in a single approach. This was first undertaken by Roberts and Weitzman (1981), who model the value of an R&D project and entry/exit boundaries as information develops. Our approach is more general than that taken by Roberts and Weitzman. The learning and its intensity are endogeneised, and we do not assume a finite time horizon for the R&D project.

Here, we aim to introduce information acquisition in a real-option framework. In the model we present, it is assumed that the firm does not know the payoff of the project under investigation (which may even be negative). It can however observe a costly, noisy signal of this payoff. Given this signal the firm has three choices: it can start the project, continue collecting signals and it can abandon the project altogether and stop collecting signals. The second *option to investigate* the project is valuable because it allows the firm to reduce the risk of undertaking a project with negative payoff.

Drawing on the theory of optimal filtering of noisy signals, the process followed by the firm's best estimate of a project's value is derived. This is then regarded as the underlying variable for a model of the value of market research along option pricing lines. The general result, apparent from considering the more general real-option pricing literature and indeed suggested in a different context by Bernanke (1983), is that the presence of a learning effect decreases option values over and above NPV.

In this modelling framework, we also consider the situation in which the firm has some control over the intensity of its signals: it can control the amount of information it buys over time. The firm's problem now begins to look like the two-armed bandit problem studied by Rothschild (1974) and Bellman (1956). The firm must decide whether to continue its project, and at what pace, or to enter the market, or drop out completely. It must therefore decide which of three "slot-machines" offer the most favourable outcome, given that obtaining more information is expensive. This analogy is not however exploited here. It is not intended to build a complete theory of optimal research behaviour, but to model this control problem specifically within the real-option pricing framework.

The structure of the paper is as follows. First, we discuss the general framework of optimal filtering. This is then applied to the valuation of market research projects. Finally, a situation in which the firm can control the "intensity" of its market research is considered, and the optimality condition derived.

2.Optimal Filtering

A great deal of consideration has been given in physics and engineering to the processing of noisy signals, to situations in which we are concerned with the value of a process which we cannot observe directly. A key example would be in tuning a radio, which receives signals which have been distorted by various other factors.

The theory of optimal filtering is one approach taken to such a problem, deriving the optimal contemporaneous estimate in a continuous time framework. The optimality criterion used in this case is to formulate the best (minimum variance) estimate of the unobserved system. The ideas in this section are clearly not new; the treatment is based on Arnold (1974) and Oksendal (1992). The basic model and some results are discussed heuristically in one dimension, using stochastic differential equations. An interpretation is also given in discrete time. Finally, it is worth mentioning that it is assumed that all key model parameters are known. The situation in which they are not is that of “adaptive filtering” (see for example Goodwin and Sin (1984)); such a situation adds little to the analysis described here.

<u>The optimal filtering model.</u>

In the general model, there is some magnitude *V* which is of interest to us, but which we cannot observe directly. We can only observe a noisy signal of *V*. Our problem is to estimate the current state of the “system” *V* on the basis of the observations we have.

At the highest level of generality, we can model our observations *H(t)* as a disturbed functional of *V*. There are then two components to this model. The first is a systematic function of *V*. The second is pure noise. This is assumed for simplicity to be white noise; extensions to coloured noise have been made (see eg. Brown and Hwang (1992)). Thus we can write the model as

(1) $$H(t) = m(V,t) + s\,(t)e \quad ,$$

where m represents the mapping from *(V,t)* to the real line, $s\,(t)$ is a variance parameter, and e is assumed to be white noise. With no loss of information, we can define the “signal representation” *Z(t)* as the cumulative past history of our observations:

(2) $$Z_t = \int_0^t H_s ds \, .$$

This then allows us to specify the stochastic differential equation which this signal must satisfy:

(3) $$dZ = m(V,t)dt + s\,(t)dW \, ,$$

where dW is a Wiener process, $Z(0) = Z_0$ given and $t \geq 0$. To enable results to be couched in terms of stochastic differential equations, the signal process is used from now on.

We now have a model for the signal we observe; to allow for cases in which the system variable also changes over time, we need a general model for the evolution of V. The model suggested is the generalised Ito process

(4) $\quad dV = a(V,t)dt + x(V,t)dX$,

where dX is a simple Wiener process, $V(0) = V_0$ and we assume that dX, dW, V_0 and Z_0 are all independent.

<u>A special case.</u>

The most famous special case of the filtering problem is the Kalman-Bucy theorem for linear systems (see Bucy and Joseph (1968), Kalman (1963)). Linearity in this application is defined as being "in the narrow sense", which boils down to the variance of the processes involved being free of V and Z. This will be important in the next section.

In this special case, we model the system with the equation

(5) $\quad dV = aVdt + xdX$,

and the signal process by the equation

(6) $\quad dZ = mVdt + sdW$.

Given the white noise assumption, it can be shown (see Oksendal (1992))that our optimal estimate of V is the mean of the conditional distribution of V, conditional on the set of observations $Z[t_0,t]$, ie.

(7) $\quad \hat{V}(t) = E[V(t) \mid \mathrm{Z}[\mathrm{t}_0,\mathrm{t}]]$.

Furthermore, if *V(0)* and *Z(0)* are normally distributed or constant, it can be shown that *V* and therefore *Z* are Gaussian processes. Therefore all conditional distributions are normal.

It therefore follows that the conditional density $f_v(Z[t_0,t])$ of *V* on the basis of our observations is a normal distribution with mean $\hat{V}$ and conditional variance

(8) $$S_t = E\left[\left(V_t - \hat{V}_t\right)^2 \middle| Z[t_0, t]\right].$$

The Kalman-Bucy theorem states that the dynamic equations for the updating of these parameters as new information arrives are

(9) $$d\hat{V} = a\hat{V}dt + S\frac{m}{s^2}\left(dZ - mVdt\right)$$, and

(10) $$\frac{dS}{dt} = 2aS + x^2 - S^2\frac{m^2}{s^2}.$$

The second equation is a deterministic Riccati equation. *S* is independent of the observations $Z[t_0,t]$, and for a non-negative initial value, it can be shown that there exists a unique global solution for *S* (see Bucy and Joseph (1968)). This equation for *S* clearly shows the emergence of a "learning effect": the precision of the optimal estimate increases over time.

<u>The Kalman-Bucy filter as the limit of a discrete time problem.</u>

To illustrate the workings of the model, consider a discrete time example. Given an initial guess $\hat{V} = E[V]$ with variance s_V^2, and a signal *H*=*V*+*e*, where *e* is the noise term, and is assumed to be white noise. Define the "precisions" $p_V = s_V^{-2}$ and $p_e = s_e^{-2}$. We are looking for the best linear approximation of *V* given our current information $\hat{V}$ and *H*. In other words, we are looking for the minimising arguments *a*, *b* such that

$$E[(a\hat{V} + bH - V)^2]$$

is minimised. Expanding, we get

$$\begin{aligned}
E[(a\hat{V} + bH - V)^2] &= E[(a\hat{V} + be - (1-b)V)^2] \\
&= E[(a\hat{V} + be)^2] - 2E[(a\hat{V} + be)(1-b)V] + (1-b)^2 E[V^2] \\
&= a^2\hat{V}^2 + b^2 s_e^2 - 2a(1-b)\hat{V}^2 + (1-b)^2 s_V^2 + (1-b)^2\hat{V}^2 \\
&= (a - (1-b))^2\hat{V}^2 + b^2 s_e^2 + (1-b)^2 s_V^2 .
\end{aligned}$$

The first order conditions are then

$$\begin{aligned}
0 &= a - (1 - b) \\
0 &= b s_e^2 - (1-b) s_V^2 .
\end{aligned}$$

The first condition means that we will have a weighted average of $\hat{V}$ and H in the new estimate $\hat{V}$[1]; substituting into the second condition, the first condition implies that

$$\begin{aligned}
b &= \frac{p_e}{p_V + p_e} \\
a &= \frac{p_V}{p_V + p_e} .
\end{aligned}$$

This gives a new best estimate $\hat{V}' = \frac{p_V}{p_V + p_e}\hat{V} + \frac{p_e}{p_V + p_e}H$, which has a precision of $p' = p_V + p_e$ (by substitution into the original equation).

The above analysis is now heuristically extended to continuous time. For simplicity, suppose the system process is given by $dV=0$. Also, suppose that signals are of the simple form

(11) $dZ = mVdt + s\,dW$, with parameters defined as before.

The estimate $\hat{V}$, its variance $S(t)$ and its precision $p(t)$ are given, and we want to update them given dZ. Write (heuristically) for the observation H,

(12) $H \equiv \frac{1}{m}\frac{dZ}{dt} = V + \frac{s}{m}\frac{dW}{dt}$, where the error e now has the variance $s_e^2 = \frac{s^2}{m^2}\frac{1}{dt}$

and the

precision $p_e = \frac{m^2}{s^2}dt$. The precision is therefore well defined as we move into continuous time, whereas the variance is not. The problem is now identical to the discrete time example, and we can substitute in directly to obtain the weights

(13) $a = 1 - \frac{Sm^2}{s^2}dt + O(dt^2)$, and

(14) b = $\frac{Sm^2}{s^2}dt + O(dt^2)$.

Finally, we reach

(15) $\hat{V}' - \hat{V} = d\hat{V} = -\frac{Sm^2}{s^2}\hat{V}dt + \frac{Sm}{s^2}dZ$,

(16) $\frac{dp}{dt} = \frac{m^2}{s^2}$ and

(17) $\frac{dS}{dt} = -\frac{m^2}{s^2}S^2$.

These equations are in exact agreement with the Kalman-Bucy filtering formulae, (9) and (10).

Thus the optimal filtering theorems represented by (9) and (10) can be seen to be simply the continuous time analogues of finding the best linear estimator of an unobserved V given a discrete set of observations. As mentioned above, under the Gaussian assumption, it can be shown (see eg. Brown and Hwang) that the Kalman-

Bucy theorem finds the conditional mean of V as well as the best linear estimator; in more general cases, it finds the best estimator, but not necessarily the conditional mean. This fact means that in order to obtain the most robust results from the use of the Kalman-Bucy theorem, we should try to model at the level of a process which satisfies this assumption: a stochastic process which is linear "in the narrow sense", and is subject to a Gaussian noise process. This will become a consideration in the next section, when a normalising transform is used to work with a variable which we would rather not assume to be normally distributed. In this next section, the theory of optimal filtering is used to model the evolution of a firm's best estimate of product value, to represent an underlying state variable for a model of the value of market research.

3. Modelling the value of market research.

This section applies the theory of optimal filtering to the valuation of market research. We assume that a firm can enter a market and receive an amount V for a fixed entry cost I . The firm is assumed to be risk neutral. Prior to entering the market, the firm cannot observe V directly, but must estimate it on the basis of a noisy market research signal. The entry cost I is known. The firm must decide whether to continue with the costly market research, enter the market, or drop the project completely. Given that its optimal estimate of V is changing over time, this flexibility offers it a certain real option value.

The optimal estimate of market value.

We make four key assumptions made in our model of market research. Firstly, that there is a deterministic component to such a signal which is proportional to the true value of the market opportunity. This corresponds to an average “reply rate” of those surveyed. We might suppose that 50% of people who are sent a market research for reply, or that 90% of people who will respond to a drug will do so during testing. Therefore if we send out 100 market research forms and receive 50 back, of which 30 are favourable, we would use 60% as our estimate of potential market share. In addition, it is assumed for simplicity that there is just one true value V; that is, $dV=0$.

Thirdly, the signal process Z observed by a firm undertaking market research is assumed to be noisy. As well as the component correlated with V, there is a disturbance term. Not exactly 50% of people, or whatever figure generally prevails, will reply at any given time.

Lastly, it is assumed that the signal Z is distributed lognormally. This is assumed because Z is a function of V, the actual value of the product. Given that V cannot be less than zero, it seems that replies to market research cannot be below zero, since this implies a respondent assigning a negative willingness to buy to the product. Moreover, the empirical literature on the returns to market research (in terms of product sales) suggests that a lognormal distribution should be appropriate (see for example Garbrowski (1991) and Garbrowski and Vernon (1983), or Sanders, Rossman and Harris (1958)). Garbrowski (1991) demonstrates the stylized fact that pharmaceutical companies undertaking R&D have tended to rely on the occasional “blockbuster” product to support a large mass which perform very poorly, while Scherer (1958) shows that the empirical distribution of the data from Sanders at al. implies a strongly skewed and leptokurtic distribution, with a possibly unbounded mean; he suggests a Pareto-Levy distribution.

The Paretian distribution is not hugely useful for modelling purposes, but the assumption that market research will be lognormally distributed at least reflects the skew and kurtosis implied by the data. It also avoids the situation in which our estimate of V is negative; which it never can be, given that the information that $V \geq 0$ is contained within our information set. In reality, if we were actually estimating V on the basis of Z, this non-negativity constraint on V would not matter, since we would be very unlikely to obtain a negative estimate of V for a given realisation of the signal process. However, given that we want in fact to solve an equation over all possible ranges of variables, from a modelling point of view we want to ensure that variables are within the correct ranges. The assumption of lognormal signals does just this.

Since the Kalman-Bucy theorems are couched in terms of normally distributed variables, we define our "variables of interest" as $v =: \ln V$, and $z =: \ln Z$. These assumptions then give rise to a model for the modified market research signal of the form

(18) $dz = mvdt + s\,dW$,

where m is the average reply rate, s is the continuous variance of the noise, and dW is a Wiener process. From standard results in probability theory, it must be recognised that $\hat{V}$ is not simply $\hat{V} = e^{\hat{v}}$, but

(19) $\hat{V}(t) = \exp\left(\hat{v}(t) + \frac{1}{2}S(t)\right)$. For the moment, we will regard the problem as being framed in terms of $\hat{v}$.

Lastly, it is assumed that when starting a project, a firm knows “almost nothing”. This is represented by the assumption that S_0^{-1} can be approximated by 0. This assumption is made in order to remove the influence of initial S on the model; it can easily be modified, and indeed must be for the optimal control model presented later.

The value of market research.

In the model presented in this section, it is assumed that the firm must maintain a constant continuous payment flow $m \geq 0$ in order to receive its market research signals, allowing $m=0$ as a special case. For cases where $m>0$, we have a problem with the same structure as a perpetual instalment American option. We must pay to keep the option open, but can terminate the option to receive either nothing, or any positive intrinsic value of the market opportunity. In other cases, we have a simple perpetual American-style option valuation problem. We write the value of the firm's real option *F*.

Assuming for simplicity that the actual value of the product is fixed, we are working with the system

(20) $dv = 0 \, , \, v_t = v_0$.

We have assumed that the firm observes the process

$$dz = mvdt + s dW .$$

By the Kalman-Bucy theorem of optimal linear filtering, the optimal estimate $\hat{v}$ of v satisfies

(21) $$d\hat{v} = -\frac{m^2}{s^2} S\hat{v}dt + \frac{m}{s^2} dz \quad ,$$

and the instantaneous variance S of $\hat{v}$ satisfies

(22) $$\frac{dS}{dt} = -S^2 \frac{m^2}{s^2} \quad .$$

This equation for S has a straightforward solution

(23) $$S_t = \frac{s^2 S_0}{s^2 + S_0 m^2 t} \quad .$$

Divide (23) through by S_0 to get

(24) $$S_t = \frac{s^2}{s^2 / S_0 + m^2 t}\ .$$

Initially, S is assumed to be close to infinity: the firm knows almost nothing. S_0^{-1} is therefore approximated by 0. Under this assumption, the first term in the denominator disappears, so that S is given by

(25) $$S_t = \frac{s^2}{m^2 t}\ .$$

This shows how the variance of the error of the optimal estimate of V increases with time, from an initial state of almost zero knowledge.

Substituting the equation for S back into that for $\hat{v}$, it follows that we can now specify the process for $\hat{v}$ as being

(26) $$d\hat{v} = \left(\frac{v}{t} - \frac{\hat{v}}{t}\right)dt + \frac{s}{mt}dW \quad .$$

If we take the expectation of $d\hat{v}$ conditional on our information set (the observations *Z[….]*), it is clear that $d\hat{v}$ has zero drift with respect to this information set. All updates to our estimate of $\hat{v}$ are unpredictable. This corresponds to the fact that if an estimate of an unmoving magnitude required a predictable update at a later date, we would simply include that update now. This follows as a necessary condition of forming an optimal estimate.

With $\hat{V}$ is defined in (19) above in terms of $\hat{v}$, it can be verified directly using Ito's Lemma that $\hat{V}$ satisfies the stochastic differential equation

(27) $$d\hat{V} = \left(\frac{v}{t} - \frac{\hat{v}}{t}\right)\hat{V}dt + \frac{s}{mt}\hat{V}dW \quad ,$$

and we have a lognormally distributed estimate of V, with an expected drift of zero. The updates to $\hat{V}$ are therefore also an innovation process with respect to our information set. With the model (27) for the evolution of $\hat{V}$, we can now regard the problem as being in terms of *V* once more.

Given that we cannot observe the ultimate state variable for the option valuation problem, *V*, the quantity $\hat{V}$ will have to be regarded as being the state variable on which the value of the market research "option" depends. With this in mind, the value of the project is defined as the expected present discounted value of its final or intrinsic value, discounting at an appropriate discount rate r. At the end of the project (assuming an optimal choice of termination date *T**), we pay *I* to receive the expected value of the project, $\hat{V}$. So we define the option value *F* as

(28) $F(\hat{V},t) \equiv E\left[F\left(\hat{V},T^*\right)e^{-r(T^*-t)}\right]$, where

(29) $F(\hat{V},T^*) = \max(\hat{V} - I, 0)$.

Time must be included in this problem, since the volatility on which option value depends is some function of time. Actual calendar time must be included, not just some relative time variable, as there is a singularity in *S* at *t=0*. Condition (29) describes the firm's optimal decision at termination time. It can either enter the market and receive *V-I*, or abandon the project (with expected payoff *0*). Notice that as the firm bases its decision on expected *V*, but on entry receives actual *V*, it is still possible that the firm could enter a market and make a loss.

We are dealing with an optimal stopping problem: the firm is currently undertaking market research, and must determine when it would be optimal to stop, either to enter the market or to abandon the project completely. There are three regions for which we must describe the behaviour of *F*, corresponding to the trade-offs the firm faces. In the "out" stopping region, the firm determines that its estimate of product value is so low that it is not worth continuing to pay to hold open its real option in the hope of favourable later information. The option value from waiting is not sufficient to justify the necessary expenditure. In the "in" stopping region, the firm decides to enter the market, since its estimate of product value sufficiently high to make waiting for further information undesirable. The option value of waiting is outweighed by the estimated profits foregone by not entering now.

In the third region, the firm simply waits for more information about the project, paying a sum (assumed to be the constant flow *m*) to hold its option open. In this region, the option of investing in the project is estimated as being sufficiently valuable to warrant continuing (costly) investigation, but not so lucrative that entering now would be optimal.

In specifying the valuation problem, it is simply observed that *F* can never fall to below its intrinsic value $\Lambda = \max(\hat{V} - I, 0)$. If it ever did, by terminating its option by entering or abandoning, the firm could reach a higher payoff, and the optimal stopping decision implied by *F* would be sub-optimal. Therefore the free boundaries are included in the problem simply by imposing the conditions $F \geq 0$, and $F \geq \hat{V} - I$ on the equation for *F*. This equation is similarly specified as a weak inequality, and a complementary slackness condition invoked.

If these conditions hold with equality, the trade-off between the costs and benefits of waiting swings in favour of stopping, to receive the estimated intrinsic value of the project. The loci where this is the case are the free boundaries which the firm determines as the outcome of its optimal stopping problem.

It follows from the imposition of these conditions that the value of this real option to the firm must satisfy

(30) $$F(\hat{V}, t) = \max(\hat{V} - I, (1 + r\,dt)^{-1} E[F + dF] - m\,dt, 0)$$

over a time-step dt. We know what F is in the “in” and “out” stopping regions. In the interesting region, the continuation region, the equation satisfied by F can again be found straightforwardly. In this region,

$$(31)\qquad (1+rdt)F = E[F(\hat{V}+d\hat{V},t+dt)]-mdt$$

must hold. This implies via an Ito expansion inside the expectation that

$$(32)\qquad (1+rdt)F = E\left[F+\frac{\partial F}{\partial t}dt+\frac{\partial F}{\partial \hat{V}}d\hat{V}+\frac{1}{2}\frac{\partial^2 F}{\partial \hat{V}^2}(d\hat{V})^2\right]-mdt\ .$$

Evaluating the expectation, this implies that

$$(33)\qquad rFdt = \frac{\partial F}{\partial t}dt+\frac{1}{2}\frac{\sigma^2}{m^2 t^2}\hat{V}^2\frac{\partial^2 F}{\partial \hat{V}^2}dt-mdt\quad .$$

Dividing by dt and rearranging, F is found to satisfy

$$(34)\qquad \frac{\partial F}{\partial t}+\frac{1}{2}\frac{\sigma^2}{m^2 t^2}\hat{V}^2\frac{\partial^2 F}{\partial \hat{V}^2}-rF-m=0\quad .$$

This equation describes the process followed by any derivative claims on V which are dependent on the expected value of V. Since we cannot observe V itself, such a derivative claim must regard $d\hat{V}$ as the underlying process.

In evaluating market research, (34) is therefore to be solved under the condition

$$(35)\qquad F \geq \Lambda = \max(\hat{V}-I,0)\,.$$

When (35) holds with equality, (34) is only satisfied as an inequality. This can be confirmed simply by observing that the intrinsic value or “payoff” Λ is not a solution to (34).

This equation, with time included, cannot be solved analytically under these conditions and must be solved by numerical techniques. One initial analytic result is however apparent from a simple change of the time variable. If we define t such that

$\frac{t^2}{dt} = \frac{1}{dt}$, ie. t =*-1/t* , then equation (34) becomes

$$(36) \qquad \frac{¶F}{¶t} + \frac{1}{2}\frac{s^2\hat{V}^2}{m^2}\frac{¶^2 F}{¶\hat{V}^2} - t^2 rF - t^2 m = 0 .$$

Imposing the condition (35) means that the problem now looks like a perpetual US instalment option, with time-dependent instalment and discount factor. As $t \to 0$, or $t \to -\infty$, the discount and cost terms disappear, and it seems that under these conditions, the option value of the project $F \to V$. However, this is somewhat misleading, as in approaching a truly infinite initial variance, the probability of being at any particular value of *V* goes to zero. Thus it could also be argued that, given an infinite diffusion of information, initial option values are everywhere infinite.

Numerical results.

The numerical solution of equation (34) subject to condition (35) is now considered. It is in fact difficult to solve numerically for two reasons. The first reason is that it is a perpetual option, whereas to solve the backwards equation requires some terminal data. In order to supply this, we impose a terminal condition on the valuation problem. The motivation for this is that we can simply say that once the uncertainty surrounding our option has come down to a certain threshold, we will regard the problem as deterministic. In other words, there is some value of *S* which means that we are no longer interested in the diffusion term in the equation for *F*. At this point, we (the firm) will either be in or out of the market for definite. All we do by waiting is incur the time cost of discounting the option value into a subsequent period, and the payment flow m necessary to take it there. This would not be optimal. At some time

t^*, S is below our threshold level of "caring" S^*, and so we define a "final condition" as:

(37) $\quad F(\hat{V}, t^*) = \max(\hat{V} - I, 0) \quad .$

As a matter of mathematical necessity, two side conditions are also imposed which accord with the situation modelled are also imposed, namely

(38) $\quad F(0,t) = 0$

(39) $\quad \dfrac{\P^2 F}{\P \hat{V}^2} \longrightarrow 0 \text{ as } \hat{V} \longrightarrow \infty .$

These then allow a solution to the equation to be derived.

The second difficulty with deriving a solution to the equation is somewhat more technical and derives from the finite difference solution method used. This method works by approximating the differential equation for F by a difference equation, which is then solved over a "mesh" of points in the t and $\hat{V}$ dimensions.

When t is large and S small, this mesh must be very fine over $\hat{V}$ in order to pick up the small diffusion term. Conversely, as we move back towards the singularity in S, the increments in option value stepping back through time become very large, so that the grid must be very fine over t to avoid instability. A solution was therefore derived by "patching together" grids of different resolution at various points. The solution was not extended completely back to $t=0$, as at this point S is infinite. The solution begins only after a "tick" of time has taken place.

Option values at various points in time are shown in *figure 1* at the end of the paper, with parameter values $m = 20$, $s = 0.3$, $m = 1$, $r = 6\%$ and $I = 100$. *Figure 1* clearly shows the "learning effect". Option values diminish over time, moving down towards the intrinsic value (which is also NPV) as time goes on.

The effect of learning on the free boundaries is demonstrated in *Figure 2*. Early on, there is a great deal of uncertainty and a large amount of value in waiting. For most estimates of $\hat{V}$, the firm should continue to wait for more information before committing itself to a decision. As the uncertainty falls however, projects with very low $\hat{V}$ will be terminated and high $\hat{V}$ entered, as their true intrinsic value becomes increasingly clear and large changes become less and less likely.

The numerical results show clearly that if a firm is learning about the prospects of a particular product, the option value of research into the product declines over time. This is to be contrasted with the standard real-option result based on a constant variance model, in which option values are independent of time, and show no tendency to converge to NPV.

The next section extends this model of market research further. The effects of changing *m* are analysed, and an optimal control problem for situations in which the firm can control the intensity or quality of its information via controlling *m* is solved.

4. Varying the intensity of market research.

An interesting extension of this model is where the amount a firm has to pay for its market research has an influence on the characteristics of the market research itself. For example, suppose that each market researcher out in the street costs one pound. Therefore by spending *m* pounds continuously, we are receiving *m* signals. Each signal is again assumed to be distributed lognormally, and it is assumed that the signal z= *ln Z* has a deterministic component which is a linear function of *v=ln V* and a pure Gaussian noise component. So each signal received from each market researcher satisfies

(40) $dz_i = mvdt + s\,dW_i$

We assume that $E(dW_i dW_j) = 0$ for $i \neq j$. We receive *m* of these signals, and therefore assuming all market researchers are identical, the overall observation $z = \sum_{i=1}^{,} dz_i$ which we obtain satisfies

(41) $$dz = mmvdt + s\sqrt{m}dW \quad .$$

There are *m* times as many replies on average in the aggregate signal process as there are in each observation itself; and the adding together of *m* independent normal processes with instantaneous variance s gives an instantaneous variance for the final constructed process of $\sqrt{m}$ times the original variance.

<u>The value of market research for different intensities.</u>

Following the initial method for describing the value of this option to the firm, initially assuming *m* to be fixed, it follows by the Kalman-Bucy theorem once more that if the outside system is

(42) $$dv = 0 \quad ,$$

and we observe

(43) $$dz = mmvdt + s\sqrt{m}dW \quad ,$$

then the optimal estimate $\hat{v}$ of *v* satisfies the stochastic differential equation

(44) $$d\hat{v} = -m\frac{m^2}{s^2}S\hat{v}dt + \frac{m}{s^2}Sdz \quad ,$$

and *S* now satisfies

(45) $$\frac{dS}{dt} = -m\frac{m^2}{s^2}S^2 \quad .$$

Solving for *S* again and substituting into the equation for $\hat{v}$, it is found that $\hat{v}$ satisfies

(46) $$d\hat{v} = \left(\frac{v}{t} - \frac{\hat{v}}{t}\right)dt + \frac{s}{mt\sqrt{m}}dW \quad .$$

By equation (19) we can again write the process followed by the optimal estimate of V, $\hat{V}$, as

(47) $$d\hat{V} = \left(\frac{v}{t} - \frac{\hat{v}}{t}\right)\hat{V}dt + \frac{s}{mt\sqrt{m}}\hat{V}dW .$$

The option value across the three possible regions (in, out, continue), defined as before, is again

(48) $$F_t = max(V\text{-}I, \left(1 + r\,dt\right)^{-1}\left(E\left[F + dF\right] - mdt\right), 0) \ .$$

Therefore using the same method as before in the continuation region, it follows that F satisfies

(49) $$\frac{\partial F}{\partial t} + \frac{1}{2}\frac{s^2}{m^2 t^2 m}\hat{V}^2\frac{\partial^2 F}{\partial \hat{V}^2} - rF - m = 0 \ .$$

The effect of changing m is clear from this equation. A higher m increases costs, but decreases variance. Two clear results emerge if this equation is solved for different but fixed values of m, subject to the same condition reflecting the structure of the project, and by assertion ensuring optimal imposition of the free boundaries:

(50) $F \geq max(V\text{-}I , 0)$

The first is that a higher m brings forward an earlier optimal entry time for given $\hat{V}$. The results from numerical solutions are shown in *figure 3;* parameter values are as before. Here, the number of time steps before the free boundaries first appear in the numerical solution is plotted for different values of m. The second result is that at any given time, from an initial state of S close to infinity, a higher m gives a lower option value. This is shown in *figure 4.* Option values over and above intrinsic value are plotted for the same parameter values, after 0.05 years, again for different m. Both these results stem from the same phenomenon, that a higher m means that S decreases more rapidly, giving a lower diffusion term. This explains both the low option value, and the early entry, which occurs because of the low option value.

Controlling the intensity of market research.

If we now assume that a firm is able to vary its level of market research, there is an interesting optimal control problem. However, in order to consider this problem, we need to alter the set-up of the model. The first change is to remove the assumption of initial variance *S* being sufficiently close to infinity that we can approximate S_0^{-1} by zero. This is in order to remove an apparent paradox. If a firm literally knows nothing about a project, then with the infinite diffusion of information, the option value of such a project is undefined, as argued above: anything could happen. With fixed *m*, this is not a problem since for $m > 0$ we move away from the singularity immediately. However, a firm with a real option and a choice over *m* could spend nothing on market research, and hold an option with an undefined value; such an option might even be regarded as being infinitely valuable. The infinite option value is in some sense illusory, since with no spending the probability of the firm entering the market is zero.

This paradox is removed if we assume that initial variance is finite. In this case, a firm will spend some money on market research, since with a finite option value (from finite variance) the firm will not be optimising if it has a zero probability of actual entry.

This is an innocuous assumption. In reality, we know that the value of a project will be, say, at least ten pounds, and less than the GDP of the world. For a finite initial *S*, the firm will be willing to spend money. Doing so will admittedly have a depressing effect on future option values, but will also bring earlier optimal entry, as discussed above. The key is that the firm cannot change its current *S*, only future values of S, and so it is not just paradoxically paying to receive a lower option value immediately. It will alter a whole range of future values: expected entry times as well as option values.

The second change to the set-up of the model is to introduce a more general cost function to describe the relationship between the number of researchers and their associated costs. This is done in order to examine what effect different cost structures have on the outcome of the maximisation problem. In particular, it will be seen that the linear case shown above causes some problems, and that in order to obtain a maximum as the outcome to the optimisation problem, marginal costs of information must be increasing in the amount of information. Finally, the problem now requires the inclusion of S as a state variable for the maximisation problem. F will depend on current S, so that in turn the value of S will have an impact on the choice of market research. Since S is a deterministic function of time, and calendar time no longer has any relevance (the optimisation problem is the same for any identical values of S, whatever the date), we remove the time dependence of F.

As before, we assume we have m researchers undertaking our market research, each one of whose observations is assumed to bear a linear relationship to true market value and also contains some noise (assumed to be NIID across researchers). Signals are again assumed to be lognormally distributed, and so again we work from the transformed variables $z=ln\ Z$ and
$v=ln\ V$.

The underlying variables therefore again satisfy

(51) $$d\hat{v} = -m\frac{m^2}{s^2}S\hat{v}dt + \frac{m}{s^2}Sdz \quad \text{and}$$

(52) $$\frac{dS}{dt} = -m\frac{m^2}{s^2}S^2 .$$

Without solving for *S*, it can be seen that $\hat{V}$ satisfies

(53) $$d\hat{V} = S\hat{V}\frac{m^2}{s^2}m(v-\hat{v})dt + S\hat{V}\frac{m}{s}\sqrt{m}dW \; .$$

Costs are represented by the function *c(m).* Alternatively, we could regard the costs as being a simple linear function of the number of researchers, but with the “quality” or "intensity" of their information as represented by a re-interpretation of *m* a more general function of these costs. Either interpretation is supported by a one-one mapping from *m* to *c*.

We define the value of the option, as a function of *m*, $\hat{V}$ and *S*, as the discounted expectation of the payoff on entry:

(54) $$F(\hat{V},S,m) \equiv E\left[(\hat{V}_{T*} - I)^{+} e^{-r(T*-t)}\right].$$

and the value of the option to a firm adopting an optimal policy with respect to the intensity of its market research, as

(55) $$F* = \sup_m F^m \; .$$

An optimal policy provides a mapping from $(S,\hat{V})$ to a particular *m**, from which this option value can then be calculated. As noted above, time is not included explicitly as the important aspect of current calendar time is captured by current *S*, and the optimal entry time will be a function of *S* and of the optimal policy for *m*.

The method used to derive the equation satisfied by the value of market research is dynamic programming, deriving the Bellman equation for the problem. We know the value of the project in the two stopping areas. In the continuation region, it must be the case that the value of the real option now is the discounted expected value of the option next under an optimal policy, minus the costs of continuing. That is, it must hold that

$$(56)\qquad (1+rdt)F^*(S,\hat{V}) = E\left[F^*\left(S + dS(m^*),\hat{V} + d\hat{V}(m^*)\right) - c(m^*)dt\right].$$

From this, by definition it follows that

$$(57)\qquad (1+rdt)F^*(S,\hat{V}) = \max_m \left[E\left[F^*(S + dS(m),\hat{V} + d\hat{V}(m)) - c(m)dt\right]\right].$$

By Ito's Lemma, this implies that

$$(58)\quad rF^*dt = \max_m E\left[\frac{\partial F}{\partial S}dS + \frac{\partial F}{\partial \hat{V}}d\hat{V} + \frac{1}{2}\frac{\partial^2 F}{\partial \hat{V}^2}(d\hat{V})^2 - c(m)dt\right] \quad .$$

Substituting in for *dS* and $d\hat{V}$ and evaluating the expectation, we have that

$$(59)\qquad rF^*dt = \max_m \left(S^2 m \frac{m^2}{s^2}\left(\frac{\hat{V}^2}{2}\frac{\partial^2 F^*}{\partial \hat{V}^2} - \frac{\partial F^*}{\partial S}\right)dt - c(m)dt \right).$$

The first order condition for a maximum is then

$$(59)\qquad \frac{m^2}{s^2}S^2\frac{\partial F^*}{\partial S} = \frac{1}{2}\frac{m^2}{s^2}S^2\frac{\partial^2 F^*}{\partial \hat{v}^2} - \frac{dc}{dm}(m^*),$$

so that $m^*(S,\hat{V})$, the optimal policy rule, solves

$$(60)\qquad \frac{dc}{dm}(m^*) = S^2\frac{m^2}{s^2}\left(\frac{\hat{V}^2}{2}\frac{\partial^2 F^*}{\partial \hat{V}^2} - \frac{\partial F^*}{\partial S}\right).$$

The condition for a global maximum is that $c''(m)<0$. This is satisfied if the marginal cost of extra information is increasing in *m*. This will be true if we are willing to make some kind of "overkill" assumption: twice as much spending will not result in twice as much information, as the researchers get in one another's way. In the case of linear costs used in the simple model of market research, *m* is in fact undefined.

For example, take the possibility of quadratic research costs. If $c(m)=0.5m^2$, then condition (60) becomes

$$(61)\qquad m^* = S^2\frac{m^2}{s^2}\left(\frac{1}{2}\frac{\partial^2 F^*}{\partial \hat{v}^2} - \frac{\partial F^*}{\partial S}\right),$$

and $m*$ is now defined. The full Bellman equation for the value of market research is then:

$$(62)\quad \frac{1}{2}\left(S^2 \frac{m^2}{s^2} \left(\frac{1}{2} \frac{\partial^2 F*}{\partial \hat{V}^2} - \frac{\partial F*}{\partial S} \right) \right)^2 - rF* = 0,$$

which is to be solved again subject to the condition

(50) $F* \geq \max(\hat{V} - I, 0)$. This equation is not solved here.

The condition (60) indicates that the optimally researching firm is in effect "reallocating" volatility over time, at a cost. It equates the marginal cost of obtaining another source of information, with the level of current volatility (captured by the second $\hat{V}$ derivative) and the effect of changing *m* on option value, captured by the *S* derivative. It "balances" the allocation of volatility over the lifetime of the project.

In a sense, by choosing a higher *m,* a firm is "squashing" time: *S* decays more quickly. It was shown above that this effect of a higher *m* leads to a quicker optimal entry time, and also a lower option value. Since *S* is a function of *(1/mt)*, two problems with different levels of *m* would look the same if viewed over time, if time for the value of the project with higher *m* is "slowed down" accordingly. A firm following an optimal policy balances the effect of this squashing of time on option value, with its marginal cost and current levels of volatility. It is for this reason that we can drop the time-dependence of *F*.

Because of the presence of discounting, this does not mean however that time is irrelevant in the sense that we would be indifferent between spending £40 per day for ten days and £4 per day for 100 days. We still have a well-defined optimal level of current expenditure in terms of a constant flow, given where we want to go in terms of current *S*.

5. Conclusion

In this paper, it has been argued that while previous models of the value of research into product opportunities assumed that the underlying state variable on which the value of the project depends follow a constant variance process, this is undesirable. There will be an impact from greater information and learning over time on the variance of estimated product value, and hence on associated option values.

A model has been presented here which includes this learning effect, based on the theory of optimal filtering. The optimal filtering theorem used allows us to specify the stochastic process followed by the optimal estimate of product value, to proxy as a state variable for situations in which actual value is unobservable. Optimal filtering is unrealistic because it assumes a continuous observation of the signal process, but demonstrates clearly how uncertainty and therefore option value decreases with time. The effects of being able to control the intensity of the signal to be filtered have also been demonstrated, and the optimal policy solved for.

This work is clearly by no means comprehensive. Further possibilities are to alter the assumptions of optimal filtering to allow for lumpier information. Also, different optimisation problems in terms of a firm’s preferences over value and uncertainty would be interesting. It is possible that such problems would lead to more intuitively interpretable results. Lastly, it is worth noting that the imposition of different boundary conditions on the partial differential equations used would enable the analysis and evaluation of projects under different decision structures; most notably, a project in which there is a single decision point for the (in, out) decision.

6. References

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