Various Passport Options and Their Valuation

Hyungsok Ahn, **Antony Penaud**, and **Paul Wilmott**
OCIAM, Oxford University

Abstract. The passport option is a call option on the balance of a trading account. The option holder retains the gain from trading, while the writer is liable for the loss. We establish pricing equations for various passport options including the multi-asset passport and those with discrete trading constraints. The results are typically known as the Hamilton-Jacobi-Bellman equations and multiple layers of free boundary partial differential equations for a sequence of optimal stopping times. Also we examine the gain by selling passport options to utility maximising investors and to investors who guess the market from imperfect information.

1 Introduction

Market participants trade risky assets for better returns, at least on average. As long as the risk remains significant, however, investment in these assets does not guarantee a positive payoff and traders are prone to become a victim of their own strategies. Nowadays, a variety of financial instruments allow traders to generate non-linear returns and hence enable them to avoid extreme financial loss to some extent. An example of such an instrument is the passport or perfect trader option.

The passport option is a call option on the balance of a trading account. The option holder trades an asset (or several assets) and takes the net profit of all the trades he made before maturity. The issuer is liable for the net loss. This option certainly makes the fund management straightforward and safer as long as fund managers are willing to pay the upfront premium for the passport option.

The first version of the passport option was launched by Bankers Trust. The option holder trades one asset with the position limit specified in the contract and with an immunity from the net loss he may incur in trading. The puzzle in the price valuation is that the option holder's trading strategy is not known *a priori.* As in the American option valuation, we surmount the uncertainty of the option holder's strategy by maximising the price over all feasible trading strategies. As a result the price of the option under the complete market assumption (see Harrison and Pliska (1981), for example) is given by

$$\sup_{|q|\leq L} \hat{E}\Big[e^{-rT}\,\max(A_T(q),0)\Big] \tag{1}$$

where $q = \{q(t),\, 0 \leq t < T\}$ is the position of the option holder on the underlying asset, A_T is the balance of the trading account at the maturity T, r is the risk-free interest rate, L is the position limit, and $\hat{E}$ is the expectation under the risk-neutral measure, also known as the equivalent martingale measure which evaluates the value of the replicating portfolio as an expectation. Hyer, Lipton-Lifschitz, and Pugachevsky (1997) describe the price (1) as a solution of the Hamilton-Jacobi-Bellman (HJB) equation of the corresponding control problem. Provided that the underlying asset pays no dividend and that the growth rate of the cash balance in the account specified in the contract coincides with the risk-free rate (so called symmetric case), the equation becomes:

$$-v_t = r(av_a + sv_s - v) + \frac{1}{2}\sigma^2 s^2 \sup_{|q|\leq L}\Big(q^2 v_{aa} + 2qv_{as} + v_{ss}\Big) \tag{2}$$

with terminal data $\max(a,0)$. The first term in the right side indicates that the return of the replicating portfolio coincides with the risk-free rate. The balance of the trading account inherits the movement of the underlying asset except that its volatility is amplified (or condensed) by the amount of assets the customer holds. Thus the meaning of the last term in (2) is that the riskless portfolio of the issuer must absorb the curvature effect (i.e., gamma) in the worst case. The only other work that we know of is by Andersen, Andreasen, and Brotherton-Ratcliffe (1998).

Various types of passport options are traded in the market, depending on the number of underlying assets and on exotic trading constraints. In this paper, we establish the

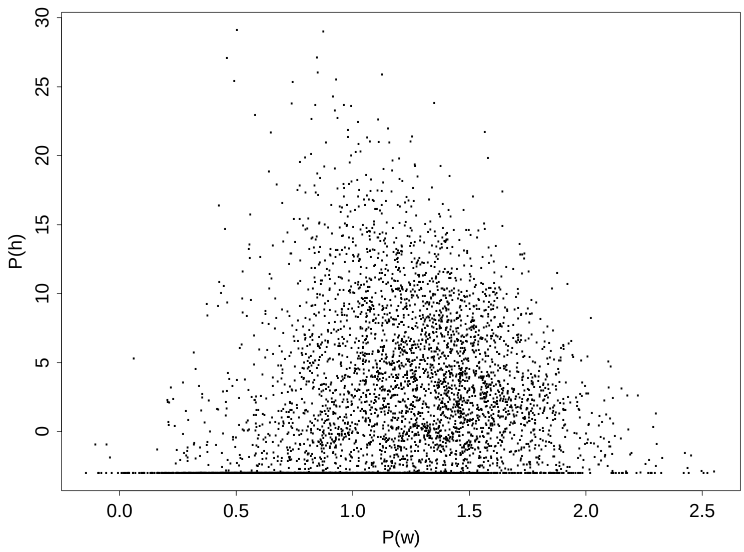


Figure 1: Profits of writer (w) and holder (h)

pricing equations for such options using stochastic control and optimal stopping times. In constructing a replicating strategy, the issuer must tune his position with the trading strategy performed by the option holder, not the one that maximises the price. We provide a description of the hedging strategy in the next section. We find that some trading constraints make the valuation procedure delicate, and that their influence to the price of the option is somewhat surprising. For example restrictions on the number of permitted trades do not make the option much cheaper. See Figure 4 in Section 3.

Another issue that we address in the paper is how market participants will benefit from buying and selling passport options. For example, the issuer of the option will gain when the option holder's strategy differs from the one that attains the maximum price (1). It is unlikely that the option holder will follow the price maximising strategy when he perceives the physical drift significantly different from the risk neutral one. In fact the option holder will try to obtain the best out of his option utilizing his view on the market movement. Thus it is conceivable that the passport option will remunerate both issuer and holder. We examine the profit of the issuer as his customer guesses the market direction from imperfect information. For a preview, Figure 1 shows a scatterplot of 5,000 simulations where the issuer hedges his position (short 1 passport option with $L = 1$), while his customer guesses the market direction. We alternate a rising market and a falling market 10 times during the life of the option (6 months), and let the option holder guess correctly 60% of the time (details in Section 4). The horizontal axis is for the issuer (writer) and the vertical is for the option holder. The asset price volatility is 20% annum and the price of the option is 2.94. The mean profit is 1.14 for the issuer and 2.16 for the option holder. Thus the mean return for the option holder is 72%. The few points at which the issuer loses money are the result of discrete hedging in simulation (100 rebalancings) and of errors in numerical computations.

The paper is structured as follows. In Section 2, we describe the pricing equation for the multi-asset passport option in terms of the HJB equation. We show that the price maximising trading strategy has its value in the boundary of the position limit. Also we reduce the number of variables using the similarity solution. This reduces the burden of severe numerical computation. In Section 3, we investigate several trading constraints. Due to the complexity of the notations and the numerical procedures we restrict our treatment to the single asset passport option. We consider two exotic trading constraints: (i) a restriction on the number of trades; (ii) a restriction on the time between trades. We describe the price maximising strategy as a sequence of optimal stopping times. The resulting pricing equations become multiple layers of free boundary partial differential equations (PDE). We provide a unified methodology for resolving further complications caused by imposing a penalty at each trade. In Section 4, we investigate the utility of the two positions, short and long, in trading passport options. We will consider two models, one with perfectly specified drift and the other with imperfect information. Section 5 contains concluding remarks.

2 Multi-Asset Passport Options

Let $S = (S^{(1)}, \ldots, S^{(n)})'$ be a set of tradable assets. While we may use a broader class of diffusion processes for describing the evolution of S, we insist on using a simple structure where each $S^{(i)}$ follows a log-normal diffusion:

$$dS_t^{(i)} = \mu_i S_t^{(i)} dt + \sigma_i S_t^{(i)} dW_t^{(i)}$$

where $W = (W^{(1)}, \ldots, W^{(n)})$ is a standard Brownian motion with correlation matrix $[\rho_{ij}]_{n \times n}$. The option holder maintains $q^{(i)}$ shares of the i-th asset, and the balance on the trading account A evolves as

$$dA_t = r\Big(A_t - \sum_{i=1}^{n} q^{(i)}(t)\, S^{(i)}\Big)\, dt + \sum_{i=1}^{n} q^{(i)}(t)\, dS_t^{(i)} \tag{3}$$

where r is the risk-free interest rate. The first term appears in (3) describes how the cash balance (i.e., deposit/withdraw) in the trading account is compounded. As mentioned in Hyer *et al.* (1997), the rate is rather an attribute of the contract and it may differ from the risk-free rate. But we will stay with the one that we have chosen for simplicity. The payoff of the passport option holder at the maturity is $\max(A_T(q), 0)$. Note that $A_T(\lambda q) = \lambda A_T(q)$ for each scalar λ, since $A_0 = 0$. Therefore we set $L = 1$ without loss of generality.

2.1 Description of the Pricing Equation

In what follows we will use rather abstract notations: $\bigtriangledown$ and $\bigtriangledown^2$ are the gradient (the first derivative) and the Hessian (the second derivative), respectively. The advantage of using these symbols is that they convey the meaning of the equations without requiring the effort of tracking double indices. First we state the equation, and then we will try to justify it financially.

We designate X to be the vector of size $n+1$ consists of A and S. For the time being, we restrict the trading strategy q to be a Markov policy (also know as a feedback control), meaning that $q(t)$ is a function of t and X_t. We define the value function of the problem as follows:

$$v(t,x) = \sup_{|q|\leq 1} \hat{E}^{t,x}\left[e^{-r(T-t)} \max(A_T(q),0)\right] \tag{4}$$

where the supremum is taken over all feasible Markov policies with position limit $L=1$. We have the following result:

Proposition 2.1. *Let $x=(a,s_1,\ldots,s_n)'$. The value function $v(t,x)$ defined in (4) satisfies the following HJB equation:*

$$-v_t = r\Big(\langle \nabla v, x\rangle - v\Big) + \frac{1}{2}\cdot \sup_{|q|\leq 1}\Big\{\, \langle q, Cq\rangle\, v_{aa} + 2\langle \nabla_0^2 v, Cq\rangle + \langle s, Ds\rangle \,\Big\}\,, \tag{5}$$

$$v(T,x) = \max(a,0)\,,$$

where $C=[\rho_{ij}\sigma_i\sigma_j s_i s_j]_{n\times n}$, $D=[\rho_{ij}\sigma_i\sigma_j\, \nabla_{ij}^2\, v]_{n\times n}$, and $\nabla_0^2 v = (\nabla_{01}^2 v,\ldots,\nabla_{0n}^2 v)'$.

If the contract permits the option holder to trade only one asset ($n=1$), the HJB equation (5) is reduced to (2). Obtaining the above result is a standard procedure in stochastic control and one can check that a unique viscosity solution to (5) exists. In addition, the existence of a classical solution to (5) provides a justification for restricting the class of trading strategies to that of Markov policies, which is known as the "verification theorem" in stochastic control theory. Discussing this in detail is beyond the scope of this paper, and we refer to Fleming and Soner (1992). The implication of the theorem is that the value function of the control problem will not be increased by a non-Markovian policy as long as the status at the moment is the same. Thus, as long as the issuer of the passport option knows the current prices of the underlying assets and the balance of the trading account of his customer, it is not important how his customer ends up with the current balance.

Here we sketch an informal derivation of the HJB equation (5), and argue that it is the pricing equation for our problem. Under the complete market assumption, the issuer of the option can construct a risk-free portfolio using the underlying assets. Hence we consider the following portfolio:

$$\Pi = \langle \Delta, S\rangle - v = \sum_{i=1}^{n} \Delta^{(i)}\, S^{(i)} - v \tag{6}$$

where v is the value of the option which is in short and Δ is the hedging strategy (i.e. the amount of assets to hold). Our first task is to determine the correct Δ to make (6) riskless.

Due to the Markovian nature, the value v of the option will depend upon the prices of $S^{(i)}$, $i = 1, \ldots, n$, and the balance of the trading account A at the moment. Applying Itô's formula to v yields:

$$dv = \langle \bigtriangledown v, dX_t \rangle + v_t dt + \frac{1}{2} \bigtriangledown^2 v\,(dX_t, dX_t) \tag{7}$$

where $\bigtriangledown^2 v\,(,)$ is the bilinear form defined by the Hessian of v. Recall that X consists of A and S. Thus we ascertain that the only random factor in the growth of Π at the moment is contained in

$$\langle \Delta, dS \rangle - \langle \bigtriangledown v, dX \rangle = \sum_{i=1}^{n} (\Delta^{(i)} - \bigtriangledown_i v - q^{(i)} v_a)\, dS^{(i)} - r\left(A - \langle q, S \rangle\right) v_a dt$$

where q is the trading strategy performed by the customer. In order to eliminate the risk caused by the random growth factor dS, we must choose $\Delta = (\bigtriangledown_i v + q^{(i)}\, v_a)'_{i=1,\ldots,n}$. Again q is the *actual* trading strategy performed by the option holder. We emphasize this point. In order to create a hedging strategy Δ, the issuer must be fully aware of the trading activities conducted by his customer. Now we mandate the rest terms in Π to grow at least at the risk-free rate r even in the worst case for the issuer:

$$-r\left(A - \langle q, S \rangle\right) v_a dt - v_t dt - \frac{1}{2} \bigtriangledown^2 v(dX, dX) \;\geq\; r\Pi\, dt$$

which is equivalent to

$$-v_t dt \geq r\left(\langle \bigtriangledown v, X \rangle - v\right) dt + \frac{1}{2} \bigtriangledown^2 v\,(dX, dX)\,. \tag{8}$$

On the other hand, the equality must hold for at least one choice of trading strategy q to avoid arbitrage. In fact, q is concealed in dX because the balance of the trading account A is an element of X and hence by (3). Therefore we conclude the result (5) ascertaining that the following must prevail regardless of the state of A and S at the moment:

$$-v_t = r\left(\langle \bigtriangledown v, X \rangle - v \rangle\right) dt + \sup_{|q| \leq 1} \frac{1}{2} \bigtriangledown^2 v\,(dX, dX)\,.$$

2.2 Properties of the Pricing Equation

We will discover several properties of the pricing equation (5). First, the equation (5) does not depend upon the risk-free rate r. Second, we construct a similarity solution in such a way that the number of space variables in the equation can be reduced. This reduces the computation time and error in solving the equation numerically. Finally, we will discuss the behavior of $q_\star$ the price maximising trading strategy.

Recall that X is a vector valued process consists of A and S. Thus X solves a linear stochastic differential equation (SDE), and consequently, for each scalar λ, λX solves the same SDE with initial condition λx. This together with the payoff $\max(a, 0)$ assures that the value function v is homogeneous of degree 1 in space variables: $v(t, \lambda x) = \lambda v(t, x)$ for each scalar λ. Differentiating this identity with respect to λ and evaluating it at $\lambda = 1$, we obtain the following:

Proposition 2.2. *The solution v of the HJB equation* (5) *satisfies* $\langle \nabla v, x\rangle = v$.

As a result, the price of the passport option doesn't depend upon the risk-free rate. As we mentioned earlier in the section, the growth rate of the cash balance in the trading account is the value specified in the contract. If it differs from the risk-free rate r then the price of the option depends upon the growth rate as well as the risk-free rate. Also if the terminal payoff is not $\max(a,0)$, the result may fail.

Next we consider a diffeomorphism ξ on the state space $\mathbb{R}\times(\mathbb{R}^+)^n$ which satisfies the following properties:

$$\begin{aligned} \xi_i(\lambda x) &= \xi_i(x), \qquad i=0,\dots,n-1, \\ \xi_n(\lambda x) &= \lambda \xi_n(x), \end{aligned}$$

for each x in $\mathbb{R}\times(\mathbb{R}^+)^n$. Define u via $v(t,x) = v(t,\xi^{-1}\circ\xi(x)) = u(t,\xi(x))$. Then we must have

$$\lambda u(t,\xi(x)) = u(t,\xi(\lambda x)) = u(t,\xi_0(x),\dots,\xi_{n-1}(x),\lambda\xi_n(x)).$$

Therefore we conclude that $u(t,\xi) = \xi_n f(t,\xi_0,\dots,\xi_{n-1})$ for some function f. For example, suppose we choose a diffeomorphism ξ on $\mathbb{R}\times(\mathbb{R}^+)^n$ defined by

$$\xi(x) = \Big(\frac{x_0}{x_n},\dots,\frac{x_{n-1}}{x_n},x_n\Big).$$

Then the value function v can be rewritten as $v(t,x) = x_n f(t,y,z)$ where $y = a/x_n$ and z is a vector of size $n-1$ with its element $z_i = x_i/x_n$. Furthermore f satisfies the following HJB equation:

$$-f_t = \frac{1}{2}\cdot\sup_{|q|\le 1}\Big\{\,\langle q,\hat{C}q\rangle\, f_{yy} + 2\langle \nabla_0^2 f,\Lambda q\rangle + \langle z,\hat{D}z\rangle\,\Big\}, \tag{9}$$

$$f(T,y,z) = \max(y,0),$$

where $\hat{D} = [(\sigma_n^2 - \rho_{in}\sigma_i\sigma_n - \rho_{jn}\sigma_j\sigma_n + \rho_{ij}\sigma_i\sigma_j)\nabla_{ij}f]_{(n-1)\times(n-1)}$, $\nabla_0^2 f = (\nabla_{01}^2 f,\cdots,\nabla_{0(n-1)}^2 f)'$ and where

$$\begin{aligned} \langle q,\hat{C}q\rangle &= \sigma_n^2(y-q_n)^2 - 2\sigma_n(y-q_n)\sum_{i=1}^{n-1}\rho_{in}\sigma_i z_i q_i + \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\rho_{ij}\sigma_i\sigma_j z_i z_j q_i q_j, \\ (\,\Lambda q\,)_i &= \sigma_n^2(y-q_n)z_i + \sum_{j=1}^{n-1}\sigma_j\left(\rho_{ij}\sigma_i - \rho_{nj}\sigma_n\right) z_i z_j q_j + \rho_{ni}\sigma_n\sigma_i z_i q_n. \end{aligned}$$

When $n=1$, z is nullified and all the sums vanish. Thus the value function for the single-asset passport option can be written as $v(t,a,s) = sf(t,a/s)$ where $f(t,y)$ satisfies

$$-f_t = \frac{1}{2}\sigma^2\cdot\sup_{|q|\le 1}(y-q)^2 f_{yy} \quad \Big(\text{ or } = \frac{1}{2}\sigma^2(1+|y|)^2 f_{yy}\Big) \tag{10}$$

with terminal data $\max(y, 0)$.

Next we discuss the behavior of $q_\star$ the price maximising trading strategy. Since the terminal data $\max(a, 0)$ is convex in a, v_{aa} stays positive as long as the solution of (5) does not explode. Also note that C is a positive definite matrix. Therefore the supremum in the HJB equation (5) is always attained in the boundary of the feasible set. Of course this will fail when the terminal data is not convex.

Proposition 2.3. *The price maximising trading strategy $q^\star$ satisfies $|q_t^\star| = 1$ for all t almost surely.*

2.3 Numerical Example : Dual Passport (n=2)

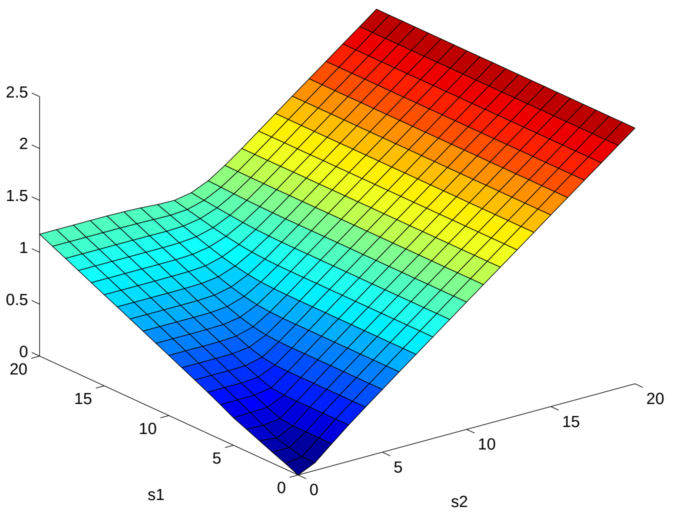


Figure 2: Price of dual passport option with constraints $q^{(1)}q^{(2)} = 0$

$\sigma_1 = 0.2$, $\sigma_2 = 0.4$, $\rho = 0.5$

Figure 2 is a graph of the price of the 6 month dual passport option with constraint $q^{(1)}q^{(2)} = 0$: i.e., the option holder may use his passport to enter into different territories, but given any time the physical presence has to be in either one of them. Intuitively this constraint reduces the effect of correlation by not allowing the two assets to coalesce each other in the trading account. Thus pricing and hedging are somewhat less sensitive to the possible misspecification of the correlation. In this example, the volatility of $S^{(2)}$ dominates that of $S^{(1)}$. Thus, when the price of $S^{(2)}$ is greater than that of $S^{(1)}$, the price the option is robust to the change of prices in $S^{(1)}$.

Figure 3 is the case when there is no constraint. Obviously the contract without the constraint $q^{(1)}q^{(2)} = 0$ is more expensive than the one with the constraint.

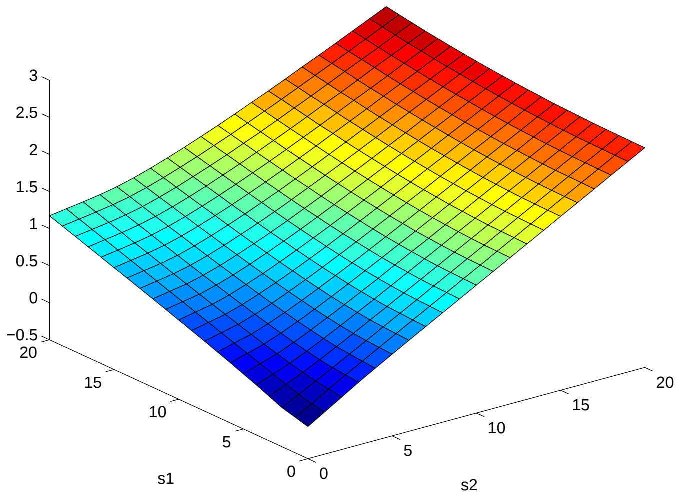


Figure 3: Price of dual passport option without constraints

$\sigma_1 = 0.2$, $\sigma_2 = 0.4$, $\rho = 0.5$

3 Discrete Trading Constraints

In this section, we discuss the trading constraints which force the option holder to trade only a finite number of times. We will confine the scope of our discussion to the single-asset passport option. As in the previous section the value function is proportional to the position limit L. Thus, we may set $L = 1$. Two constraints to be considered here are a specification of the total number of trades and a restriction on the time between trades. In both cases, it is clear that the price maximising strategy has its value (i.e., the amount of holding) equal to one of the limits ± 1. Then the price maximising strategy can be identified by a sequence of optimal stopping times. We will describe the option price with multiple layers of free boundary PDE's. We refer to Van Moerbeke (1976) for the justification of the equivalence of the optimal stopping problems and free boundary PDE's.

When the option holder is permitted to trade only once, an analytical approach is available. We demonstrate this case first. We will start by arguing that the value of the option is maximised when the option holder trades at time 0. Suppose that the option holder buys one share of the underlying at time $\tau \in [0, T)$. Then the balance of his trading account at the maturity becomes $A_T = S_T - S_\tau e^{r(T-\tau)}$, and hence the value of $\max(A_T, 0)$ at time τ is the same as that of the European call with strike $S_\tau e^{r(T-\tau)}$:

$$S_\tau \Big(N(\frac{1}{2}\sigma\sqrt{T-\tau}) - N(-\frac{1}{2}\sigma\sqrt{T-\tau}) \Big) \;=\; S_\tau \phi(T-\tau) \tag{11}$$

where N is the distribution function of the standard normal random variable. If the option holder shorts the underlying at time $\tau \in [0, T)$, the balance ends up $A_T = S_\tau e^{r(T-\tau)} - S_T$

at the maturity. In this case, the value of $\max(A_T, 0)$ at time τ coincides with that of the European put with strike $S_\tau e^{r(T-\tau)}$, which is identical to (11) by the put-call parity. Note that $t \to \phi(t)$ is a decreasing function. Therefore

$$\hat{E}\Big[e^{-r\tau} S_\tau \phi(T-\tau)\Big] \;\leq\; \hat{E}\Big[e^{-r\tau} S_\tau\Big]\phi(T) \;=\; S_0\phi(T)\,. \tag{12}$$

The last equality is obtained from the fact that $e^{-rt}S_t$ is a martingale under the risk neutral measure and hence by the optional sampling theorem (see Karatzas and Shreve (1988), for example). As a result, the stopping time that maximises the option value must be identically 0, and the price of the option is $s\phi(T)$. This coincides with that of the European call with strike se^{-rT}, or equivalently the European put with the same strike.

The hedging strategy is straightforward in this case. At time 0, the issuer buys $\phi(T)$ shares of the underlying asset and stays with the position until the option holder demands a trade. This costs the issuer exactly the price of the option he wrote. Suppose the option holder wants to trade at time τ. The issuer cashes $\phi(T-\tau)$ shares of his assets, and pays for a call or put at strike $S_\tau e^{r(T-\tau)}$, depending on whether the option holder wants to be long or short in the underlying asset. From then on, the vanilla option take cares of the rest. Therefore when τ is greater than zero, the issuer gains $\phi(T) - \phi(T-\tau)$ shares of the underlying which he may cash in anytime. A risk neutral choice is to cash $-\phi'(T-t)dt$ shares of the underlying assets during $[T-t, T-t+dt]$ until the customer demands a trade.

3.1 Limited Number of Trades

Our goal is to describe the price maximising value function when only a finite number of trades permitted. As we discovered earlier in the case when only one trade is permitted, the worst case for the issuer is when his customer demands a trade at the very moment he signs the contract. So for the time being, we will assume that one trade is made at time 0.

We consider $v^{(n+)}$ and $v^{(n-)}$, where n is the number of trades to be made and $+/-$ indicates the current position, long or short. Then the price of the option is the maximum of $v^{(n+)}(0,0,s)$ and $v^{(n-)}(0,0,s)$, if $n+1$ trades are allowed, because we assumed one trade is made at time 0. These functions are homogeneous of degree 1 in space variables, and hence $av_a^{(n\pm)} + sv_s^{(n\pm)} = v^{(n\pm)}$ regardless of n. If the option holder is not allowed to trade any more ($n = 0$), the value functions evolve without an obstacle:

$$\begin{aligned}
\mathcal{L}^+ v^{(0+)} &= v_t^{(0+)} + \frac{1}{2}\sigma^2 s^2\left(v_{ss}^{(0+)} + 2v_{as}^{(0+)} + v_{aa}^{(0+)}\right) &= 0\,,\\
\mathcal{L}^- v^{(0-)} &= v_t^{(0-)} + \frac{1}{2}\sigma^2 s^2\left(v_{ss}^{(0-)} - 2v_{as}^{(0-)} + v_{aa}^{(0-)}\right) &= 0\,,
\end{aligned}$$

with terminal data $v^{(0\pm)}(T,a,s) = \max(a,0)$. Now we investigate $v^{(n+)}$ for $n > 0$. Suppose that $\Pi = \Delta S - v^{(n+)}$ is the riskless portfolio for the issuer. As we described in Section 2, the issuer must choose $\Delta = v_s^{(n+)} + qv_a^{(n+)}$ where q is the actual trading strategy performed by his customer. Since the riskless portfolio must grow at least at the risk-free rate r,

$$\mathcal{L}^{+} v^{(n+)} \;=\; v_t^{(n+)} + \frac{1}{2}\sigma^2 s^2 \left(v_{aa}^{(n+)} + 2v_{as}^{(n+)} + v_{ss}^{(n+)} \right) \;\le\; 0 \,. \tag{13}$$

The equality must hold at least in one case to avoid arbitrage, and when it does, it is the worst case for the issuer. Suppose that $\mathcal{L}^{+}v^{(n+)}$ is strictly less than 0 in a situation. This means that the option holder staying with long position is no longer the worst case for the issuer. Thus the value $v^{(n+)}$ must coincide with the residual value $v^{((n-1)-)}$. On the other hand, if $v^{(n+)}$ exceeds $v^{(n-1)-}$, the trade demanded by the option holder does not provoke the worst case for the issuer. Thus $\mathcal{L}^{+}v^{(n+)}$ vanishes. Combining these, we obtain:

$$\mathcal{L}^{+} v^{(n+)} \cdot \left(v^{(n+)} - v^{((n-1)-)} \right) = 0 \,, \qquad \mathcal{L}^{+} v^{(n+)} \le 0 \,, \qquad v^{(n+)} \ge v^{((n-1)-)} \tag{14}$$

with the terminal condition $v^{(n+)}(T, a, s) = \max(a, 0)$. This is known as a variational formulation of a free boundary PDE (i.e., linear complementary problem). For the theoretical aspects of the problem such as regularity conditions, we refer to Friedman (1988). Similarly, $v^{(n-)}$ satisfies the following:

$$\mathcal{L}^{-} v^{(n-)} \cdot \left(v^{(n-)} - v^{(n-1)+} \right) = 0 \,, \qquad \mathcal{L}^{-} v^{(n-)} \le 0 \,, \qquad v^{(n-)} \ge v^{(n-1)+} \tag{15}$$

with terminal data $v^{(n-)}(T, a, s) = \max(a, 0)$. If the contract specifies a fixed amount of penalty, say p, on each trade the option holder engages, then we replace the free boundary conditions in (14) and (15) by

$$v^{(n+)} \ge v^{((n-1)-)} + p \qquad \text{and} \qquad v^{(n-)} \ge v^{((n-1)+)} + p \,.$$

Now we discuss how the issuer hedges the option. Suppose that n trades are allowed so that the maximum of $v^{((n-1)+)}$ and $v^{((n-1)-)}$ is the price. The price is proportional to that of the underlying asset, and hence we may write it as $s\phi_n(T)$. When we construct $v^{(k\pm)}$, $k \ge 0$, we assumed that the option holder trades at time 0 while he may not have to. If the option holder indeed trades at time 0, the issuer follows $v^{(k\pm)}$, $k = n-1, \ldots, 0$, tracking his customer's position ($+/-$ as long/short) and the number of trades to be made. For example, if the customer holds 0.7 shares of the underlying and if he is allowed to engage 3 more trades, the issuer holds $\Delta = v_s^{(3+)} + 0.7v_a^{(3+)}$. Next, suppose that the option holder is not sure about the market direction at time 0 and he waits until time $\tau > 0$. In this case, the issuer buys $\phi_n(T)$ shares of the underlying asset at time 0. Since $S_\tau \phi_n(T - \tau)$ is sufficient for the issuer to hedge the option from the the moment (i.e., τ) his customer makes the first trade and since ϕ_n is decreasing, the issuer gains $\phi_n(T) - \phi_n(T-\tau)$ shares of the underlying asset. As before the issuer may cash in $\phi_n(T) - \phi_n(T - \tau)$ shares gradually until his customer decides to make the first trade.

3.2 Restrictions on Time between Trades

Now the contract states that the option holder is allowed to trade only after a specified time, say ω, has elapsed since the last trade. We introduce the idea of a clock which keeps track of time since the last trade. The clock is reset to zero immediately after each time the option holder trades, and it keeps ticking until its hand reaches ω, where it remains until the next trade. To model this, we introduce an additional time variable θ:

$$\theta(t) = \begin{cases} t - \tau_i\,, & \text{if } \tau_i \le t < \tau_i + \omega\,, \\ \omega\,, & \text{if } \tau_i + \omega \le t < \tau_{i+1} \end{cases}$$

where τ_i and τ_{i+1} are adjacent trading times. The option holder is allowed to trade only when the clock is dormant, i.e. $\theta = \omega$.

We will describe the value of the option using price maximising value functions, $v^{(+)}$ and $v^{(-)}$. The status described by $v^{(+)}(t,a,s,\theta)$, for $\theta \in (0,\omega)$ is that the option holder is currently long in asset, but not allowed to trade. $v^{(+)}(t,a,s,\omega)$ is for the case when the option holder is currently long in asset and he is allowed to trade which puts him in a short position in asset. $v^{(+)}(t,a,s,0)$ describes the very moment the option holder puts himself in a long position in asset. $v^{(-)}$ describes the opposite case. As before these functions are homogeneous of degree 1 in space variables, and hence $av_a^{\pm} + sv_s^{\pm} = v^{\pm}$.

The evolution of the price maximising value functions $v^{(\pm)}$ will depend on the status of the clock, active or dormant. First we consider the case when the clock is active ($\theta < \omega$). The option holder is not allowed to trade and all he can do is to watch the clock ticking anxiously. The price maximising value functions evolve naturally (i.e., without any obstacles) as the clock ticks:

$$\begin{aligned} v_t^{(+)} + v_\theta^{(+)} + \frac{1}{2}\sigma^2 s^2 \left(v_{ss}^{(+)} + 2v_{as}^{(+)} + v_{aa}^{(+)}\right) &= 0\,, \\ v_t^{(-)} + v_\theta^{(-)} + \frac{1}{2}\sigma^2 s^2 \left(v_{ss}^{(-)} - 2v_{as}^{(-)} + v_{aa}^{(-)}\right) &= 0\,. \end{aligned}$$

Next, we suppose that the clock is dormant ($\theta = \omega$), and hence the option holder is allowed trade. Then the value functions evolve with several rules. First, the price maximising value functions must stay above the residual values:

$$v^{(+)}(t,a,s,\omega) \ge v^{(-)}(t,a,s,0) \quad \text{and} \quad v^{(-)}(t,a,s,\omega) \ge v^{(+)}(t,a,s,0)\,. \tag{16}$$

Second, the riskless portfolio for the issuer grows at least at the risk free rate:

$$\mathcal{L}^+ v^{(+)} = v_t^{(+)} + \frac{1}{2}\sigma^2 s^2 \left(v_{ss}^{(+)} + 2v_{as}^{(+)} + v_{aa}^{(+)}\right) \le 0\,, \tag{17}$$

$$\mathcal{L}^- v^{(-)} = v_t^{(-)} + \frac{1}{2}\sigma^2 s^2 \left(v_{ss}^{(-)} - 2v_{as}^{(-)} + v_{aa}^{(-)}\right) \le 0\,. \tag{18}$$

where each function is evaluated at (t,s,a,ω). Here we dropped $v_\theta^{(\pm)}$ as the clock is no longer ticking. The third rule combines the first two in the following sense. $\mathcal{L}^+ v^{(+)} < 0$ indicates that trading demand does not provoke the worst case. This happens only when $v^{(+)}(t,a,s,\omega) = v^{(-)}(t,a,s,0)$. On the other hand, if $v^{(+)}(t,a,s,\omega)$ exceeds $v^{(-)}(t,a,s,0)$, waiting does provoke the worst case. In summary,

$$\mathcal{L}^+ v^{(+)}(t,a,s,\omega) \cdot \left(v^{(+)}(t,a,s,\omega) - v^{(-)}(t,a,s,0)\right) = 0\,. \tag{19}$$

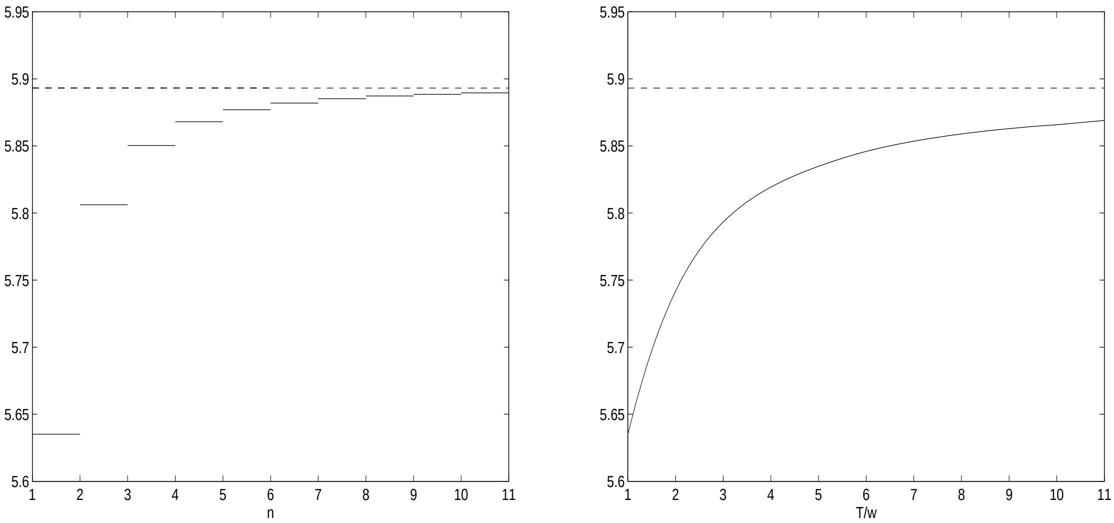
In the opposite case, we have

$$\mathcal{L}^- v^{(-)}(t,a,s,\omega) \cdot \Big(v^{(-)}(t,a,s,\omega) - v^{(+)}(t,a,s,0) \Big) = 0\,. \tag{20}$$

The conditions (16) to (20) define a system of linear complementary problem. As in the case without clock, the price is the maximum of $v^{(+)}(0,0,s,0)$ and $v^{(-)}(0,0,s,0)$, and hedging is a matter of tracking the position of the option holder. If there is a fixed penalty p on each trade, we replace the free boundary conditions in (16) by

$$v^{(+)}(t,a,s,\omega) \geq v^{(-)}(t,a,s,0) + p \quad \text{and} \quad v^{(-)}(t,a,s,\omega) \geq v^{(+)}(t,a,s,0) + p\,.$$

3.3 Numerical Results



(a) Limited number of trades (b) Clock

Figure 4 : Discrete trading constraints

The price of the single-asset passport option is proportional to the price of the underlying asset, even with discrete trading constraints. Let $\phi(n)$ be the proportional constant when n trades are permitted (i.e., the maximum of $v^{(n-1)+}$ and $v^{(n-1)-}$) and $\phi_c(\omega)$ for the clock. Figure 4 (a) shows the value of $100 \cdot \phi$, thus the price of the option when the underlying asset price is 100, for $n = 1, \ldots, 10$. The dotted line is the price of the passport option without trading constraints, i.e., $n = \infty$. The volatility is 20% annum and the maturity is six months. Figure 4 (b) is the price of the passport option with clock, $100 \cdot \phi_c$. Parameters are the same as (a). We have chosen the variable T/ω for the horizontal axis. Thus the number of permitted trades is the greatest integer not more than the variable T/ω. Again the dotted line is the case when there is no clock, i.e., $\omega = 0$. Note that the option with clock is less expensive. This is because it has more restriction even with the same number of trades are allowed.

Also it is noteworthy that the price of the option with 10 trades is already close to that of the unconstrained passport option. In practice, however, it is customary that the contract compels the option holder to refrain from frequent trades. In order the maintain a delta-neutral position, the issuer of the passport option needs to trade the underlying asset at least as often as his customer. When the customer trades very often, the issuer is burdened with excessive transaction cost.

4 Utility of Trading Passport

In this section we examine how the option holder utilizes his option and how much the issuer gains by selling the option. The investor who owns a passport option may construct his trading strategy to maximise his utility, predicting the market movement. When the physical trend of the market differs from the risk-neutral drift significantly, the option holder will benefit as long as he has a correct view on the market. At the same time, the issuer will gain from the difference between the price maximising trading strategy and the trading strategy performed by his customer. We will focus on the case for the single asset passport option with the position limit $L = 1$.

4.1 Perfectly Specified Drift Model

Modeling the gain by investors may be fictitious when one assumes a perfectly specified drift μ. However, the presumption allows us to evaluate the gain by selling passport options to a transcendental investor who trades ideally. Therefore, if the gain turns out significant in this case, then the result remains persuasive.

First we assume that the option holder finds his strategy by solving the value of the maximum expected utility of the payoff:

$$u(t, a, s) = \sup_{|q| \leq 1} E^{t,a,s}\Big[e^{-r(T-t)} U\Big(\max(A_T(q), 0)\Big)\Big] \tag{21}$$

where E is the expectation under the physical measure and U is the option holder's utility function which is increasing in its argument. One can check that u satisfies the following HJB equation:

$$-u_t = rau_a + \mu s u_s - ru + \sup_{|q| \leq 1} \Big(qs(\mu - r)u_a + \frac{1}{2}\sigma^2 s^2 \{q^2 u_{aa} + 2q u_{as} + u_{ss}\}\Big), \tag{22}$$

$$u(T, a, s) = U\Big(\max(a, 0)\Big)$$

where μ is the physical drift of the underlying asset. If U is non-convex (in fact, most of the popular utility functions are concave because of risk aversion), u_{aa} need not be positive, and consequently, the utility maximising strategy can have its value anywhere inside the position limit. If $U(x) = x$, u_{aa} stays positive, as in the case the value function of the option, and hence the utility maximising strategy has its value either ± 1. The interpretation of the linear utility is that the option holder maximises expected return, which can be transformed from (21) by scaling it with the price of the option. A motivation for studying such utility is that the investor's portfolio is already insured by the passport option he owns and that it is tractable. In this case u is homogeneous of degree 1 in space variables, and hence it has a similarity solution of the form $u(t, a, s) = sh(t, a/s)$. Furthermore $h(t, y)$ satisfies the following HJB equation:

$$-h_t = (\mu - r)(h - yh_y) + \sup_{|q|\leq 1} \Big(\frac{1}{2}\sigma^2 (y-q)^2 h_{yy} + q(\mu - r)h_y\Big) \tag{23}$$

with the terminal data $\max(y, 0)$. From this, we obtain the option holder's trading strategy:

$$q = \operatorname{sign}\Big(\frac{\mu - r}{\sigma^2} \cdot \frac{h_y}{h_{yy}} - y\Big). \tag{24}$$

When μ coincides with the risk-free rate r, (23) agrees with the price maximising value function for the option (10), and q in (24) coincides with the price maximising strategy q_*. If μ differs from r, then the option holder's choice will be different from the price maximising strategy.

Next we discuss the issuer's hedging strategy, Δ. In Section 2, we explained that the hedging strategy must be in tune with the actual trading strategy performed by the option holder, and that it is given by $\Delta = v_s + qv_a$ where v solves the HJB equation (2). Then the profit of the issuer becomes:

$$P = v(0, 0, S_0) + \int_0^T e^{-rt}\Delta\Big(dS - rSdt\Big) - e^{-rT}v(T, A_T, S_T). \tag{25}$$

The first term is the price of the option he collects in cash, the second is the result of the delta hedging, and the third is the present value of the potential liability. Applying Itô's formula to v yields:

$$\begin{aligned} P &= -\int_0^T dt\, e^{-rt} \cdot \Big(v_t + \frac{1}{2}\sigma^2 s^2 \{v_{ss} + 2qv_{as} + q^2 v_{aa}\}\Big)(t, A_t, S_t) \\ &= \frac{1}{2}\sigma^2 \int_0^T dt\, e^{-rt} S_t^2 \cdot \Big((q_\star^2 - q^2)v_{aa} + 2(q_\star - q)v_{as}\Big)(t, A_t, S_t) \end{aligned} \tag{26}$$

where $q_\star$ is the price maximising strategy and q is the strategy performed by the option holder. Here we have exploited (2) as well as Proposition 2.3, i.e., $av_a + sv_s - v = 0$. Recall that $v(t, a, s)$ has a similarity solution $sf(t, a/s)$ where f is defined in (10). In particular, we have

$$v_{as} = -\frac{a}{s^2} f_{yy} \qquad \text{and} \qquad v_{aa} = \frac{1}{s} f_{yy}.$$

Also recall that $q_\star = -\operatorname{sign}(y)$. Thus we have a further reduction in the integrand of (26):

$$s^2 \cdot \Big((q_\star^2 - q^2)v_{aa} + 2(q_\star - q)v_{as}\Big) = f_{yy}(t, \frac{a}{s}) \cdot \Big(2|a| + 2qa + (1-q)^2 s\Big) \tag{27}$$

Now suppose that the option holder finds his strategy by maximising the expected return. Then, as we computed earlier in (24), the option holder's strategy q depends on a and s only through the ratio a/s and has its value either ± 1. Hence the last term in (27) drops out, and the profit of the issuer becomes:

$$P = \sigma^2 \int_0^T dt\, e^{-rt} S_t \cdot \Big(|Y_t| + q(t, Y_t)Y_t\Big) f_{yy}(t, Y_t) \tag{28}$$

where Y is the ratio A/S. To obtain the expected profit $E[P]$ of the issuer, we define

$$g(t, a, s) = \sigma^2\, E^{t,a,s}\Big[\int_t^T d\tau\, e^{-r\tau} S_\tau \cdot \Big(|Y_\tau| + q(\tau, Y_\tau)Y_\tau\Big) f_{yy}(\tau, Y_\tau)\Big].$$

Again we observe that g has a similarity solution of the form $g(t, a, s) = s\psi(t, a/s)$ and that $\psi(t, y)$ satisfies the following PDE:

$$-\psi_t = (\mu - r)(1-y)\psi_y + \mu\psi + \frac{1}{2}\sigma^2(1-y)^2\psi_{yy} + \sigma^2 e^{-rt}\Big(|y| + yq(t, y)\Big)f_{yy}.$$

subject to $\psi(T, y) = 0$. To solve this equation, we need to obtain f from (10) and q from (23).

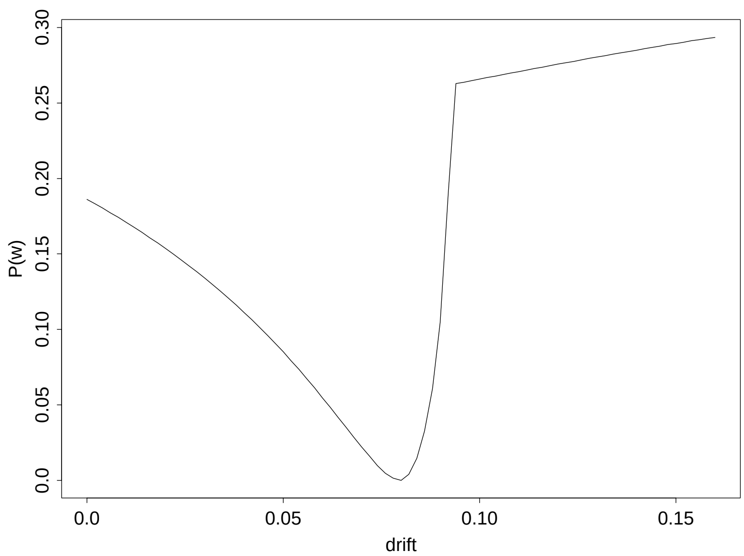


Figure 5: Issuer's expected gain versus the drift of the underlying asset.

Figure 5 shows the expected gain by the issuer as a function of μ, the physical drift, that is $\psi(0,0)$ agains μ. The asset volatility is 20% annum and the maturity of the option is 6 months. We calculate the profit $100 \cdot \psi$ at 81 different values of the physical drift from zero to 16%. When the drift coincides with the risk-free rate $r = .08$ (i.e., 8% annum), the gain vanishes. As we explained earlier, the issuer gains more as the gap between the drift and the risk-free rate become larger.

4.2 Imperfect Information Model

When the physical drift is positive, the price of the asset increases in the long run. In a short period, however, the volatility dictates the price behavior and the effect of the drift is reduced. Thus, even if the drift is positive, the price may fall in a short period. We use the term "market direction" for the direction of the price in a short period to distinguish it from the drift.

We investigate how investors benefit from buying passport options in the environment where he must guess the market direction from imperfect information. Thus the strategy performed by the option holder will be different from the one that maximises the price and the issuer gains as long as he hedges well. In the following, we describe our simulation model. Suppose that the price of the asset is a log normal diffusion with physical drift μ and volatility σ. Then the probability that the price at time τ is greater than the price at time 0 is given by

$$N\Big(\sqrt{\tau}(\frac{\mu}{\sigma} - \frac{1}{2}\sigma)\Big) \tag{29}$$

where N is the distribution function for the standard normal random variable. For example, if $\sigma = 0.2$ (i.e., 20% annum) and $\tau = 0.05$ (18 days, roughly), then $\mu_+ = 0.2466$ yields 60% chance of rise and $\mu_- = -0.2066$ yields 60% chance of fall. Thus, if the investor guesses

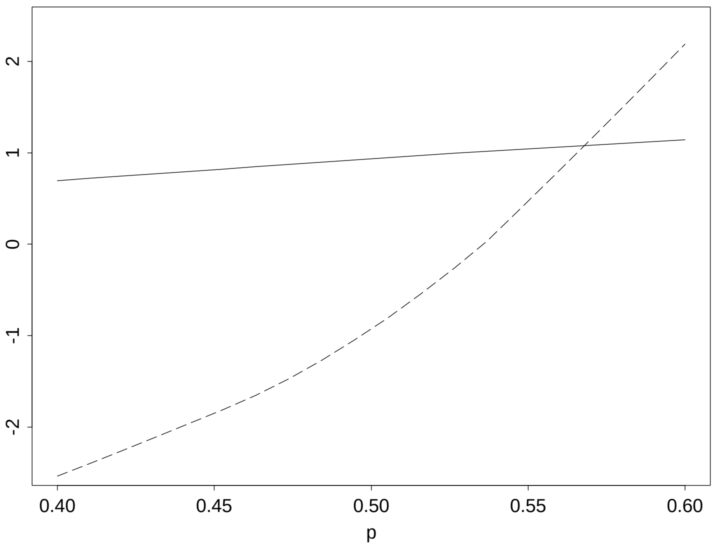


Figure 6: Profits and correct guessing probabilities.

a rising market when $\mu_+ = 0.2466$ (or a falling market when $\mu_- = -0.2066$), then he will be correct only 60% chance of the time. In general, we may choose $\mu_+(p)$ and $\mu_-(p)$ so that the probability of a correct guess becomes p. As we alternate μ_+ and μ_-, the investor guesses the market direction correctly only $100 \cdot p\%$ of the times.

Figure 6 shows the mean profit made by the issuer (the thin line) and the mean profit made by the option holder (the broken line), as a function of the guessing probability. We alternate μ_+ and μ_- 10 times during the life of the option which we set at six months. As before, the volatility is 20% annum and the initial asset price \$100. The curves in the picture are in fact the present values of the mean profits (i.e., it is already discounted). Thus the issuer beats the risk-free rate regardless of the correct guessing probability. If the option holder guesses correctly 53.8% of the time, then he beats the risk-free rate as well.

5 Concluding Remarks

Leland (1980) explained that the investors whose risk tolerance grows with wealth more rapidly than that of the average investor would benefit from portfolio insurance. Nowadays, a variety of options serve market participants as portfolio insurance as well as investment tools. Unlike people who buy-and-hold vanilla options to protect their static portfolio, investors equipped with passport options can actively rectify their market position more aggressively as well, with a limited downside. Thus passport options could attract the investors with below average risk aversion as well.

In this paper, we established pricing equations for various passport options in a unified methodology. We have also addressed that how traders gain from selling passport options and how the option holder maximises his utility. It is doubtful that the issuer is always capable of hedging his position well against all the risks including those which are not calibrated in the model. Also there is no guarrantee that the option holder has a correct view on the market movement every time. Nevertheless it is meaningful to analyse the gain by selling the options to ideal investors, as most investors may not be any near ideal. We have also included simulation results for the gain by selling passport options to non-ideal investors.

REFERENCES

Andersen, L., J. Andreasen, and R. Brotherton-Ratcliffe (1998), The passport option, Preprint.

Fleming, W. H. and H. M. Soner (1992), Controlled Markov Processes and Viscosity Solutions, Springer-Verlag, New York.

Friedman, A. (1988) Variational Principles and Free Boundary Problems, Robert E. Kreiger Publishing, New York.

Harrison, J. M. and S. R. Pliska (1981), Martingales and stochastic integrals in the theory of continuous trading, *Stoch. Proc. and Their Appl.*, **11**., pp 215-260.

Hyer, T., A. Lipton-Lifschitz, and D. Pugachevsky (1997), Passport to success, *Risk Magazine*, **10**., No. 9., pp 127-131.

Karatzas, I. and S. E. Shreve (1988), Brownian Motion and Stochastic Calculus, Springer-Verlag, New York.

Leland, H. E. (1980), Who should buy portfolio insurance?, *J. of Finance*, **35**, No. 2, pp 581-594.

Van Moerbeke, P. (1976), On optimal stopping and free boundary problems, *Arch. Rational Mech. Anal.*, **60**., pp 101-148

Wilmott, P., J. N. Dewynne, and S. D. Howison (1993), Option Pricing: Mathematical Models and Computation, Oxford Financial Press, Oxford.