

# Uncertain Parameters, an Empirical Stochastic Volatility Model and Confidence Limits

by

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## Abstract

In this paper we build upon the recently developed uncertain parameter framework for valuing derivatives in a worst-case scenario. We start by deriving a stochastic volatility model based on a simple analysis of time-series data. We use this stochastic model to examine the time evolution of volatility from an initial known value to a steady-state distribution in the long run. This empirical model is then incorporated into the uncertain parameter option valuation framework to provide 'confidence limits' for the option value.

## Section 1: Introduction

### 1.1 Motivation

The Black-Scholes option pricing equation for a stock (Black & Scholes, 1973) assumes that the risk-free interest rate and the volatility of the underlying asset remain at predetermined and constant levels over the life of the option. Although this may be a valid simplifying assumption for short maturity options, it becomes increasingly less plausible as the maturity increases. There have been numerous efforts to address this simplification by either allowing rates and volatility to be time dependent or by introducing stochastic interest rate and volatility. Both of these approaches suffer from practical and theoretical problems. Not only are the models for the volatility and interest rate rarely accurate but if the variables are stochastic then one must model a 'market price of risk' associated with the non-traded stochastic variables.

In this paper we develop a process for dealing with the volatility and interest rate, although only explicitly show the modelling and analysis for volatility. We start by describing the approach of Avellaneda, Levy & Paras (1995), and Lyons (1995). Following them, we assume that we do not know exactly where each parameter is going to be over the life of the option, instead we state a 'certainty band,' within which we are sure it will trade. In this way we generate a worst price and a best price for the option by assigning values to the parameters in such a way as to minimise or maximise the change in the value of the hedged portfolio. This generates a nonlinear partial differential equation (PDE) which can be solved using finite-difference methods. We refine our pricing by using optimum amounts of standard traded options to statically hedge out as much of the payoff as possible; thereafter the residual is delta hedged in order to obtain the tightest worst/best price spread for the option. This procedure yields a 'certainty interval' for the price of the option, driven by the input volatility and interest rate bands. Hence we know that, for example, if we could access the option in the market below our worst price then within the limits of our 'certainty band' assumptions, we are guaranteed a profit. The uncertain parameter approach does not, in itself, provide any information about the plausible behaviour of a random parameter, on the contrary, the central assumption is that the behaviour is unknown. This part of the paper is simply a recapitulation of the work of Avellaneda, Levy & Paras (1995), and Lyons (1995). The more classical method of dealing with difficult-to-predict variables is to model them stochastically. This

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we do for volatility, deriving a stochastic model such that drift and variance, as well as the steady-state mean and dispersion of volatility are compatible with historical data. We then solve the equation governing the probability density function for volatility (the Fokker-Planck equation) to observe the evolution of volatility from the known value today (modelled as a delta function) to the steady-state distribution. We then solve the backward Kolmogorov equation in order to calculate the probability that the volatility will remain within a given region: specifically the ‘certainty band.’ Hence having made the underlying assumption that the variable is uncertain, and having defined a ‘certainty band’ which will not be breached, we superimpose on this framework a model for the evolution of the variable that we have gained from empirical data. This enables us to attach a confidence limit to the worst/best price spread we have calculated for the option. The issue of bounds for the values of options when volatility is stochastic was originally addressed by El Karoui, Jeanblanc-Picque & Viswanathan (1991). See also Zhu & Avellaneda (1997) for an E-ARCH model for volatility and bounds on volatility using Monte Carlo simulations.

In this way we have developed an option valuation framework that is essentially different from the conventional probability based approach. We acknowledge that parameters are uncertain and derive a spread of option prices corresponding to extreme scenarios. We then superimpose on this the information gained through empirical modelling of the parameters and derive a likelihood of the option price remaining within the calculated spread. This approach should have many applications, ranging from a robust risk-measurement approach for portfolio management, a departure from the traditional mean/standard deviation valuation, to valuation of exotic options, either for pricing or trading purposes.

Avellaneda, Levy & Paras (1995), and Lyons (1995) are the inspiration behind the uncertain parameter approach, which they applied to volatility for single stock options. This concept is an attempt to reduce the dependence of an option value on the underlying model and as such is an important new direction for financial modelling.

The procedure followed in Section 5 for fitting a stochastic model to empirical data for volatility was first used by Apabhai, Choe, Khennach & Wilmott (1995) in deriving a stochastic model for the US spot interest rate. The method examines short-term data to model the variance in the stochastic process and long-term data to model the drift. The method thus acknowledges the timescales over which the two determinants of the model (the deterministic drift and the random variance) act.

## 1.2 Outline

This paper is organised as follows. Section 2 briefly derives the Black-Scholes equation and introduces some terms and definitions to be used in the paper. Section 3 introduces the model of Avellaneda, Levy & Paras (1995) and Lyons (1995) and the rationale behind the uncertain parameter approach and describes the methodology of worst-case and best-case valuations. Further, the use of optimal static hedging and residual delta hedging to tighten the worst/best price spread is described. Section 4 derives a stochastic model for volatility which fits empirical data for the Dow Jones Index over the last 20 years. The method is applicable to many financial time series. Section 5 uses the values calculated for the drift and variance of volatility in Section 4 to solve the Fokker-Planck equation for the probability density function of volatility, and hence analyse the evolution of volatility from an initial value to the steady-state distribution. Section 6 uses the input ‘certainty band’ as well as the empirical stochastic model of Section 4 to calculate the probability of volatility remaining within the band, so that we can attach a confidence limit to the worst/best price spread for the option derived using the uncertain parameter approach in Section 3. Section 7 summarises the work of the paper. Finally there is a list of references.

## **Section 2: The Black-Scholes Framework**

In this section we remind the reader of the original Black-Scholes analysis. In order to develop the Black-Scholes pricing equation on a stock paying no dividends, we start by modelling the return on the underlying asset,  $\frac{dS}{S}$ , as a stochastic process

$$\frac{dS}{S} = \mathbf{m}dt + \mathbf{s}dX \quad (2.1)$$

where  $\mathbf{m}$  is the drift,  $dt$  is the time step,  $dX$  is a random variable drawn from a normal distribution with mean zero and variance  $dt$ ,  $\mathbf{s}$  is the volatility of the returns, and is constant for the life of the option.

We define  $V(S,t)$  to be the value of an option where  $0 \leq t \leq T$ ,  $t$  is time and  $T$  is the exercise date. Ito's Lemma, which gives the stochastic process followed by  $V$ , shows that

$$dV = \left( \frac{\partial V}{\partial t} + \mathbf{m}S \frac{\partial V}{\partial S} + \frac{1}{2} \mathbf{s}^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \mathbf{s}S \frac{\partial V}{\partial S} dX . \quad (2.2)$$

Inspection of (2.1) and (2.2) shows that by constructing a portfolio  $\Pi$ , consisting of  $V(S,t)$  and  $S$  in some ratio, it is possible to eliminate the underlying source of uncertainty:

$$\Pi = V - \Delta S , \quad (2.3)$$

with  $\Delta = \frac{\partial V}{\partial S}$  to eliminate the random component. Hence  $\Pi$  is instantaneously risk free and as such it must have a risk-free return:

$$d\Pi = dV - \Delta dS = r\Pi dt \quad (2.4)$$

where  $r$  is the risk-free rate of return, and is taken to be constant over the life of the option.

Substituting from (2.1), (2.2), (2.3) into (2.4) we arrive at the Black-Scholes PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2} \mathbf{s}^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0 . \quad (2.5)$$

This is a linear second-order one-dimensional backward parabolic PDE. In order for (2.5) to be a well-posed problem, with a unique, well-behaved solution, we need to impose one final condition (the option payoff) and two boundary conditions. For example, for a vanilla call on an asset we have:

$$\text{Final condition:} \quad V(S,T) = \max(S - K, 0) \quad (2.6)$$

$$\begin{aligned} \text{Boundary conditions:} \quad V(S,t) &\rightarrow 0 \quad \text{as } S \rightarrow 0 \\ V(S,t) &\sim S - Ke^{-r(T-t)} \quad \text{as } S \rightarrow \infty . \end{aligned} \quad (2.7)$$

The Black-Scholes equation, (2.5) can be solved in closed form when the parameters  $\mathbf{s}$  and  $r$  are constant for all  $0 \leq t \leq T$ .

### Section 3: Uncertain Parameters and Certainty Bands

#### 3.1 Methodology for dealing with uncertain parameters

We now develop the uncertain parameter methodology by assuming that the parameters are uncertain and that the best we can do is to specify a band for each parameter, within which we are sure it will lie for the life of the option: 'certainty bands.' This idea was originally due to Avellaneda, Levy & Paras (1995), and Lyons (1995). By choosing the values that the parameters take as functions of the underlying and time, we derive a worst-case price and a best-case price for the option, where the worst case and the best case represent respectively a minimum change and a maximum change in the value of the hedged portfolio,  $d\Pi$ . In this way we obtain a band of prices for the option dependent on the defined certainty bands.

We first set up our delta-hedged portfolio, as in Section 2. The return on this risk-free portfolio is still given by

$$d\Pi = \left( \frac{\mathcal{V}}{\mathcal{I}_t} + \frac{1}{2} \mathbf{s}^2 S^2 \frac{\mathcal{V}^2}{\mathcal{I}S^2} \right) dt. \quad (3.1)$$

Now we define our parameters to lie within their ‘certainty bands’

$$\begin{aligned} \mathbf{s}^- &\leq \mathbf{s} \leq \mathbf{s}^+ \\ r^- &\leq r \leq r^+ \end{aligned}$$

The parameters  $\mathbf{s}^-$ ,  $\mathbf{s}^+$  and  $r^-$ ,  $r^+$  must be deterministic. We will assume that they are constant.

We define the worst-case price to be the price of the option associated with the scenario in which the parameters, within the limits of their bands, combine in such a way as to always produce the lowest price. The lowest price corresponds to a minimum change, the least increase and the most decrease, in  $d\Pi$  at each timestep. This procedure, described below, produces the least positive and the most negative value for  $\Pi$ . From (3.1)

$$\min\{d\Pi\} = \min_{\mathbf{s}^- \leq \mathbf{s} \leq \mathbf{s}^+} \left\{ \frac{\mathcal{V}}{\mathcal{I}_t} + \frac{1}{2} \mathbf{s}^2 S^2 \frac{\mathcal{V}^2}{\mathcal{I}S^2} \right\} \quad (3.2)$$

Examining (3.2) we see that the value we choose for the volatility will be a function of the sign of  $\frac{\mathcal{V}^2}{\mathcal{I}S^2}$ . That is, if  $\frac{\mathcal{V}^2}{\mathcal{I}S^2} \leq 0$  then a minimum in  $d\Pi$  will require  $\mathbf{s} = \mathbf{s}^+$  for that timestep, and if  $\frac{\mathcal{V}^2}{\mathcal{I}S^2} > 0$  then a minimum in  $d\Pi$  will require  $\mathbf{s} = \mathbf{s}^-$ .

Since  $r$  does not enter into the calculation for  $d\Pi$ , to find the minimising value for  $r$ , we must look at the value of the portfolio,  $\Pi$ . If  $\Pi \leq 0$  then we use the smallest value to discount,  $r = r^-$  since this will yield the lowest value for the portfolio as a whole, and conversely if  $\Pi > 0$  then  $r = r^+$ . Although in the Black-Scholes world of known parameters the option replication is self financing, one must still buy/sell the underlying. This involves the borrowing or lending of cash. The amount of cash held is simply the negative of the portfolio value and it is this quantity that is exposed to the uncertainty in the interest rate. If one is borrowing money to pay for the hedge then high interest rates correspond to the worst-case scenario. Hence  $r$  is a function of the sign of  $\Pi$ .

This analysis yields the uncertain parameter option pricing equation where the parameters  $\mathbf{s}$  and  $r$  are functions of the signs of  $\frac{\mathcal{V}^2}{\mathcal{I}S^2}$  and  $\Pi$  respectively

$$\frac{\mathcal{V}}{\mathcal{I}_t} + \frac{1}{2} \mathbf{s}(\Gamma)^2 S^2 \frac{\mathcal{V}^2}{\mathcal{I}S^2} + r(\Pi)S \frac{\mathcal{V}}{\mathcal{I}S} - r(\Pi)V = 0, \quad (3.3)$$

$$\text{where } \Gamma = \frac{\mathcal{V}^2}{\mathcal{I}S^2}.$$

The best-case scenario involves the same analysis as above but selecting values of parameters to yield the maximum  $d\Pi$  and  $\Pi$ .

### 3.2 Optimal static hedging

Static hedging, so called because unlike delta hedging it does not require constant readjustment of the hedge amount, is the name given to hedging an option with another option. It is the only way to hedge an option without incurring gamma risk. That is, it is the only way to hedge volatility. Here we only briefly describe optimal static hedging, see Avellaneda & Paras (1996) for further details.

The idea of optimal static hedging is best explained by a simple example. Suppose that we hold a call option on a stock with a strike price of  $K$ , and a payoff  $P$  defined by  $P = \max(S - K, 0)$ . We are going to hedge this with a short position in another call, with a different strike price,  $K_1$ . This hedging option has a payoff  $P_1$  defined by  $P_1 = \max(S - K_1, 0)$ . The 'residual' payoff is hence  $R = \max(S - K, 0) - I \max(S - K_1, 0)$ , where  $I$  is the quantity of the second option. Why have we statically hedged the original contract? We have done this because the residual payoff for the hedged position is much smaller than the payoff for the original contract, making the valuation less sensitive to the volatility estimate and, indeed, less sensitive to the model for the underlying. The payoff to be delta hedged is now a smaller residual payoff, with as much of the initial payoff as possible having been statically hedged. Since the price of the portfolio of two options is the sum of the prices of the standard option and the residual, and the price of the residual is a small constituent of the total, the worst/best spread for the basket option will be tighter.

Most importantly, we can choose the quantity  $I$  to give our original contract the best possible value. This is what is meant by 'optimal' static hedging. We iteratively optimise the procedure described above by selecting the amount of option used to statically hedge such that we maximise the worst-case price and minimise the best-case price. In this way we collapse the worst/best spread to a level closer to that where market participants are willing to trade. Now

$$\text{Price} = \text{Price}_R + I\text{Price}_1 \quad (3.4)$$

where  $\text{Price}_P$  is the price of the option with payoff  $P$

$\text{Price}_R$  is the price of the residual, payoff  $R$

$\text{Price}_1$  is the price of the option with payoff  $P_1$ .

In this simple expository example we have unrealistically hedged a vanilla option with another vanilla option. In practice, the optimal static hedging is used to hedge an exotic contract with a portfolio of vanillas. The more vanillas that are available for hedging, the narrower the worst/best price spread. If the cashflows of the exotic contract are similar to cashflows of traded contracts then we can reduce spreads significantly, almost eliminating dependence on the model. If the cashflows of exotic are far removed from cashflows of traded contracts then we may be left with large spreads. But this is only to be expected, the value of such an exotic contract will be highly model dependent and if we are being conservative in our pricing we would naturally accommodate this uncertainty by increasing our spreads. Examples of how well the model can reduce spreads can be found in Avellaneda & Paras (1996).

## Section 4: Deriving an Empirical Stochastic Volatility Model

### 4.1 Introduction

In this paper we shall be working with daily Dow Jones Index spot data from 7-Jan-75 to 10-Aug-95. However, the techniques are applicable to many financial time series. *Figure 4.1* below shows the calculated monthly volatility of daily returns over this period.

Previous attempts at modelling volatility stochastically are numerous. However, since adding another stochastic variable complicates option pricing, the emphasis to date has been on deriving a tractable model. In this paper we try to determine a stochastic model for volatility by fitting the drift and variance functions to empirical data and letting tractability take a secondary role

We will assume that the stochastic process for volatility is given by

$$ds = a(s)dt + b(s)dX. \quad (4.1)$$

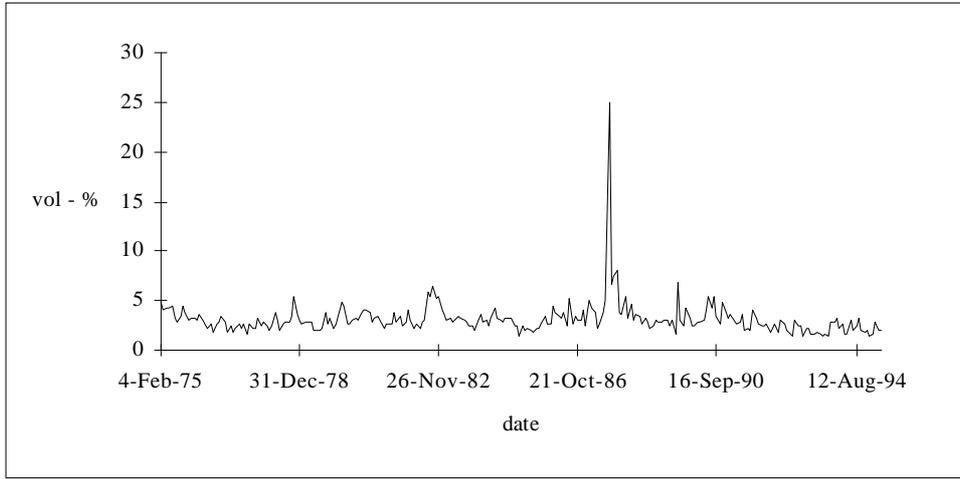


Figure 4.1: Monthly Volatility of Daily Returns for the Dow Jones Index over the last 20 years

where the drift,  $\mathbf{a}(\mathbf{s})$  and volatility,  $\mathbf{b}(\mathbf{s})$  of volatility are both functions of volatility. The main assumption in this is that the terms are independent of time. Not only does this make the data analysis far simpler, it also reduces the scope for serious modelling errors. We will try to model the average behaviour of volatility over the last twenty years. This means that we do not exclude extreme market conditions such as the 1987 crash since this was, of course, unpredictable.

We shall also assume that the volatility is uncorrelated with the underlying. Again, this assumption greatly simplifies the analysis of the data.

We use twenty years of daily Dow Jones Index closing prices (in total over 5,000 observations) to calculate daily returns and monthly volatility of these daily returns.

#### 4.2 Estimating the Variance of Volatility

We start by assuming that the volatility function  $\mathbf{b}(\mathbf{s})$  takes the form

$$\mathbf{b}(\mathbf{s}) = \mathbf{f}\mathbf{s}^g \quad (4.2)$$

where  $\mathbf{f}, \mathbf{g}$  are constants and are to be determined. We assume the volatility of volatility to have this simple form to reduce the problem to finding only the two parameters. The method that we adopt below is not restricted to any functional form for  $\mathbf{b}(\mathbf{s})$  and our assumption could easily be dropped; this point will be made clearer below.

From (4.1),

$$(d\mathbf{s})^2 = \mathbf{b}(\mathbf{s})^2 dt \mathbf{n}^2$$

to leading order, where  $\mathbf{n}$  is a standardised Normal random variable. Using (4.2)

$$(d\mathbf{s})^2 = \mathbf{f}^2 \mathbf{s}^{2g} dt \mathbf{n}^2. \quad (4.3)$$

From our time series for volatility we calculate a time series for  $d\mathbf{s}$ . Taking expectations and then the natural logarithm of (4.3) yields

$$\ln(E[(d\mathbf{s})^2]) = \ln(\mathbf{f}^2 dt) + 2g \ln(\mathbf{s}). \quad (4.4)$$

Hence if (4.2) were true we would expect a plot of  $\ln(E[(d\mathbf{s})^2])$  versus  $\ln(\mathbf{s})$  to yield a straight line with slope  $2g$  and intercept with the y-axis  $\ln(\mathbf{f}^2 dt)$ . If we had not chosen the functional form (4.2) we could at this stage fitted whatever seemed to be the most accurate and plausible curve.

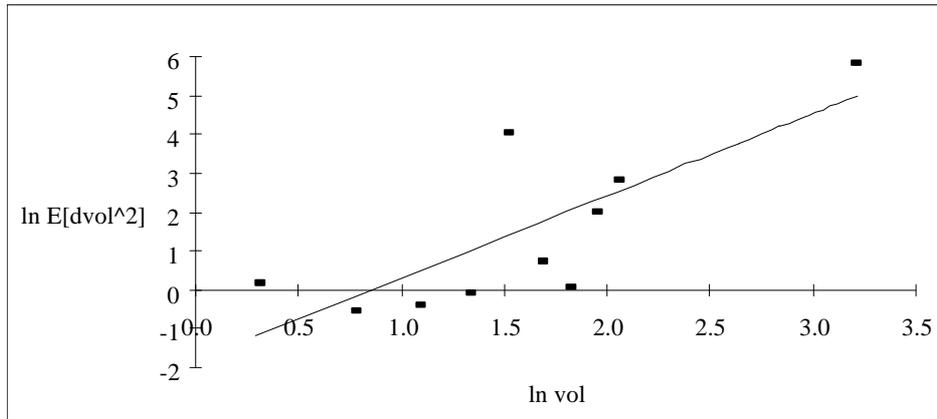
In order to fit a line to the data we split  $\mathbf{s}$  into  $b$  equal size buckets spanning all values of  $\mathbf{s}$

$$b_i = \mathbf{s}_{\min} + i \frac{(\mathbf{s}_{\max} - \mathbf{s}_{\min})}{b} \quad \text{for } i = 0, 1, \dots, b$$

and calculate the mean of  $(d\mathbf{s})^2$  for each  $\mathbf{s}$  falling in a particular bucket:

$$E[(d\mathbf{s})^2]_i = \frac{1}{n} \sum_{\mathbf{s}_i \leq \mathbf{s} < \mathbf{s}_{i+1}} (d\mathbf{s})^2$$

where  $n$  is the number of observations falling within the bucket. *Figure 4.2* shows the plot of  $\ln E[(d\mathbf{s})^2]_i$  versus  $\ln(b_i)$  with a fitted line, for monthly volatility of daily returns.



*Figure 4.2: Plot of  $\ln E[(d\mathbf{s})^2]$  versus  $\ln(b)$  and the fitted line*

We fit the line by linear regression analysis using a least squares method. The intercept with the y-axis of the fitted line is estimated to be -1.8, and the slope 2.10. Hence

$$\begin{aligned} \ln(\mathbf{f}^2 dt) = -1.8 & \quad \rightarrow \quad \mathbf{f} = 1.45 \\ & \quad \rightarrow \quad \mathbf{b}(\mathbf{s}) = 1.45\mathbf{s}^{1.05} \\ 2\mathbf{g} = 2.1 & \quad \rightarrow \quad \mathbf{g} = 1.05 \end{aligned} \tag{4.5}$$

Hence we have derived an expression for the volatility of volatility that is consistent with empirical data.

### 4.3 Estimating the Drift of Volatility

We are *not* going to use the same binning procedure for finding  $\mathbf{a}(\mathbf{s})$  as we used for finding  $\mathbf{b}(\mathbf{s})$ . This is because we want to ensure that the volatility model has realistic long-term properties. It is natural to determine the volatility term  $\mathbf{b}(\mathbf{s})$  by examining the fine details of the time series, looking at the shortest possible timescales. However, the drift is not seen over such short timescales, and only becomes apparent over longer timescales. Both the drift and the volatility of volatility determine the long-term behaviour of the volatility. In particular, between them they determine the steady-state distribution of volatility, *if it exists*. If we can determine this distribution then we can back out the drift structure using the Fokker-Planck equation. On the other hand, if we were to estimate  $\mathbf{a}(\mathbf{s})$  from the short-term details of the volatility time series, there would be no guarantee about the properties of the long-term behaviour of volatility. We could easily end up with a model that shows unrealistic long-term behaviour. Our approach to the analysis of the data is crucially important and must be stressed.

We will assume that the steady-state distribution exists. If it does exist, then it can be determined from the Fokker-Planck equation. But it can also be determined empirically from the data.

The equation governing the PDF for  $\mathbf{s}$  is the Fokker-Planck equation (or forward Kolmogorov equation, see Schuss, 1980)

$$\frac{\partial P}{\partial t} = \frac{1}{2} \frac{\partial^2 (\mathbf{b}(\mathbf{s})^2 P)}{\partial \mathbf{s}^2} - \frac{\partial (\mathbf{a}(\mathbf{s}) P)}{\partial \mathbf{s}} \quad (4.6)$$

where  $P(\mathbf{s}, t)$  is the PDF for  $\mathbf{s}$ .

As  $t \rightarrow \infty$  there are a limited number of possibilities for the distribution. Given that we are assuming the governing stochastic differential equation is time homogeneous, either there exists a theoretical steady-state distribution, or the volatility grows or decays in the long run (as would be the case for the underlying lognormal asset). However, we can deduce a steady-state distribution from the data. Assuming that we have found such a the steady-state distribution,  $P_\infty$ , empirically then it must satisfy the steady-state version of (4.6):

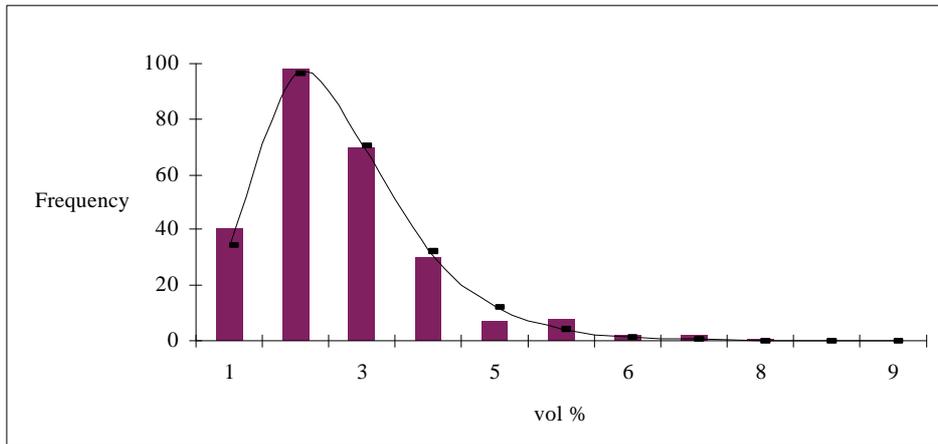
$$\frac{d(\mathbf{a}(\mathbf{s}) P_\infty)}{d\mathbf{s}} = \frac{1}{2} \frac{d^2 (\mathbf{b}(\mathbf{s})^2 P_\infty)}{d\mathbf{s}^2}$$

Integrating once we get

$$\mathbf{a}(\mathbf{s}) = \frac{1}{2P_\infty} \frac{d(\mathbf{b}(\mathbf{s})^2 P_\infty)}{d\mathbf{s}} + \frac{c}{P_\infty} \quad (4.7)$$

where  $c$  is a constant of integration. This constant can be shown to be zero by considering the behaviour of  $\mathbf{s}$  for large and small values, see Oztukel (1996).

From (4.7), we see that in order to find the drift term, we need to find  $P_\infty$ . This we can do empirically by plotting the frequency distribution of  $\mathbf{s}$  versus buckets of  $\mathbf{s}$ , that is, how many observations fall into each bucket. This graph is shown in *Figure 4.3* for the same periods of return and volatility as previously used.



*Figure 4.3: The Steady-State Distribution of  $\mathbf{s}$  and the Fitted Curve*

The empirical distribution closely resembles a lognormal curve. Hence we shall assume this to be the form for  $P_\infty$ . The generic lognormal distribution is described by

$$P_\infty = \frac{1}{a\mathbf{s}\sqrt{2\pi}} e^{-\left(\frac{\log \frac{\mathbf{s}}{\bar{\mathbf{s}}}}{a}\right)^2 / 2} \quad (4.8)$$

where  $\log \bar{\mathbf{s}}$  represents the mean of the distribution  $\log \mathbf{s}$ , and  $a$  describes the dispersion of the distribution about the mean.

We find from the data that,  $a = 0.35$  and  $\bar{s} = 2.5\%$ . From (4.7) and (4.8) we have that

$$\mathbf{a}(s) = f^2 s^{2g-1} \left( g - \frac{1}{2} - \frac{\ln \frac{s}{\bar{s}}}{2a^2} \right). \quad (4.9)$$

## Section 5: The Time Evolution of Stochastic Volatility

### 5.1 Introduction

In Section 4 we derived a stochastic model for the volatility. It will be important in what follows to make use of this model to derive the probabilistic evolution of volatility from an initial value. We turn once again to the forward Kolmogorov equation, given by (4.6) which describes the evolution over time of the probability density function of a random variable defined by a stochastic differential equation. We use the functional forms of  $\mathbf{b}(s)$  and  $\mathbf{a}(s)$  derived in Sections 4.2 and 4.3 respectively and use explicit finite-differences to solve for the PDF of volatility over a specified time horizon. Hence we observe the evolution of the distribution of volatility. By further computing the cumulative density function (CDF) for each timestep, we calculate contours of probability. These define the different regions in which volatility will lie with equal probability over time. Equation (4.6) is

$$\frac{\partial P}{\partial t} = \frac{1}{2} \frac{\partial^2 (\mathbf{b}(s)^2 P)}{\partial s^2} - \frac{\partial (\mathbf{a}(s) P)}{\partial s}.$$

Now instead of letting  $\frac{\partial P}{\partial t} = 0$  and deriving the steady-state probability, we apply a delta function as an initial condition, meaning that we know the value of today's volatility with certainty, and solve for the resulting PDF as time evolves. Noting that (4.6) is a forward equation, we approximate it using forward differences in time and central differences in space.

### 5.2 Time evolution and contours

Inputting the variance and drift values derived Sections 4.2 and 4.3 and solving the Fokker-Planck equation numerically, we observe the evolution of the distribution of volatility from an initial state  $s = 4\%$ .

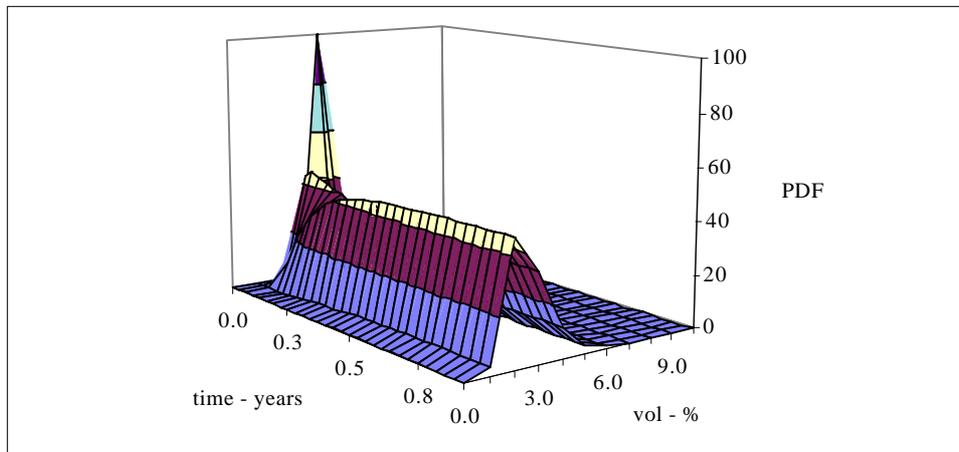


Figure 5.1: The time evolution of the distribution of monthly volatility of daily returns

Figure 5.1 is a three-dimensional representation of the evolution of the probability density function of volatility over time. Figure 5.2 shows a cross-section of the distribution of volatility at time = 0.03 years, time = 0.30 years and time = 1 year, from which it is possible to see more clearly how volatility evolves. Note how there is

little change between times 0.3 and 1, the volatility is very quickly settling down to its steady state, and is forgetting about its initial value at time 0.

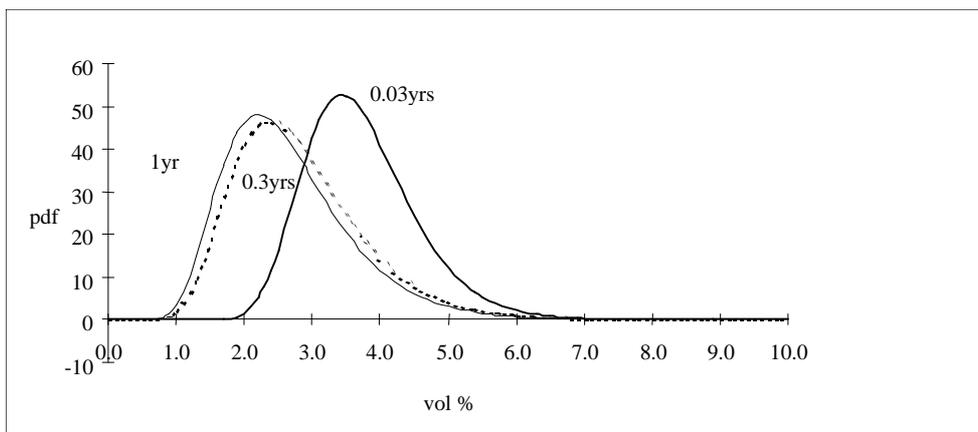


Figure 5.2: A cross-section of the volatility distribution at two different times

A method of analysing the evolution of a probabilistic distribution is to compare how the levels of volatility which have an equal likelihood of occurrence evolve over time. This can be done by calculating the cumulative density function. We then define contours that represent equal numbers of observations, that is, equal probabilities of occurrence for each timestep. This yields a contour map, as shown in Figure 5.3 which maps the value that volatility takes with equal probability for each point in time. Hence the contour CDF=0.8 tells us that 80% of the observations for volatility lie below this contour, and similarly the contour CDF=0.2 tells us that 20% of the observations have volatility less than defined by this contour.

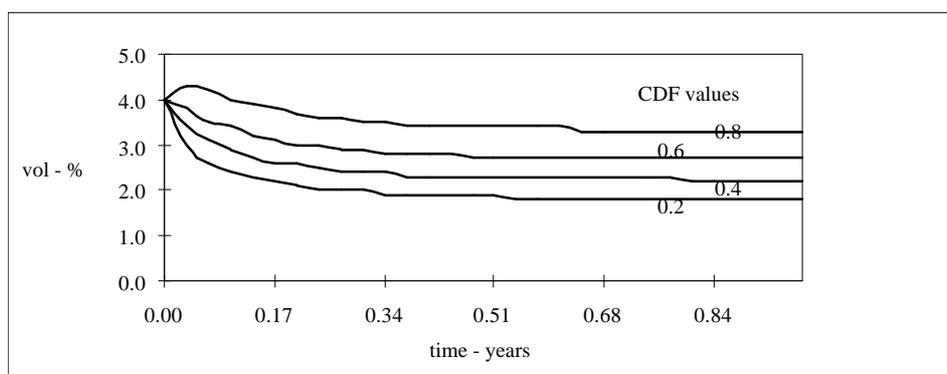


Figure 5.3: Contour Map for Volatility

## Section 6: Stochastic Volatility, Certainty Bands and Confidence Limits

### 6.1 Introduction

We are now in a position to incorporate into one framework the ideas, calculations and analyses developed in the preceding Sections. Our aim is to make a statement that portrays the confidence with which, according to the stochastic volatility model we have developed, the volatility will lie within the given ‘certainty band,’ and, as a direct consequence thereof, the confidence with which the uncertain parameter option price will lie within the worst/best spread we calculate with the nonlinear uncertain parameter model. In other words, we wish to calculate the probability of the random variable remaining within the prescribed bands, and hence the probability of the ‘certainty interval’ being certain. The methodology is as follows.

A ‘certainty band’ was initially described as being a band defining the range of values that the variable under consideration could take over the time of interest. Having developed a stochastic model for volatility, we will no longer say with complete confidence that volatility will remain within any two given bands. However, we can calculate the likelihood of the variable remaining within these bands.

Introduce the function  $C(\mathbf{s}, t)$  as the probability that the volatility leaves the certainty band before time  $T$ . This function satisfies the Kolmogorov backward equation, see Wilmott et al (1993),

$$\frac{\partial C}{\partial t} + \mathbf{a}(\mathbf{s}) \frac{\partial C}{\partial \mathbf{s}} + \frac{1}{2} \mathbf{b}(\mathbf{s})^2 \frac{\partial^2 C}{\partial \mathbf{s}^2} = 0. \tag{6.1}$$

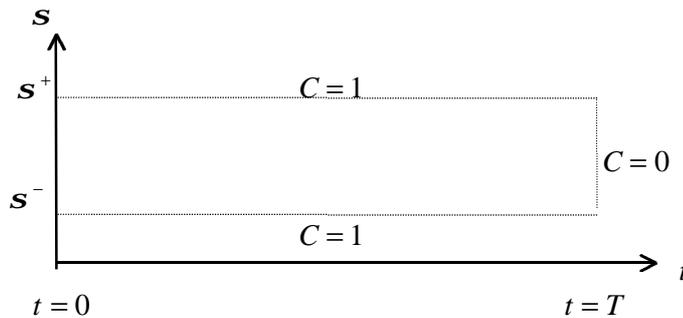
The final condition is given by

$$C(\mathbf{s}, T) = 0 \text{ for all } \mathbf{s} \tag{6.2}$$

since at  $t = T$  the likelihood of  $\mathbf{s}$  breaching the certainty band is zero, and the boundary conditions are given by

$$C(\mathbf{s}^+, t) = C(\mathbf{s}^-, t) = 1 \text{ for all } t \tag{6.3}$$

where  $\mathbf{s}^-$ ,  $\mathbf{s}^+$  are the lower and upper barriers respectively, since if  $\mathbf{s} = \mathbf{s}^+$  or  $\mathbf{s} = \mathbf{s}^-$  then the likelihood of breaching the certainty band is one. The final and boundary conditions are shown schematically below in *Figure 6.1*



*Figure 6.1*

### 6.2 Confidence limits

We define the confidence limit to be the probability we attach to the ‘certainty interval’ for the option price and derive it by calculating the function  $C(\mathbf{s}, t)$ . The required probability is then  $1 - C(\mathbf{s}, t)$ . This problem was solved numerically by a simple finite-difference scheme. Looking at *Figure 6.2* we can see that for an option with one year to expiry, using the above derived terms for drift and variance, if the current volatility level is 5% then the probability that bands defined as 1% - 9% will not be breached is 79%. Hence we can make the statement that we are 79% certain of the worst/best spread for the price of the option. This probability will obviously increase as the time to expiry decreases, and will decrease as the level of volatility approaches the barriers.

vol / time	0.00	0.20	0.40	0.60	0.80	1.00
1.00	0.00	0.00	0.00	0.00	0.00	0.00
2.00	0.74	0.78	0.82	0.87	0.92	1.00
3.00	0.78	0.82	0.87	0.92	0.97	1.00
4.00	0.79	0.83	0.88	0.93	0.98	1.00
5.00	0.79	0.83	0.88	0.93	0.97	1.00
6.00	0.77	0.82	0.86	0.91	0.95	1.00
7.00	0.72	0.76	0.80	0.84	0.88	1.00
8.00	0.54	0.57	0.60	0.63	0.66	1.00
9.00	0.00	0.00	0.00	0.00	0.00	0.00

*Figure 6.2: Probability of option price remaining within worst/best spread*

## Section 7: Conclusion

In this paper we have approached the pricing of a contingent liability from an uncertain parameter stand-point by assuming parameters are uncertain and by specifying bands within which we are confident the parameters will lie. We have thus derived a worst/best price spread for the option and refined this spread by using static hedging. We then derived an empirical stochastic model for volatility, used this to calculate a probability for the variable remaining within the specified bands, and hence derived a confidence limit on the uncertain parameter option price. In this way we have created a framework for valuing options that uses different inputs and assumptions from the traditional Black-Scholes approach. The advantage of this framework essentially lies in the lack of a probabilistic argument. That is, once we have traded an option at a certain price we know that if the parameters do not leave some specified bands we are guaranteed to make a profit. We can then superimpose on this structure the empirical stochastic model for volatility and attach a confidence limit to our option price.

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