# A Nonlinear Non-probabilistic Spot Interest Rate Model

by David Epstein and Paul Wilmott
Mathematical Institute, Oxford University

Address for correspondence:
Paul Wilmott
Mathematical Finance Group
Mathematical Institute
24-29 St Giles
Oxford
OX1 3LB
UK

Email: paul@wilmott.com

Abstract
We show how to use 'uncertainty' in place of the more traditional Brownian 'randomness' to model a short-term interest rate. The advantage of this model is principally that it is difficult to show statistically that it is wrong. Whether the model is useful for pricing fixed-income products is less clear. We discuss the pros and cons of the model, showing how to price and hedge various contracts, saying which are easy and which are hard.

## 1. Introduction

The following table describes reasons why the traditional interest rate models may be inadequate. By 'traditional' we mean the stochastic differential equation models, whether they be single or multi factor, and whether or not they are from the Heath, Jarrow & Morton family.

| **Assumption (A) or Consequence (C)** | **Reason for Inadequacy** |
|---|---|
| Normal increments (A) | There is plenty of statistical evidence that changes in interest rates are not Normally distributed. Typically there are greater chances of a small or very large move than predicted by a Normal distribution. And, of course, financial variables are known to be discontinuous. |
| Correlation between rates of different maturities (C) | Although rates are undoubtedly correlated across maturities, that correlation is notoriously unstable. Dynamic hedging is therefore not as simple as the theory makes out. Even if dynamic hedging were possible, the effects of discrete hedging are enormous. |
| Known parameters/functional forms (A) | All financial parameters, such as volatility, are unstable. Most models result in prices that are not robust to these parameters. Practitioners have various *ad hoc* ways of getting around this problem, some are sensible others are not. |
| Linearity (C) | The value of any contract is independent of what it is hedged with. Although this is financially nonsense it does make pricing easy. |
| Yield curve fitting (A) | The desire to dynamically hedge requires traditional models to correctly price hedging instruments. These days this is done via yield curve fitting. Impossible to justify theoretically, and rarely tested empirically, this is the universal practice. |
| Single price (C) | Almost all models result in a single price for a contract. The concept of one 'fair value' is popular. However, given all the uncertainty surrounding models and parameters, it is foolhardy to believe that a value is correct in any but a theoretical sense. |
| Analytical solutions (A or C?!) | Most popular models are chosen for their tractability i.e. that simple contracts have simple formulae for their prices. Should the desire for a nice formula drive the modelling process? |

Having dismissed all current models, can we replace them with something better? The answer is a resounding 'Yes' if our only concern is with validity of the model. A more rigorous test, and one which we may fail in many cases, is whether our model is 'useful.'

Our model is the next step in the evolution of financial models that began with Black & Scholes (1973). They presented a theory of options based on delta hedging and no arbitrage. This foundation was adapted for interest rates (see Vasicek, 1977) and later generalised by Heath, Jarrow & Morton (1992). The latter took the whole yield as an input so that they correctly 'modelled' today's discount factors. The main problem with these models is in the accurate estimation of parameters. (Apabhai, Choe, Khennach & Wilmott, 1995, show just how unstable some of the interest rate parameters can be.) In the equity world, Avellaneda, Levy & Paras (1995) and Lyons (1995) showed how to incorporate uncertainty in parameters to get around this problem. Their approach cannot be applied directly to the interest rate world because the variable that we choose to model is the spot interest rate and is not a traded quantity. Taking ideas from all of the approaches and modifying them to be applicable in the interest rate world, we came up with the present model.

## 2. The model

Our model is very simple. The independent variables are time, $t$, and a short-term interest rate, $r$. Other, multi-factor models are possible but do not add significantly to the accuracy of the model. We do not specify the process for $r$, only stating what it is not allowed to do. (See Epstein & Wilmott, 1997, 1998 for more details.)

First of all, we restrict the spot interest rate to lie in a given band

$$r^- \le r \le r^+.$$

For instance, we may say that a rate of greater than 20% or less than 3% is not attainable. To narrow down the possible evolution of $r$ we further impose a constraint on its growth rate,

$$c^- \le \frac{dr}{dt} \le c^+.$$

The short-term interest rate is not allowed to grow or decay faster than 4% *per annum* for example. The above are two restrictions on the path of $r$. We will now loosen the constraints, allowing the spot rate to jump discontinuously. We do this by subtly changing the definition of $r$. Now $r$ is a representative level for the spot interest rate, but the real rate $r'$ lies no more than a distance $\varepsilon$ away from $r$. We say that the spot rate $r'$ shadows $r$. Thus

$$|r - r'| \le \varepsilon.$$

This completes the spot rate model. Figure 1 shows a possible path for $r'$. Note that very extreme behaviour is possible, behaviour not permitted by other models.

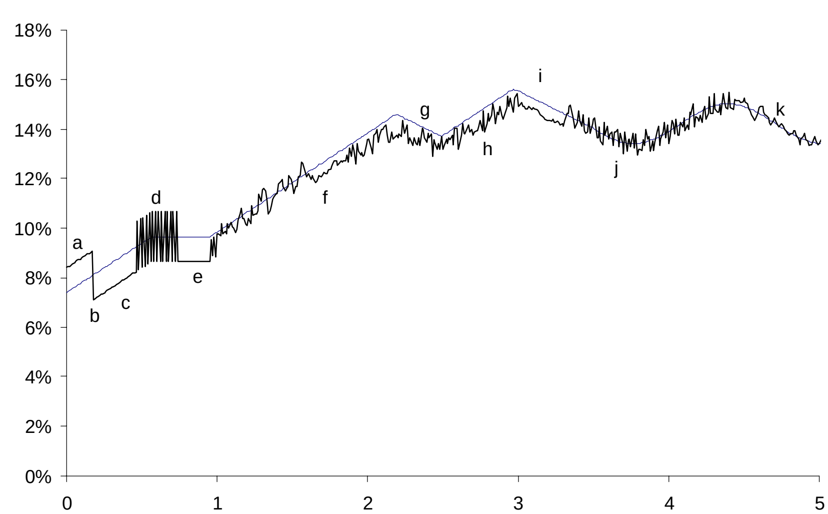

Figure 1: A possible evolution of the short-term interest rate. See text for an explanation.

In this figure are shown an evolution of *r* and of *r'*. The latter, the spot interest rate, is the volatile line, although the word 'volatile' does not have its usual, or indeed any precise, meaning here. We have deliberately plotted a rather bizarre evolution, demonstrating the rich structure that the model allows. Working from left to right, we see (a) a steady rate increase followed by (b) a jump, (c) a further smooth increase then followed by (d) a period during which the rate jumps discontinuously every day from one extreme to another. There is a period (e) where the rate is constant, followed by (f) a Brownian-looking spell with an upward trend. There is then another Brownian-looking period but with a downward trend, (h) another rising period, followed by (i) a low volatility period. There is then (j) a very volatile time with sinusoidal periodicity, followed by a (k) calmer spell.

Having set up the model, we now turn our attention to pricing various fixed-income contracts. In the next section we derive the governing equation.

## 3. The pricing equation

To set the scene, assume that we want to 'value' a contract consisting of a stream of known cashflows arriving at given times in the future. These cashflows may be positive or negative. Rather than find a single value for these cashflows today we shall find a range of values, from the worst up to the best. In practice, being pessimistic or conservative, we will be more interested in the worst possible value. We introduce $V(r,t)$ as the worst value of the contract at time $t$, when the spot rate (or rather the quantity which the real spot rate is shadowing) is $r$. The governing equation for $V$ comes from examining the smallest possible change in $V$ from one moment to the next, and then discounting at the spot rate.

The change in $V$ from time $t$ to time $t+dt$ is given by

$$V(r+dr,t+dt)-V(r,t) \qquad \text{(A)}$$

where $dr$ is the small increment in $r$ during the timestep $dt$. We do not know the amount of this increment until it happens. From a Taylor series expansion we find that (A) becomes

$$\frac{\partial V}{\partial t}dt+\frac{\partial V}{\partial r}dr. \qquad \text{(B)}$$

If we are being pessimistic then we must ask what could be the worst incremental value for our contract. We must therefore find the smallest value of (B) over all the possible values for $dr$. Trivially, we find that the worst case is for $dr$ to be as small as possible if $\partial V/\partial r$ is positive, and as large as possible if $\partial V/\partial r$ is negative. Thus the worst increment in $V$ is

$$\left(\frac{\partial V}{\partial t}+c\left(\frac{\partial V}{\partial r}\right)\frac{\partial V}{\partial r}\right)dt$$

where

$$c(x)=\begin{cases}c^- & x\geq 0\\ c^+ & x<0\end{cases}$$

Had we invested an amount $V$ in a risk-free account it would have grown by an amount

$$r'V\;dt$$

during the same timestep. However, $r'$ is unknown. All we now is that it lies within a certain distance of the shadow rate $r$. Being pessimistic, we assume that this account grows by the largest amount, after all we are not investing in this risk-free account. (So not only does our contract change by the least amount, we also missed an excellent risk-free opportunity.) This increment is simply

$$(r + e(V))V\,dt$$

where

$$e(x) = \begin{cases} \varepsilon & x \geq 0 \\ -\varepsilon & x < 0 \end{cases}$$

The final stage in deriving the governing equation is to equate the worst-case increment of our contract with the best case obtained from investing an amount *V* in a 'risk-free' account earning the spot rate. After dividing through by the timestep we arrive at

$$\frac{\partial V}{\partial t} + c\left(\frac{\partial V}{\partial r}\right)\frac{\partial V}{\partial r} - (r + e(V))V = 0 \qquad \text{(C)}$$

where, to summarize,

$$c(x) = \begin{cases} c^- & x \geq 0 \\ c^+ & x < 0 \end{cases}$$

$$e(x) = \begin{cases} \varepsilon & x \geq 0 \\ -\varepsilon & x < 0 \end{cases}$$

This is our governing pricing equation. It is first-order nonlinear hyperbolic. As far as the mathematics and numerical analysis are concerned, the equation differs greatly from the traditional second-order linear parabolic.

As we said earlier, the model predicts a range of values for a contract. The best price is obtained by solving the same equation but with the inequalities reversed in the definitions of *c* and *e*.

## 4. Consequences of the nonlinearity

The most important feature of (C) is that it is nonlinear. Nonlinearity of the pricing equation results in the following properties of the prices of contracts:

- There is no such thing as 'the' price of a contract
- Long and short positions have different values
- The value of a portfolio of contracts is generally not the same as the sum of the values of the individual components
- Hedging a 'target' contract with other, market-traded contracts will change the marginal value of the target contract

(Some recent nonlinear credit risk models have other interesting properties as well (Ahn, Khadem & Wilmott, 1998).)

Because of the nonlinearity, the value of a portfolio of contracts is not necessarily the same as the sum of the values of the individual components. This is a very important point to understand: the value of a contract depends on what else is in the portfolio. These two points are key to the importance of nonlinear pricing equations: they give us a bid-offer spread on prices, and they allow optimal static hedging.

One of the interesting points about nonlinear models is the prediction of a spread between long and short prices. If the model gives different values for long and short then this is in effect a spread on prices. This can be seen as either a good or a bad point. It is good because it is realistic, spreads exist in practice. It only becomes bad when this spread is too large to make the model useful. The following idea of static hedging for spread reduction was originally due to Avellaneda & Paras (1996).

Suppose that we want to sell a contract with some payoff that does not exist as a traded contract, an over the counter (OTC) contract. We want to determine how low a price can we sell it for (or how high a price we can buy it for), with the constraint that we guarantee that we will not lose money as long as our range for volatility is not breached. By adding and subtracting traded contracts to and from our OTC contract we can modify its marginal value, either decrease or increase it, after allowing for the quoted prices of the traded contracts. So, generally speaking, we expect a different OTC value if we choose a different static hedge portfolio. Of course, we now ask if we get different values for an option depending on what other contracts we hedge it with then is there a best static hedge? Details are contained in Epstein & Wilmott (1997, 1998).

One of the inevitable consequences of this optimization concerns the optimal static hedging of a portfolio of traded contracts. If we try to price a traded contract in isolation we will get a spread of prices that differ greatly from the market prices. However, if we then statically hedge this contract we find that it is optimal to hedge it one-for-one with the traded contract itself. In other words, it is optimal to close the position and we find that the contract price is then the same as the market price. Fitting or calibration is a consequence of the optimization, we do not have to fudge any parameters to match market prices.

## 5. Examples

**Coupon-bearing bonds**

Contracts with known, fixed cashflows at prescribed dates, such as coupon-bearing bonds, are the easiest contracts to value in the above framework. Each cashflow is represented by a jump in the value of the contract across the payment date. For example, if the holder of a contract receives an amount *q* on date *T* then we have

$$V(r,T-) = V(T+) + q. \tag{D}$$

Here *T*- means just before the payment is made and *T*+ just after. Financially, this jump condition represents the loss of value of the contract after the coupon has been paid.

Portfolios of coupon-bearing bonds are treated as the sum of all the individual cashflows with a discontinuity in the contract value at each coupon date.

**Range notes**

The range note pays a fixed amount, say $1, for every day that a specified interest rate lies within a given range. If the relevant interest is a very short rate then it can be approximated by *r*. In this case, the governing equation is

$$\frac{\partial V}{\partial t} + c\left(\frac{\partial V}{\partial r}\right)\frac{\partial V}{\partial r} - \left(r + e(V)\right)V + I(r) = 0$$

Where $I(r)$ is the function taking the value 1 for values of $r$ in the range.

**Swaps**

Vanilla swaps are an exchange of a fixed and a floating interest rate on a fixed principal. They are easily incorporated into the framework by first decomposing them into a series of zero-coupon bonds. This is a model-independent decomposition, see Wilmott (1998) for details.

**Caps and floors etc.**

Caps and floors put a bound on periodic payments of interest, either bounding them above or below. Each cap is made up of a series of caplets and each floor a series of floorlets. The cashflow in a caplet or floorlet is a function of a usually short-term interest rate. If this interest rate can be approximated by the rate $r$ then we can price caps and floors using Equation (C) with similar jump conditions to (D), the only difference being that the $q$ is now a function of $r$.

**Index amortising rate swaps (IARS)**
The index amortizing rate swap differs from the vanilla swap in the amount of the principal. In the vanilla swap the principal remains fixed at its initial, agreed value. In the IARS the level of the principal amortizes, decreases, according to some schedule that depends on the level of an index at the time of the payment of the interest. The index could be any quantity, but a particularly popular IARS has an index that is the short-term interest rate itself.

Because the principal amortizes according to a sophisticated schedule, the IARS is path dependent. Yet this path dependency is easily accommodated within the present model. The trick to valuing this contract is to introduce a new state variable. This new state variable is the current level of the principal, denoted by $P$. The value of the IAR swap is $V(r,P,t)$.

The variable $P$ is deterministic and jumps to its new level at each resetting. Since $P$ is piecewise constant, the governing differential equation for the value of the swap is, in the present model, still (C).

At each reset date there is an exchange of interest payments and an amortization of the principal. If we use $t_i$ to denote the reset dates and $r_f$ for the fixed interest rate, then the swap jumps in value by an amount $(r-r_f)P$. Subsequently, the principal $P$ becomes $g(r)P$ where the function $g(r)$ is the representation of the amortizing schedule. This gives us the jump condition

$$V(r,P,t_i^-) = V(r,g(r)P,t_i^+) + (r - r_f)P.$$

At the maturity of the contract there is one final exchange of interest payments, thus

$$V(r,P,T) = (r - r_f)P.$$

The problem is nonlinear, and must be solved numerically. The structure of this particular IAR swap is such that there is a similarity reduction, just look for a solution of the form

$$V(r,P,t) = PH(r,t).$$

We are lucky that the similarity reduction is not affected by the nonlinearity.

**Bond options**
Options on bonds can also be priced with the framework. However, some ingenuity is required. It is very easy to price contracts for which cashflows are functions of the spot interest rate. It is much harder to price contracts that are derivatives of derivatives of the spot rate; bonds are spot rate derivatives and bond options are therefore derivatives of derivatives.

Let us call the maturity of the underlying bond $T$ and the expiry of the option $T_O$. We will probably want to hedge our bond option with other bonds so we must be able to value a portfolio consisting of one option and any number of bonds. We introduce two functions $U$ and $V$, both functions of $r$ and $t$. The former is the value of the whole portfolio if we do not exercise the options and the latter is the value if we do exercise the option. The difference between the former and the latter problems is that the former only has cashflows associated with the vanilla bonds whereas the latter has two extra sets of cashflows; one is the cashflow at the date $T_O$ corresponding to the strike of the options and other cashflows corresponding to payments due to the new bond into which we have exercised.

We solve (C) for both $U$ and $V$, working backwards from the last cashflow date until we get to $T_O$. We therefore know how much the portfolio is worth in the worst case for both possibilities, exercise or non exercise. Since we have a choice whether or not to exercise we continue solving (C) but with

$$V(r,T_O) = \max(U(r,T_O), V(r,T_O))$$

as the final condition at time $T_O$.

This example is of particular interest since it gives us an idea of difficulties that may be encountered in pricing with the nonlinear worst-case model. For example, if we have $n$ bond options in our portfolio we must introduce $2^n$ functions to allow for all the possible combinations of exercise and non exercise.

## 6. Conclusions

This paper was a summary of a new, non-probabilistic paradigm for modelling short-term interest rates. The model has many nice features such as ranges for prices and exact matching of market-traded prices. Prices are also more robust than in traditional models. The nonlinearity which results in these nice properties also makes the pricing of some contracts quite complicated. To get the full benefit of the model, one must combine all contracts into one portflio, but this will inevitably lead to a computationally expensive problem. But to price contracts independently and therefore quickly completely misses the point of the model.

## Acknowledgements

DE thanks the Smith Institute and PW thanks the Royal Society for their financial support.

## References

Ahn, H, Khadem, V & Wilmott, P 1998 The pricing of risky bonds: current models and future directions. MFG Working Paper, Oxford

Apabhai, MZ, Choe, K, Khennach, K & Wilmott, P 1995 Spot-on modelling. Risk magazine, December **8** (11) 59-63

Avellaneda, M, Levy, A & Paras, A 1995 Pricing and hedging derivative securities in markets with uncertain volatilities. *Applied Mathematical Finance* **2** 73-88

Avellaneda, M & Paras, A 1996 Managing the volatility risk of derivative securities: the Lagrangian volatility model. *Applied Mathematical Finance* **3** 21-53

Black, F & Scholes, M 1973 The pricing of options and corporate liabilities. *Journal of Political Economy* **81** 637-59

Epstein, D & Wilmott, P 1997 Yield envelopes. *Net Exposure* **2** August www.netexposure.co.uk

Epstein, D & Wilmott, P 1998 Fixed income security valuation in a worst-case scenario. *International Journal of Theoretical and Applied Finance*

Heath, D, Jarrow, R & Morton, A 1992 Bond pricing and the term structure of interest rates: a new methodology. Econometrica **60** 77-105

Lyons, TJ 1995 Uncertain volatility and the risk-free synthesis of derivatives. *Applied Mathematical Finance* **2** 117-133

Vasicek, OA 1977 An equilibrium characterization of the term structure. Journal of Financial Economics **5** 177-188

Wilmott, P 1998 *Derivatives: the theory and practice of financial engineering*. John Wiley & Sons Ltd