Modelling Market Crashes: the Worst-case Scenario

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1 Introduction

Jump diffusion models (Merton, 1976) have two weaknesses: they don't allow you to hedge and the parameters are very hard to measure. Nobody likes a model that tells you that hedging is impossible (even though that may correspond to common sense) and in the classical jump-diffusion model of Merton the best that you can do is a kind of average hedging. It may be quite easy to estimate the impact of a rare event such as a crash, but estimating the probability of that rare event is another matter. In this paper we discuss a model for pricing and hedging a portfolio of derivatives that takes into account the effect of an extreme movement in the underlying but we will make *no assumptions about the timing of this 'crash' or the probability distribution of its size*, except that we put an upper bound on the latter. This effectively gets around the difficulty of estimating the likelihood of the rare event. The pricing follows from the assumption that the worst scenario actually happens i.e. the size and time of the crash are such as to give the option its worst value. And hedging, delta and static hedging, will continue to play a key role. The optimal static hedge follows from the desire to make the best of this worst value. This, latter, static hedging follows from the desire to optimize a portfolio's value. We also show how to use the model to evaluate the value at risk for a portfolio of options.

More details of the model are contained in Hua (1997), Hua & Wilmott 91997) and the text Wilmott (1998).

2 Value at risk

The true business of a financial institution is to manage risk. The trader manages 'normal event' risk, where the world operates close to a Black–Scholes one of random walks and dynamic hedging. The institution, however, views its portfolio on a 'big picture' scale and focuses on 'tail events' where liquidity and large jumps are important.

Value at Risk (VaR) is a measure of the potential losses due to a movement in underlying markets. It usually has associated with it a timeframe and an estimate of the maximum sudden change thought likely in the markets. There is also a 'confidence interval'; for example, the daily VaR is $15 million with a degree of confidence of 99%.

A more general and more encompassing definition of VaR will give a useful tool to both book runners and senior management. A true measure of the risk in a portfolio will answer the question 'What is the value of any realistic market movement to my portfolio?'

The approach taken here in finding the value at risk for a portfolio is to model the cost to a portfolio of a crash in the underlying. We show how to value the cost of a crash in a worst-case scenario, and also how to find an optimal static hedge to minimize this cost and so reduce the value at risk.

3 A simple example: the hedged call

To motivate the problem and model, consider this simple example. You hold a long call position, delta hedged in the Black–Scholes fashion. What is the worst that can happen, in terms of crashes, for the value of your portfolio? One might naively say that a crash is bad for the portfolio, after all, look at the Black–Scholes value for a call as a function of the underlying, the lower the underlying the lower the call value. Wrong. Remember you hold a *hedged* position; the position is currently delta neutral and the portfolio's value is currently at its minimum; a sudden fall (or, indeed, a rise) will result in a higher portfolio value, a crash is beneficial. If we are assuming a worst-case scenario, then the worst that could happen is that there is no crash. Changing all the signs to consider a short call position we find that a crash is bad, but how do we find the worst case? If there is going to be one crash of 10% when is the worst time for this to happen? This is the motivation for the model below. Note first that, generally speaking, a positive gamma position benefits from a crash, while a negative gamma position loses.

4 A mathematical model for a crash

The main idea in the following model is simple. We assume that the worst will happen. We value all contracts assuming this, and then, unless we are very unlucky and the worst does happen, we will be pleasantly surprised. In this context, 'pleasantly surprised' means that we make more money than we expected. We can draw an important distinction between this model and the jump diffusion models. In the latter we make bold statements about the frequency and distribution of jumps and finally take expectations to arrive at a value for a derivative. Here *we make no statements about the distribution of either the jump size or when it will happen.* At most the number of jumps is limited. Finally, we examine the worst-case scenario so that no expectations are taken.

We will model the underlying asset price behaviour as the classical binomial tree, but with the addition of a third state, corresponding to a large movement in the asset. So, really, we have a trinomial walk but with the lowest branch being to a significantly more distant asset value. The up and down diffusive branches are modeled in the usual binomial fashion. For simplicity, assume that the crash, when it happens, is from S to $(1-k)S$ with k given; this assumption can easily be dropped to allow k to cover a range of values, or even to allow a dramatic rise in the value of the underlying. Introduce the subscript 1 to denote values of the option before the crash i.e. with one crash allowed, and 0 to denote values after. Thus V_0 is the value of the option position after the crash. This is a function of S and t and, since we are only permitting one crash, V_0 must be exactly the Black–Scholes option value.

If the underlying asset starts at value S it can go to one of three values: uS, if the asset rises, call this state A; vS, if the asset falls, state B; $(1-k)S$, if there is a crash, state C. The values for uS and vS are chosen in the usual manner for the traditional binomial model.

Before the asset price moves, we set up a 'hedged' portfolio, consisting of our option position and $-\Delta$ of the underlying asset. At this time our option has value V_1. We must find both an optimal Δ and then V_1.

A time δt later the asset value has moved to one of the three states, A, B or C and at the same time the option value becomes either V_1^+ (for state A), V_1^- (for state B) or the Black–Scholes value V_0 (for state C).

The change in the value of the portfolio, between times t and $t+\delta t$ (denoted by $\delta\Pi$) is given by the following expressions for the three possible states:

$$\delta\Pi_A = V_1^+ - \Delta uS + \Delta S - V_1 \quad \text{(diffusive rise)}$$

$$\delta\Pi_B = V_1^- - \Delta vS + \Delta S - V_1 \quad \text{(diffusive fall)}$$

$$\delta\Pi_C = V_0 + \Delta kS - V_1 \quad \text{(crash)}.$$

Our aim in what follows is to choose the hedge ratio Δ so as to minimize the pessimistic, worst outcome among the three possible.

There are two cases to consider. The first, Case I, is when the worst-case scenario is not the crash but the simple diffusive movement of S. In this case V_0 is sufficiently large for a crash to be beneficial:

$$V_0 \geq V_1^+ + (S - uS - kS)\frac{V_1^+ - V_1^-}{uS - vS}. \tag{1}$$

If V_0 is smaller than this, then the worst scenario is a crash; this is Case II.

4.1 Case I: Black–Scholes hedging

The maximal-lowest value for $\delta\Pi$ occurs at the point where

$$\delta\Pi_A = \delta\Pi_B,$$

that is

$$\Delta = \frac{V_1^+ - V_1^-}{uS - vS}. \tag{2}$$

This will be recognised as the expression for the hedge ratio in a Black–Scholes world.

Having chosen Δ, we now determine V_1 by setting the return on the portfolio equal to the risk-free interest rate. Thus we set

$$\delta\Pi_A = r\Pi\,\delta t$$

to get

$$V_1 = \frac{1}{1 + r\,\delta t}\left(V_1^+ + (S - uS + rS\,\delta t)\frac{V_1^+ - V_1^-}{uS - vS}\right). \tag{3}$$

This is the equation to solve if we are in Case I. Note that it corresponds exactly to the usual binomial version of the Black–Scholes equation, there is no mention of the value of the portfolio at the point C. As δt goes to zero, (2) becomes the $\partial V/\partial S$ and Equation (3) becomes the Black–Scholes partial differential equation.

4.2 Case II: Crash hedging

In this case the value for V_0 is low enough for a crash to give the lowest value for the jump in the portfolio. We therefore choose Δ to maximise this worst case. Thus we choose

$$\delta\Pi_A = \delta\Pi_C,$$

that is,

$$\Delta = \frac{V_0 - V_1^+}{S - uS - kS}. \tag{4}$$

Now set

$$\delta\Pi_A = r\Pi\,\delta t$$

to get

$$V_1 = \frac{1}{1 + r\,\delta t}\left(V_0 + S(k + r\,\delta t)\frac{V_0 - V_1^+}{S - uS - kS}\right). \tag{5}$$

This is the equation to solve when we are in Case II. Note that this is different from the usual binomial equation, and does not give the Black–Scholes partial differential equation as δt goes to zero. Also (4) is not the Black–Scholes delta.

5 An example

All that remains to be done is to solve equations (3) and (5) (which one is valid at any asset value and at any point in time depends on whether or not (1) is satisfied). This is easily done by working backwards down the tree from expiry in the usual binomial fashion.

As an example, examine the cost of a 15% crash on a portfolio consisting of the call options in Table 1.

Strike	**Expiry**	**Bid**	**Ask**	**Quantity**
100	75 days			-3
80	75 days			2
90	75 days	11.2	12	0

Table 1: Available contracts.

At the moment the portfolio only contains the first two options. Later we will add some of the third option for static hedging, that is when the bid-ask prices will concern us. The volatility of the underlying is 17.5% and the risk-free interest rate is 6%.

The value of the portfolio assuming the worst is 21.2 when the spot is 100. This is significantly lower than the Black–Scholes value of 30.5. This large difference is due to the portfolio's gamma being highly negative. When the gamma is positive, a crash is beneficial to the portfolio's value. When the gamma is close to zero, the delta hedge is very accurate and the option is insensitive to a crash. If the asset price is currently 100, the difference between the before and after portfolio values is $30.5 - 21.2 = 9.3$. This is the 'Value at Risk' under the worst-case scenario.

6 Optimal static hedging: VaR reduction

The 9.3 value at risk is due to the negative gamma around the asset price of 100. An obvious hedging strategy that will offset some of this risk is to buy some positive gamma as a 'static' hedge. In other words, we should buy an option or options having a counterbalancing effect on the value at risk. We are willing to pay a premium for such an option. We may even pay more than the Black–Scholes fair value for such a static hedge because of the extra benefit that it gives us in reducing our exposure to a crash. Moreover, if we have a choice of contracts with which to statically hedge we should buy the most 'efficient' one. To see what this means consider the above example in more detail.

Recall that the value of the initial portfolio under the worst-case scenario is 21.2. How many of the 90 calls should we buy (for 12) or sell (for 11.2) to make the best of this scenario? Suppose that we buy λ of these calls. We will now find the optimal value for λ.

The cost of this hedge is

$$\lambda C(\lambda)$$

where $C(\lambda)$ is 12 if λ is positive and 11.2 otherwise. Now solve Equations (3) and (5) with the final total payoffs

$$V_0(S,T) = V_1(S,T) = 2\max(S-80,0) - 3\max(S-100,0) + \lambda\max(S-90,0).$$

This is the payoff at time T for the statically hedged portfolio. The *marginal* value of the original portfolio (that is, the portfolio of the 80 and 100 calls) is therefore

$$V_1(100,0) - \lambda C(\lambda,) \tag{6}$$

i.e. the worst-case value for the new portfolio less the cost of the static hedge. The arguments of the before-crash option value are 100 and 0 because they are today's asset value and date. The optimality in this hedge arises when we choose the quantity λ to maximize the value, expression (6). With the bid-ask spread in the 90 calls being 11.2–12, we find that buying 3.5 of the calls maximizes expression (6). The value of the new portfolio is 70.7 in a Black–Scholes world and 65.0 under our worst-case scenario. The value at risk has been reduced from 9.3 to $70.7 - 65 = 5.7$. The optimal static hedge is known as the 'Platinum Hedge.'

7 Continuous-time limit

If we let $\delta t \to 0$ in Equations (1), (2), 3), (4) and (5) we find that the Black–Scholes equation is still satisfied by $V_1(S,t)$ but we also have the constraint

$$V_1(S,t) - kS\frac{\partial V_1}{\partial S}(S,t) \leq V_0(S(1-k),t). \tag{7}$$

Such a problem is similar in principal to the American option valuation problem, where we also have a constraint on the derivative's value. Here the constraint is more complicated. To this we must add the condition that the first derivative of V_1 must be continuous for $t < T$.

8 Conclusion

We have presented a model for the effect of an extreme market movement on the value of portfolios of derivative products. This is an alternative way of looking at value at risk. We have shown how to employ static hedging to minimise this VaR. In conclusion, note that the above is not a jump-diffusion model since we have deliberately not specified any probability distribution for the size or the timing of the jump: we model the worst-case scenario.

One further thought is that we have not allowed for the rise in volatility that accompanies crashes. This can be done with ease. There is no reason why the after-crash model (V_0 in the simplest case above) cannot have a different volatility from the before-crash model. See Derman & Zou (1997) for relevant data.

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