The Pricing of Derivatives in Illiquid Markets

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Abstract

This paper develops a parameterised model for liquidity effects arising from trading an asset. The liquidity effect is defined as an individual transaction cost and a price slippage impact, that is felt by all participants in the market. The liquidity model is based on the CRR binomial and is applied to the pricing and hedging of options. It derives natural bid-ask spreads for an option that are based on the liquidity of the market for the underlying. We also mention further applications of our model like portfolio trading, liquidity options and strike detection.

1 Introduction

Two of the underlying assumptions of, amongst other, the basic Black-Scholes or CAPM economies are, firstly, frictionless markets and, secondly, that every agent is a price-taker. But real world markets substantially deviate from these assumptions, because for virtually all traded assets there exist both bid-ask spreads and a limited market depth. The effects of the two on asset dynamics are loosely referred to as liquidity, meaning the more of an asset is tradeable at tight spreads, the more liquid, and thus attractive, a market. Currently, a vast amount of research is conducted on how to measure, parameterise, price and manage liquidity in most fields of finance. This includes the extension of basic arbitrage or equilibrium models to cover the case of finite liquidity.

The latter has two main effects. Firstly, it represents a random transaction cost, which is correlated with the market's dynamics. In general, a market consists of competing buyers and sellers, who quote an asset's transaction directions, prices and quantities. The most common exchange structures are respectively, a monopolistic market maker, an oligopoly of market makers or an order-driven market. In all cases, there will be layers of bid and ask quotes with the respective quantities. The width and depth of the spreads primarily represent a transaction cost for market makers, since they will buy low and sell high and, secondly, an insurance again asymmetric information. Generally, if there exist many competing market participants that want to trade, bid-ask spreads tend to be narrow and market depth substantial, because low transaction cost will attract large volume. But, whereas it may be possible for agents to trade small quantities of an asset at the best possible price, the larger the trade size, the more levels of market depth will have to be tapped. Hence the average transaction price will be a, in general, nonlinear monotonously increasing function of trade size.

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Secondly, liquidity is directly responsible for the degree of market slippage. This means that, since every participant can observe the same market depth, every trade of any one agent is felt throughout. If a large trader removes a certain price level in its entirety, called the slippage, then subsequently market makers may adjust their prices. Comparatively large individual transactions or influential market participants hence push the asset price in a certain direction, in some cases deliberately (see e.g. [Tal]). Hence there is a market manipulation effect associated with liquidity, that goes beyond pure transaction costs. The two effects may have opposite signs, but for the market to be free of arbitrage, transaction costs have to be higher than the gains from market manipulation.

Generally, however, there is no consensus approach to the parameterisation and measurement of the liquidity of a market. The papers of [Lon] and [CRS] use combinations of bid-ask spreads, volume and open interest as a proxy to empirically investigate the effects on returns and distributions of the underlying and options on it. The papers of [Jar1], [Schö], [Frey], [A&C] and [H&S1] propose liquidity models that feature a reaction function that models the immediate impact of a trade and the average price paid per asset. It is also a function of both an liquidity scaling parameter and the trade size. A possible proxy for the former is explicitly given by [Krak] as the ratio of notional traded to the relative change in the price of the underlying asset. This choice of estimator has the advantage that at the time of the trade the liquidity parameter is observable and predictable. The papers by [A&C], [H&S1] and [H&S2], further consider a permanent slippage effect on the asset, by making its new equilibrium price a function of both the previous and the average transaction price. But they only apply their models to optimal portfolio trading strategies.

One area of finance where liquidity is a significant factor is the valuation and hedging of options. Even if no single trader has the intention to push the market in a certain direction, there exist agents who have to trade certain quantities of the underlying in order to hedge their exposure to a portfolio of derivatives. If, as in the Black-Scholes theory, they try and Delta-hedge, then for options with non-smooth or even-discontinuous payoffs, the Delta and Gamma, i.e. the amount of the underlying they have to hold and add/remove, respectively, may become very large close to expiry or close to payoff discontinuities. Since, in reality, markets only have limited liquidity, they will thereby automatically move the market in a certain direction. To avoid any mis-hedging, the respective ratios have to be adjusted for this feedback effect. This in turn will affect the price of the portfolio, since the risk-free amount that can be earned on a replicating portfolio changes as well. Moreover, the price of a portfolio of options is not the sum of the individual options.

To incorporate the effects of finite liquidity into option prices and hedging strategies, we employ a discrete time model, based on binomial trees. For the transaction cost effect we make the observable asset price an exponential function of the trade size, scaled by a liquidity parameter. For the permanent slippage effect, we take a geometric average of the last observed and the average transaction price. This makes the model nonlinear. We show that under certain realistic assumptions the trees become recombining and can be implemented. By changing the sign of the option payoff, we derive natural bid-ask spreads of the option, that arise from the degree of illiquidity of the market for the underlying. Finally, we mention some further extensions to and applications of the basic model.

2 The basic model

The main building block for the pricing framework of derivatives and portfolio trades is a suitable model for the underlying asset. We thus commence our analysis in a discrete time finite horizon economy where trading in assets takes place at times $\{t_0, t_1, \ldots, t_n = T\}$. The state of the economy is given by the finite set $\Omega = \{\omega_1, \ldots, \omega_m\}$ and the revelation of the true state by the increasing sequence of algebras $(\mathcal{F}_t)_{t\in\{t_0,\dots,T\}}$. The initial set of states is $\mathcal{F}_{t_0} = \Omega$, the eventual true state of the economy is revealed as $\mathcal{F}_T = \omega_j$, $\forall\omega_j \in \Omega$. There are two assets, namely a risky "stock" $S_t(\omega)$ and a riskless "bond" B_t, whose respective processes are adapted to the filtration $(\mathcal{F}_t)_{t\in\{t_0,\dots,T\}}$ and valued in $\mathbb{R}_+$.

Resorting to the widely used binomial model of [CRR], we will model randomness, which represents the arrival of information and agents trading in the stock, by making the risky asset go up by a fraction $u-1$ with probability p or down by a fraction $1-d$ with probability $1-p$ over one time step. Therefore

$$S_{t_{i+1}}(\omega_j) = \begin{cases} uS_{t_i} & \text{if } \omega_j = \omega_u \\ dS_{t_i} & \text{if } \omega_j = \omega_d \end{cases}, \tag{1}$$

where $u > d$. The bond on the other hand will always yield the riskless return r, namely

$$B_{t_{i+1}} = (1+r)B_{t_i}. \tag{2}$$

Moreover we can set the initial values of stock and bond equal to $S_{t_0} = S$ and $B_{t_0} = 1$, without loss of generality. Two key properties of the model are, firstly, the absence of arbitrage provided that $0 < d < 1 + r < u$ and, secondly, that for appropriate choice of u, d and the risk-neutral probability p the model's first two moments, approximately, can be fitted to the corresponding moments of continuous geometric Brownian motion

$$dS_t = rS_t dt + \sigma S_t dX_t, \tag{3}$$

where σ is the assets volatility and dX_t increments of standard Brownian motion. The same model is employed in the seminal paper by [B&S] as the model for the underlying asset.

On top of this random process for the underlying we construct a controlled process that represents the effect of a large or influential trader on the market. We denote this trader's holding process in the stock by $(H_t(\omega))_{\forall t,\omega}$ and in the bond by $(\hat{H}_t(\omega))_{t\in\{\forall t,\omega}$. Both processes are adapted to the filtration $(\mathcal{F}_t)_{t\in\{t_0,\dots,T\}}$ and one step ahead predictable with respect to it. The latter point entails, that the trader's portfolio can be rebalanced in between the random jumps of the underlying asset. If we now assume that S represents the mid-market price, then the best buying and selling prices will be above and below, respectively. Also, if the quantity traded is larger than the quantity offered at the best price, then more than one quote has to be filled in order to complete the trade. This means that the average transaction price $\bar{S}$ is a monotonously increasing function of the trade size. We define its process $(\bar{S}_t)_{\forall t,\omega}$ as a function $f(\cdot)$ of the current spot S_t, liquidity λ and trade size $(H_{t_{i+1}} - H_{t_i})$. Intuitively, the trade-reaction or price-impact function, in addition to being monotonous and positively sloped with respect to the trade size, should have the properties that

$$\lim_{H_{t_{i+1}}-H_{t_i}\downarrow -\infty} f = 0, \qquad \lim_{H_{t_{i+1}}-H_{t_i}\uparrow \infty} f = \infty, \qquad f(H_{t_{i+1}} - H_{t_i} = 0) = S_{t_i}.$$

One possible function as already noted in [Jar1] and [Frey] is

$$\bar{S}_{t_i} = S_{t_i} e^{\lambda(H_{t_{i+1}} - H_{t_i})}, \tag{4}$$

where $\lambda \geq 0$ is a liquidity scaling parameter and we suppressed the explicit dependence on the trajectory ω. As an example figure 1 shows the exact average transaction price as a function of trade size for an order book with homogeneous equidistant market depth and compares it with an estimate obtained from (4). The total cash flow and implicit transaction cost are given by $(H_{t_{i+1}} - H_{t_i})\bar{S}_{t_i}$ and $-(H_{t_{i+1}} - H_{t_i})(\bar{S}_{t_i} - S_{t_i})$, respectively. Unlike the transaction cost functions of [B&V] and [ENU] (4) is asymmetric, but does not require the modulus sign, which, as we will see, makes it possible to remove the path dependence and make the resulting tree recombining.

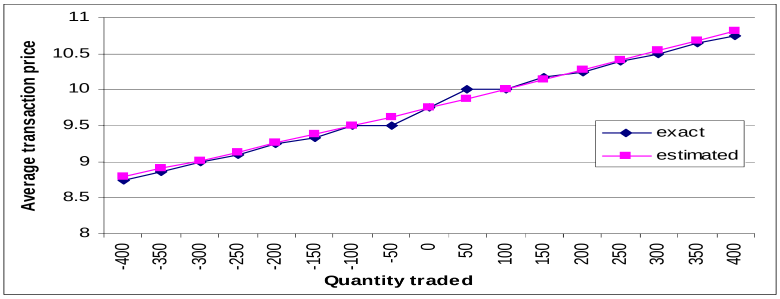


Figure 1: Average transaction prices

Now, in addition to the transaction cost effect, there is a market manipulation effect that is felt by all participants, since the best quotes have been removed from the order book. Unlike [Jar1], [Jar2], [Schö], [Frey] and [Krak] the papers by [A&C], [H&S1] and [H&S2] treat the reaction function as an instantaneous price impact and they distinguish a permanent price update effect, which is a function of both the previous equilibrium and the average transaction price. An intuitive explanation is that large trades may not contain fundamental new information and hence push the market to an untenable price level. A mathematically convenient model for this effect is to make the new equilibrium log-price a linear combination of the two previous equilibrium and average transaction log-prices or, equivalently, a geometric average of the two prices:

$$S_{t_{i+1}}(\omega_j) = \begin{cases} uS_{t_i}^{\alpha}\bar{S}_{t_i}^{1-\alpha} & \text{if } \omega_j = \omega_u \\ dS_{t_i}^{\alpha}\bar{S}_{t_i}^{1-\alpha} & \text{if } \omega_j = \omega_u \end{cases} . \tag{5}$$

If $0 \leq \alpha \leq 1$ and constant, then the new observable price $S_{t_{i+1}}$ is a convex combination. But α, realistically, can be negative, since in general the average transaction price is, depending on the trade direction, below or above the last price traded, unless only one level of market depth was filled. Combining the instantaneous trade-reaction (4), the permanent slippage (5) and reverting to the binomial representation (1) we obtain $\forall t_i$ the price dynamics

$$S_{t_i} \to \bar{S}_{t_i} = S_{t_i} e^{\lambda(H_{t_{i+1}} - H_{t_i})} \to \begin{cases} uS_{t_i}^{\alpha}\bar{S}_{t_i}^{1-\alpha} = uS_{t_i} e^{\lambda(1-\alpha)(H_{t_{i+1}} - H_{t_i})} \\ dS_{t_i}^{\alpha}\bar{S}_{t_i}^{1-\alpha} = dS_{t_i} e^{\lambda(1-\alpha)(H_{t_{i+1}} - H_{t_i})}. \end{cases} \tag{6}$$

This model setup, albeit structured similarly, is different from those of [A&C], [H&S1] and [H&S2], who in their respective papers, resort to arithmetic Brownian motion

$$dS_t = \mu dt + \sigma dX_t, \tag{7}$$

as the model for the dynamics of the underlying. Even though the latter is a computationally convenient model for high-dimensional portfolio trading applications, it may cause concerns when applied to the pricing of derivatives, mainly due to the fact that the spot of the underlying may become negative with positive probability. In the standardly used geometric Brownian motion this is only possible with zero probability.

3 The hedging and pricing of vanilla options under the basic model

Contingent claims are valued in reference to the initial value V_{t_0} of a portfolio strategy in the underlying risky and riskless assets. This self-financing hedging strategy $(H_t^*(\omega), \hat{H}_t^*(\omega))_{\forall t,\omega}$, with $H_{t_0}^* = \hat{H}_{t_0}^* = 0$[1] will exactly replicate or super-replicate any payoffs of the claim $C_t(\omega), \forall \omega \in \Omega$. For a discrete time economy the valuation of European vanilla type contingent claims under our finite liquidity model can then be formulated as a non-linear programme with objective function

$$\min_{(H_t(\omega),\hat{H}_t(\omega))_{t\in\{t_0,\ldots,T\}}} V_{t_0} = H_{t_1}\bar{S}_{t_0} + \hat{H}_{t_1}B_{t_0} \tag{8}$$

subject to initial holding, the self-financing and payoff super-replication constraints

$$H_{t_0} = \hat{H}_{t_0} = 0, \tag{9}$$

$$(\hat{H}_{t_i}(\omega) - \hat{H}_{t_{i-1}}(\omega))B_{t_{i-1}} + (H_{t_i}(\omega)$$
$$-H_{t_{i-1}}(\omega))e^{\lambda(H_{t_i}(\omega)-H_{t_{i-1}}(\omega))}S_{t_{i-1}}(\omega) = 0, \tag{10}$$

$$V_T(\omega) = H_T(\omega)S_T(\omega) + \hat{H}_T(\omega)B_T \geq C_T(\omega), \tag{11}$$
$$\forall \omega \in \Omega, \forall t_i \in \{t_0, \ldots, T\},$$

respectively, where the processes of $(B_t, S_t)_{\forall t,\omega}$ are given by (2) and (6). Because, in general, $S_t(\omega)$, $\bar{S}_t(\omega)$ and thus $C_T(\omega)$ are functions of the present and past stock-holdings the problem is path-dependent and the number of variables as well as constraints is exponentially growing as the number of time steps increases. As an example, we consider the three period economy with the set of states $\Omega = \{\omega_{uuu}, \omega_{uud}, \ldots, \omega_{ddd}\}$ and the filtration[2] $\mathcal{F}_{t_0} = \{\Omega\}$, $\mathcal{F}_{t_1} = \{\omega_u = \{\omega_{uuu}, \ldots, \omega_{udd}\}, \omega_d\}$, $\mathcal{F}_{t_2} = \{\omega_{uu}, \ldots, \omega_{dd}\}$ and $\mathcal{F}_{t_3} = \{\{\omega_{uuu}\}, \ldots, \{\omega_{ddd}\}\}$. Then the asset's dynamics are

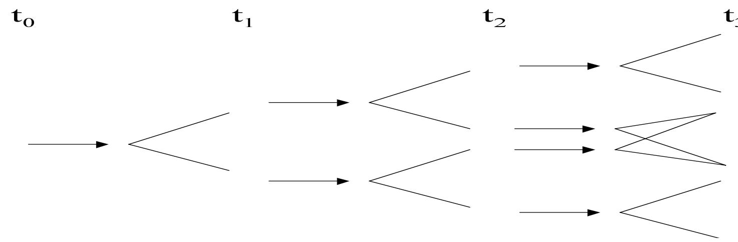
$$\begin{array}{ll}
t_0: & S_{t_0} \to S_{t_0}e^{\lambda H_{t_1}} \\
t_1: & \left\{\begin{array}{ll} uS_{t_0}e^{\lambda(1-\alpha)H_{t_1}} & \to uS_{t_0}e^{\lambda(H_{t_2}(\omega_u)-\alpha H_{t_1})} \\ dS_{t_0}e^{\lambda(1-\alpha)H_{t_1}} & \to dS_{t_0}e^{\lambda(H_{t_2}(\omega_d)-\alpha H_{t_1})} \end{array}\right. \\
t_2: & \left\{\begin{array}{ll} u^2S_{t_0}e^{\lambda(1-\alpha)H_{t_2}(\omega_u)} & \to u^2S_{t_0}e^{\lambda(H_{t_3}(\omega_{uu})-\alpha H_{t_2}(\omega_u))} \\ udS_{t_0}e^{\lambda(1-\alpha)H_{t_2}(\omega_u)} & \to udS_{t_0}e^{\lambda(H_{t_3}(\omega_{ud})-\alpha H_{t_2}(\omega_u))} \\ duS_{t_0}e^{\lambda(1-\alpha)H_{t_2}(\omega_d)} & \to duS_{t_0}e^{\lambda(H_{t_3}(\omega_{du})-\alpha H_{t_2}(\omega_d))} \\ d^2S_{t_0}e^{\lambda(1-\alpha)H_{t_1}(\omega_d)} & \to d^2S_{t_0}e^{\lambda(H_{t_3}(\omega_{dd})-\alpha H_{t_2}(\omega_d))} \end{array}\right. \\
t_3: & \left\{\begin{array}{l} u^3S_{t_0}e^{\lambda(1-\alpha)H_{t_3}(\omega_{uu})} \\ u^2dS_{t_0}e^{\lambda(1-\alpha)H_{t_3}(\omega_{uu})} \\ u^2dS_{t_0}e^{\lambda(1-\alpha)H_{t_3}(\omega_{ud})} \\ ud^2S_{t_0}e^{\lambda(1-\alpha)H_{t_3}(\omega_{ud})} \\ u^2dS_{t_0}e^{\lambda(1-\alpha)H_{t_3}(\omega_{du})} \\ ud^2S_{t_0}e^{\lambda(1-\alpha)H_{t_3}(\omega_{du})} \\ ud^2S_{t_0}e^{\lambda(1-\alpha)H_{t_3}(\omega_{dd})} \\ d^3S_{t_0}e^{\lambda(1-\alpha)H_{t_3}(\omega_{dd})} \end{array}\right. \quad \cdots
\end{array}$$

It becomes apparent that a one period model has two variables/constraints[3], a two period model 6 variables/constraints and an n-period model $2^n - 2$ variables/constraints. The controlled process

[1] This condition is mainly for simplicity and can easily be relaxed to an arbitrary value H_0. The total replication cost will actually be less, if $H_0 = H_{t_1}$.

[2] Strictly speaking, the filtration $(\mathcal{F}_t)_t$ is given by the σ-algebra of the given partition at every t_i, i.e. all the unions and complements of the elements of $\mathcal{F}_{t_i}$.

[3] We do not count the t_0 holdings or constraints.



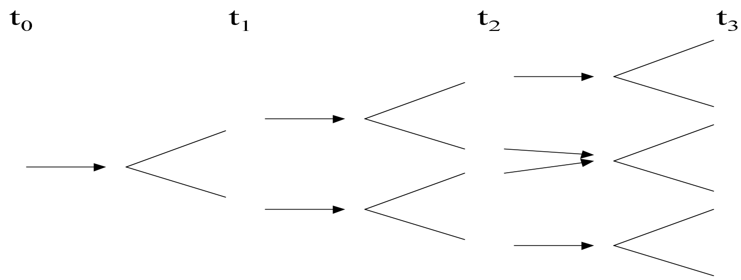


Figure 3: Asset tree for $\alpha = 0$

makes the asset tree bushy and thus hard to implement. But when we turn the inequality (11) into an equality for Markovian contingent claims two distinct trajectories with identical number of up and down moves at a time t_i will result in identical holdings in stock and bond, e.g. $H_{t_3}(\omega_{ud}) = H_{t_3}(\omega_{du})$. We refer to this condition as the "justified manipulation" effect. Because normally market manipulation or front-running are illegal, the condition gives the large trader a valid reason to exactly hedge his position. The asset tree becomes recombining and thus feasible to implement. The asset's dynamics are visualised in figure 2. Under the special case where $\alpha = 0$ the process reduces to figure 3. The case $\alpha = 0$ entails that any price impact due to large trades is a permanent effect in its entirety. Still it represents a possibly large scale nonlinear optimisation problem. To solve for the holding process $(H_t(\omega), \hat{H}_t(\omega))_{\forall t,\omega}$ we have to resort to an optimisation algorithm that will converge quickly. One standard possibility is the Newton method:

$$J([H^{(i+1)} \ \hat{H}^{(i+1)}]^T - [H^{(i)} \ \hat{H}^{(i)}]^T) = -[g_1 \ g_2],$$

where

$$J = \begin{bmatrix} \nabla_H g_1 & \nabla_{\hat{H}} g_1 \\ \nabla_H g_2 & \nabla_{\hat{H}} g_2 \end{bmatrix}$$

is the Jacobian matrix. For the terminal condition (11)we have to solve the system of implicit nonlinear functions

$$\begin{aligned} g_1(H, \hat{H}, \omega_{2j}, T) &= H_T u^{n-j} d^j S_{t_0} e^{\lambda(1-\alpha)H_T} + \hat{H}_T B_T \\ &\quad -C(u^{n-j} d^j S_{t_0} e^{\lambda(1-\alpha)H_T}) = 0 \\ g_2(H, \hat{H}, \omega_{2j-1}, T) &= H_T u^{n-j+1} d^{j-1} S_{t_0} e^{\lambda(1-\alpha)H_T} + \hat{H}_T B_T \\ &\quad -C(u^{n-j} d^i S_{t_0} e^{\lambda(1-\alpha)H_T}) = 0, \end{aligned}$$

$\forall j = 0 \ldots n$, where j and n are the number of down and time steps, respectively, and $H_{t_i}(\omega_{2j}) = H_{t_i}(\omega_{2j-1}) = H_{t_i}, \forall i$ for notational convenience. Furthermore, the intermediate self-financing conditions (11) span the system

$$\begin{aligned} g_1(H, \hat{H}, \omega_{2j}, t_i) &= (H_{t_i} - H_{t_{i+1}}) e^{\lambda(1-\alpha) H_{t_{i+1}}} u^{n-j} d^j S_{t_0} \\ &\quad + (\hat{H}_{t_i} - \hat{H}_{t_{i+1}}) B_{t_i} = 0 \\ g_2(H, \hat{H}, \omega_{2j-1}, t_i) &= (H_{t_i} - H_{t_{i+1}}) e^{\lambda(1-\alpha) H_{t_{i+1}}} u^{n-j+1} d^{j-1} S_{t_0} \\ &\quad + (\hat{H}_{t_i} - \hat{H}_{t_{i+1}}) B_{t_i} = 0, \end{aligned}$$

$\forall j = 0 \ldots n$, $i = 1 \ldots n-1$ and suppressing the explicit dependence on ω_j.

The so calculated prices represent the seller's price, i.e. how much a writer would require or a buyer would need to pay for. By multiplying the payoffs by -1 we obtain the buyer's price, i.e. how much the customer would obtain for entering into this position. These two prices, which due to the nonlinearity of the model will not be the same, represent natural bid-ask spreads, that are founded on the degree of illiquidity of the market for the underlying. Tables 1 and 2 show the bid-ask spreads for different scenarios for call and put options with time to expiry of 1 year, 50 time steps, strike of 50, annualised riskless rate of 5% and volatility of 20%.

The case $\alpha = 1$ implies, that there is no permanent slippage effect. The other market participants did not consider the trade to be based on fundamental information. In this case the model resembles the pure transaction cost models of [B&V], [BLPS] and [ENU]. In fact we can deduce the value of the manipulation effect of illiquid markets, by subtracting the result of a particular choice of α from the result for $\alpha = 1$.

3.1 Distinct bid and ask liquidity

Typically the market depth on the bid and ask side and thus liquidity is not equal. If there exist large imbalances this usually leads to increased volatility and to price movements. In that case i.e. buying when everybody is selling and vice versa, the liquidity for the transaction will be good, the converse holds if one follows the market. The reaction function (4) offers only one scaling parameter and has a linear approximation for small changes. Thus it may not offer enough flexibility to account for distinct bid and ask liquidity. One simple modification would be to replace (4) by

$$\bar{S}_{t_i} = S_{t_i}(e^{\lambda_a(H_{t_{i+1}} - H_{t_i})} I_{\mathbb{R}_+}(H_{t_{i+1}} - H_{t_i}) + e^{\lambda_b(H_{t_{i+1}} - H_{t_i})} I_{\mathbb{R}_-}(H_{t_{i+1}} - H_{t_i}))$$

where λ_a, λ_b are the bid and ask liquidity, respectively, and $I_A(x)$ is the indicator function. Figure 4 shows an actual order book snapshot and a two parameter estimate. This modification, however makes the model path-dependent and we cannot solve it through the tree structure any longer. Instead a large scale dynamical programming algorithm, possibly with approximations would have to be employed (see e.g. [ENU]).

3.2 Parameterisation and calibration of the model

[Krak] explicitly defines liquidity as the reciprocal of $\frac{\Delta H}{\Delta S}$, i.e. the sensitivity of the stock price to the quantity traded. However in this form the parameter is not dimensionless and depends on the absolute size of both the quantity and nominal stock price. A better measure would be to treat

European Calls

r=.05 sigma=.2 T=1 steps=50 H0=0

ASK PRICES

alpha=1

moneyness	lambda 0.01	0.001	0.0001	perf. Liq. 0	Black-Scholes exact
0.7	0.2850	0.2623	0.2601	0.2598	0.2649
0.8	1.1745	1.1058	1.0987	1.0979	1.1156
0.9	3.1670	3.0607	3.0495	3.0483	3.0547
1	6.3770	6.2594	6.2470	6.2456	6.2703
1.1	10.7023	10.6195	10.6110	10.6100	10.5978
1.2	15.7611	15.7078	15.7023	15.7017	15.7014
1.3	21.2970	21.2698	21.2671	21.2668	21.2642

alpha=0.5

moneyness	liquidity 0.01	0.001	0.0001
0.7	0.2848	0.2623	0.2601
0.8	1.1767	1.1059	1.0987
0.9	3.1708	3.0609	3.0495
1	6.3795	6.2594	6.2470
1.1	10.7008	10.6192	10.6109
1.2	15.7627	15.7079	15.7023
1.3	21.2963	21.2697	21.2671

alpha=0

moneyness	liquidity 0.01	0.001	0.0001
0.7	0.2838	0.2622	0.2600
0.8	1.1781	1.1060	1.0987
0.9	3.1732	3.0610	3.0496
1	6.3794	6.2594	6.2470
1.1	10.6950	10.6189	10.6109
1.2	15.7626	15.7080	15.7023
1.3	21.2936	21.2696	21.2670

alpha=-0.5

moneyness	liquidity 0.01	0.001	0.0001
0.7	0.2816	0.2621	0.2600
0.8	1.1786	1.1060	1.0987
0.9	3.1742	3.0612	3.0496
1	6.3762	6.2594	6.2470
1.1	10.6830	10.6185	10.6109
1.2	15.7607	15.7081	15.7023
1.3	21.2884	21.2695	21.2670

BID PRICES

alpha=1

lambda 0.01	0.001	0.0001
0.2339	0.2573	0.2596
1.0139	1.0899	1.0971
2.9149	3.0357	3.0470
6.0953	6.2316	6.2442
10.5098	10.6004	10.6090
15.6383	15.6956	15.7011
21.2358	21.2637	21.2665

alpha=0.5

lambda 0.01	0.001	0.0001
0.2358	0.2574	0.2596
1.0196	1.0899	1.0971
2.9167	3.0356	3.0470
6.1187	6.2333	6.2444
10.5153	10.6007	10.6091
15.6381	15.6955	15.7011
21.2370	21.2638	21.2665

alpha=0

lambda 0.01	0.001	0.0001
0.2368	0.2575	0.2596
1.0280	1.0898	1.0971
2.9225	3.0354	3.0470
6.1328	6.2348	6.2445
10.5165	10.6010	10.6091
15.6357	15.6953	15.7010
21.2364	21.2639	21.2665

alpha=-0.5

lambda 0.01	0.001	0.0001
0.2374	0.2575	0.2596
1.0336	1.0898	1.0971
2.9315	3.0353	3.0470
6.1404	6.2363	6.2447
10.5142	10.6013	10.6091
15.6356	15.6952	15.7010
21.2342	21.2639	21.2665

Table 1: Call options

European Puts

r=.05 sigma=.2 T=1 steps=50 H0=0

ASK PRICES

	moneyness	lambda	0.01	0.001	0.0001	perf. Liq. 0	Black-Scholes exact
alpha=1	0.7		15.3602	15.3375	15.3353	15.3350	15.3386
	0.8		10.2497	10.1810	10.1739	10.1731	10.1894
	0.9		6.2421	6.1359	6.1247	6.1235	6.1285
	1		3.4522	3.3346	3.3222	3.3208	3.3441
	1.1		1.7775	1.6947	1.6861	1.6852	1.6715
	1.2		0.8363	0.7830	0.7775	0.7769	0.7752
	1.3		0.3722	0.3450	0.3423	0.3420	0.3379

	moneyness	liquidity	0.01	0.001	0.0001
alpha=0.5	0.7		15.3607	15.3376	15.3353
	0.8		10.2461	10.1801	10.1738
	0.9		6.2359	6.1349	6.1246
	1		3.4549	3.3346	3.3222
	1.1		1.7807	1.6949	1.6862
	1.2		0.8344	0.7826	0.7774
	1.3		0.3733	0.3451	0.3423

	moneyness	liquidity	0.01	0.001	0.0001
alpha=0	0.7		15.3604	15.3376	15.3353
	0.8		10.2458	10.1791	10.1737
	0.9		6.2303	6.1339	6.1245
	1		3.4558	3.3347	3.3222
	1.1		1.7829	1.6950	1.6862
	1.2		0.8302	0.7822	0.7774
	1.3		0.3741	0.3451	0.3423

	moneyness	liquidity	0.01	0.001	0.0001
alpha=-0.5	0.7		15.3592	15.3376	15.3353
	0.8		10.2435	10.1780	10.1736
	0.9		6.2297	6.1328	6.1244
	1		3.4547	3.3346	3.3222
	1.1		1.7841	1.6952	1.6862
	1.2		0.8272	0.7818	0.7774
	1.3		0.3745	0.3452	0.3423

BID PRICES

alpha=1

lambda	0.01	0.001	0.0001
	15.3091	15.3325	15.3347
	10.0891	10.1651	10.1723
	5.9901	6.1109	6.1222
	3.1705	3.3068	3.3194
	1.5850	1.6756	1.6842
	0.7135	0.7707	0.7763
	0.3110	0.3389	0.3416

alpha=0.5

lambda	0.01	0.001	0.0001
	15.3089	15.3324	15.3347
	10.1024	10.1661	10.1724
	6.0056	6.1120	6.1223
	3.1944	3.3085	3.3196
	1.5859	1.6755	1.6842
	0.7197	0.7712	0.7763
	0.3112	0.3389	0.3416

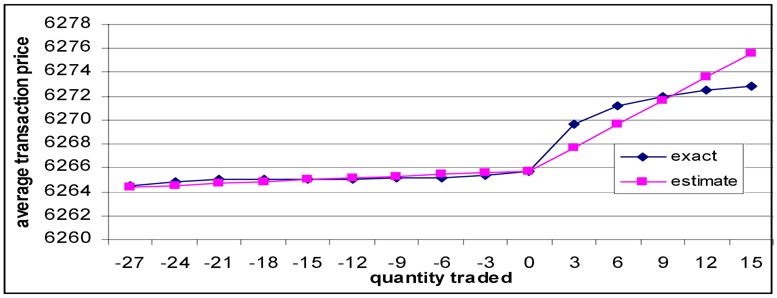
alpha=0

lambda	0.01	0.001	0.0001
	15.3078	15.3324	15.3347
	10.1097	10.1671	10.1725
	6.0144	6.1130	6.1224
	3.2097	3.3100	3.3197
	1.5854	1.6753	1.6842
	0.7236	0.7715	0.7763
	0.3110	0.3388	0.3416

alpha=-0.5

lambda	0.01	0.001	0.0001
	15.3058	15.3323	15.3347
	10.1127	10.1680	10.1726
	6.0186	6.1139	6.1225
	3.2195	3.3115	3.3199
	1.5839	1.6752	1.6842
	0.7260	0.7719	0.7764
	0.3104	0.3388	0.3416

Table 2: Put options



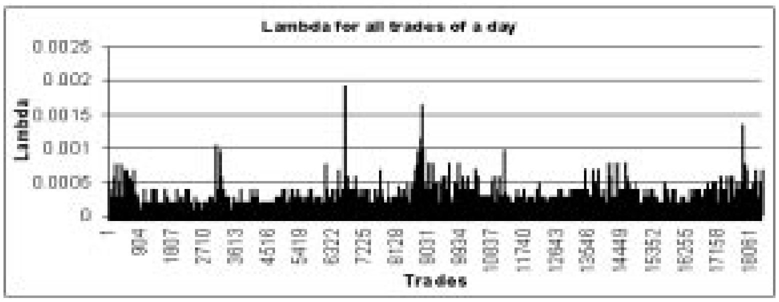


Figure 5: Lambda on a particular trading day

the product $\lambda(H_{t_i} - H_{t_{i-1}})$ as a dimensionless variable. In this case $\lambda = \frac{\Delta S/S}{\Delta H}$. In this case the liquidity parameter becomes observable at the time of the trade, since the market depth is visible. Figure 5 shows λ for all the subsequent trades in one particular trading day. To further make liquidity comparable across different stocks and markets we would need to make the denominator dimensionless as well. This could be done by dividing it by the total quantity traded across the time interval in question. That means, that one's own trades are treated as a fraction of the total market. However the total trade size in general is not predictable.

4 Conclusion

We believe that our model offers a flexible, simple but realistic approach to parameterising liquidity. It relies on inputs that are either directly observable or possible to estimate. Moreover, the speed of calculation entirely depends on the choice of optimisation algorithm employed. Also this model may offer the framework for a number of related applications, that primarily depend on liquidity.

Portfolio trading
Portfolio trading is the liquidation or rebalancing of a large portfolio of one or more stocks. In general the portfolio is assumed to be large enough to substantially move a market, so that it has to be broken up into smaller chunks. Sometimes an agent guarantees a client the liquidation price in advance, usually in terms of a spread around the volume weighted average price (vwap) over a period of time. Depending if it is necessary to return any outperformance of the vwap to the client or not, the initial agreement represents an option. The papers of [A&C] and [H&S2] deal with this

problem by resorting to arithmetic Brownian motion, optimising on objective function that trades off return against variance, scaled by a risk-aversion parameter. The implementation with our model would be straightforward.

Liquidity options
[Scho] defines liquidity options as the right or obligation to buy or sell a certain amount of an asset at the quoted spot price, exercisable within a prespecified time window. Under perfect liquidity, this amounts to a call or put option with a strike price of zero. Hence it would theoretically amount to the forward price of the asset. However when liquidity is not perfect this valuation does not hold any longer, since it may not be the cheapest alternative to take a static hedge up-front.

Exotic options in illiquid markets
[Tal] mentions, possibly illegal, practices of a large trader front-running the client that holds positions in the market. The author moreover mentions that clients require a liquidity rebate when entering into positions in illiquid markets, especially when exposed to knock-out Barriers. Our model may be extended to exotic, possibly non-Markovian payoffs so that the manipulation effect can be extracted.

Strike detection
Finally, our model may proof useful for the inverse problem: given that certain large trades are observed, is it possible to deduce where the trader wants the asset price to be or what position (strike, barrier) is defended. [L&W] show how options should be priced when asset returns are correlated. Another possibility would be to perform a maximum likelihood analysis, after having observed a sequence of large trades.

All these applications form part of current research and development. Hopefully some interesting results can be expected in the future.

References

[A&C] Almgren & Chriss, 1999, *Value under liquidation*, Risk, Dec.

[BLPS] Bensaid, Lesne, Pagès & Scheinkman, 1992, *Derivative Asset Pricing with Transaction Costs*, Mathematical Finance, 2

[B&S] Black & Scholes, 1973, *The Pricing of Options and Corporate Liabilities*, Journal of Political Economy, 2

[B&V] Boyle & Vorst, 1992, *Option Replication in Discrete Time with Transaction Costs*, The Journal of Finance, 1

[CRS] Chordia, Roll & Subrahmanyam, 2000, *Order Imbalance, Liquidity and Market Returns*, UCLA, Working Paper

[CRR] Cox, Ross & Rubinstein, 1979, *Option Pricing: A Simplified Approach*, Journal of Financial Economics, Sept.

[ENU] Edirisinghe, Naik & Uppal, 1993, *Optimal Replication of Options with Transactions Costs and Trading Restrictions*, March

[Frey] Frey, 1996, *The Pricing and Hedging of Options in Finitely Elastic Markets*, University of Bonn, Discussion Paper

[F&S] Frey & Stremme, 1995, *Market Volatility and Feedback Effects from Dynamic Hedging*, University of Bonn, Discussion Paper

[H&S1] Huberman & Stanzl, 2000, *Arbitrage-free Price-update and Price-Impact Functions*, Yale, Working Paper

[H&S2] Huberman & Stanzl, 2000, *Optimal Liquidity Trading*, Yale, Working Paper

[Jar1] Jarrow, 1992, *Market Manipulation, Bubbles, Corners, and Short Squeezes*, Journal of Financial and Quantitative Analysis, Sept.

[Jar2] Jarrow, 1994, *Derivative Security Markets, Market Manipulation and Option Pricing Theory*, Journal of Financial and Quantitative Analysis, June

[Krak] Krakovsky, 1999, *Pricing liquidity into derivatives*, Risk, Dec.

[L&W] Lo & Wang, 1995, *Implementing Option Pricing Models When Asset Returns Are Predictable*, The Journal of Finance, 1

[Lon] Longstaff, 1995, *Option Pricing and the Martingale Restriction*, Review of Financial Studies, 4

[Scho] Scholes, 1999, *Liquidity options*, Risk, Nov.

[Schö] Schönbucher, 1993, *The feedback effects of hedging in illiquid markets*, University of Oxford, MSc Thesis

[S&P] Sircar & Papanicolaou, 1998, *General Black-Scholes models accounting for increased market volatility from hedging strategies*, Applied Mathematical Finance, 1

[Tal] Taleb, 1997, *Dynamic Hedging*, Wiley