A note on the pricing and hedging of volatility derivatives[1]

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Abstract

We consider the pricing of volatility products and especially volatility and variance swaps. Under risk-neutral valuation we provide closed form formulae for volatility-average and variance swaps. Also we provide a general partial differential equation for derivatives that have an extra dependence on an average of the volatility. We give approximate solutions of this equation for volatility products written on assets for which the volatility process fluctuates on a timescale that is fast compared with the lifetime of the contracts.

1 Introduction

There has recently been much interest in products that provide exposure to the realised volatilities or variances of asset returns (or covariances between asset returns), while avoiding direct exposure to the underlying assets themselves. These products are attractive to investors who either wish to hedge volatility risk or who wish to take a view on future realised volatilities. Indeed, much of the investor interest in volatility products seems to have been provided by the LTCM collapse in 1998, which was accompanied by a dramatic increase in volatilities. As a result, a number of recent papers [2, 5, 6, 9] address the evaluation of volatility products.

Like several of these authors, we take a stochastic volatility model as our starting point. The fact that stochastic volatility models are able to fit skews and smiles, while simultaneously providing sensible Greeks, have made these models a popular choice in the pricing of exotic options. Under this framework, we present a number of formulae for the 'fair' delivery price for volatility and variance swaps, and show

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how other related contracts can be priced. We introduce a general pricing equation for derivatives depending on four state variables: the asset value S, the time t, the volatility σ, and a running average, denoted by I, which represents our knowledge to date of the average that will determine the payoff. We focus on volatility and variance swaps, in which case the average is of the volatility or the variance respectively. We also consider an asymptotic analysis under which we derive approximate solutions to this equation. The main motivation here is the empirical evidence that volatility is fast mean-reverting, compared with the typical lifetime of options and other contracts. That is, when considering the time-scale of months, stock and index volatility is observed to fluctuate rapidly, see for example the discussion in [7] or [13] and references therein. The model that we choose in this context is the mean-reverting lognormal; others can also be considered, see for example [12] for the case of an Ornstein–Uhlenbeck model. We then narrow the choice of derivatives to those whose payoffs do not depend on the underlying asset S. In this case, the analysis is simpler. In particular, having a solution for the value of the volatility swap to first order, we are able to compare with the explicit results obtained before.

The rest of the paper is organized as follows: we begin section 2 by briefly discussing the contracts, while the stochastic volatility framework is introduced in 2.1, and the general pricing equation is given. In 2.2 we present the analysis and the results for risk-neutral valuation for the volatility-average swap and the variance swap, together with formulae that allow us to hedge. We can evaluate the vega, namely the derivative with respect to volatility, and therefore have an estimate for our hedging parameter. We can then hedge efficiently using liquid contracts. In section 3 we present the asymptotic analysis which concludes with first order approximations for the volatility derivatives of interest. In section 4 we give a brief summary and motivation for further work.

2 Variance and volatility swaps

The variance swap is a forward contract in which the investor who is long pays a fixed amount $K^{var}/\$1$ nominal value at expiry and receives the floating amount $(\sigma^2)_R/\$1$ nominal value, where K^{var} is the strike and $(\sigma^2)_R$ is the realized variance. The entering price must be zero, that is, it costs nothing to enter the contract; we use this condition to find the fair value K^{var}. The measure of realized variance to be used is defined at the beginning of the contract; a typical formula for it is

$$\frac{1}{T}\sum_{i=1}^{M}\left(\frac{S_i - S_{i-1}}{S_{i-1}}\right)^2, \tag{1}$$

which in continuous time we approximate by

$$(\sigma^2)_R = \frac{1}{T}\int_0^T \sigma^2\left(t, \cdots\right) dt.$$

The corresponding payoff is then

$$(\sigma^2)_R - K^{var}. \tag{2}$$

We shall also consider other contracts, one being the realised volatility swap with payoff

$$\left(\frac{1}{T}\sum_{i=1}^{M}\left(\frac{S_i - S_{i-1}}{S_{i-1}}\right)^2\right)^{\frac{1}{2}} - K^{vol},$$

derived from the standard deviation of the averaged variance, with continuous-time limit

$$\sigma_R^{vol} - K^{vol} = \left(\frac{1}{T}\int_0^T \sigma_t^2 dt\right)^{\frac{1}{2}} - K^{vol}, \tag{3}$$

or a variation that represents the average of the local volatility, with payoff

$$\sqrt{\frac{\pi}{2MT}}\sum_{i=1}^{M}\left|\frac{S_i - S_{i-1}}{S_{i-1}}\right| - K^{vol\text{-}ave}, \tag{4}$$

which in continuous time is

$$\sigma_R^{vol\text{-}ave} - K^{vol\text{-}ave} = \frac{1}{T}\int_0^T \sigma_t \, dt - K^{vol\text{-}ave}. \tag{5}$$

In addition, we shall consider products based on an average of a suitable implied volatility, for example the implied volatility σ_t^i of the at-the-money call options with the same expiry as the volatility derivative; this implied volatility swap has continuous-time payoff

$$\sigma_R^i - K^{i\text{-}vol} = \frac{1}{T}\int_0^T \sigma_t^i \, dt - K^{i\text{-}vol}. \tag{6}$$

We could also use a single option throughout the life of the contract, for example the option that is initially at-the-money; we could further construct implied variance swaps, and so on.

Generalizing further, we can contemplate volatility options, a typical payoff being

$$\max(\sigma_R^{vol} - K, 0). \tag{7}$$

We can also consider contracts whose payoff depends on both a realised volatility or variance and the asset; for example, the payoff

$$\max(Se^{-\sigma_R^{vol}} - K, 0) \tag{8}$$

is a call option which pays more if the asset rises steadily without much volatility than if it rises in a volatile way.

The asset S is considered to follow the usual log normal process

$$\frac{dS_t}{S_t} = \mu\,(t, \cdots)\, dt + \sigma\,(t, \cdots)\, dW_t, \tag{9}$$

where W_t is Brownian motion. For the rest of the paper we fix notation as follows: the conditional expectation at time t is denoted by $E_t = E[\,.\,|\mathcal{F}_t]$ where $\mathcal{F}_t$ is the filtration up to time t and E_0 is thus the initial value of the expectation. All expectations are considered with respect to the risk-neutral probability measure.

Discounting the payoff (2) we have for the present value of a variance swap

$$F^{var} = E_0[e^{-rT}\left((\sigma^2)_R - K^{var}\right)].$$

This is zero and we derive for the fair variance forward price

$$K^{var} = \frac{1}{T} E_0[\int_0^T \sigma_t^2\, dt]. \tag{10}$$

However, we need a model for the evolution of the volatility, and this we now consider.

2.1 Stochastic volatility models

In this section we consider a stochastic volatility model. That is, the asset S satisfies the risk-neutral stochastic differential equation

$$\frac{dS_t}{S_t} = r\, dt + \sigma_t\, dW_t,$$

where volatility is now a stochastic process satisfying the risk-adjusted equation

$$d\sigma_t = (M - \lambda Q)dt + Q\, d\hat{W}_t, \tag{11}$$

where M and Q depend on the specific model we use and λ is the market price of volatility risk. The Brownian motions W_t and $\hat{W}_t$ have correlation coefficient ρ, where $-1 < \rho < 1$. We consider specifically the mean-reverting lognormal model, that is we take

$$M - \lambda Q = \alpha(\bar{\sigma} - \sigma_t), \qquad Q = \beta\sigma_t.$$

This not only ensures that volatility stays positive but also captures the empirically observed *mean-reversion property*: volatility tends to return to the mean $\bar{\sigma}$ on characteristic time-scale $1/\alpha$. We also note that it is not difficult to generalize to other types of models.

We begin by pricing derivatives whose payoff depends not only on the asset and the volatility, but on some average I of the volatility during the life of the contract. This average is defined as

$$I_t = \int_0^t F(\sigma_s)\, ds. \tag{12}$$

Using this notation the contracts described above have payoffs

$$P^{var} = \frac{I_T^{var}}{T} - K^{var}, \tag{13}$$

corresponding to (2),

$$P^{vol} = \Big(\frac{I_T^{var}}{T}\Big)^{1/2} - K^{vol}, \tag{14}$$

corresponding to (3),

$$P^{vol\text{-}ave} = \frac{I_T^{vol}}{T} - K^{vol\text{-}ave}, \tag{15}$$

corresponding to (5), and

$$P^{i} = \frac{I_T^{i\text{-}vol}}{T} - K^{i\text{-}vol}, \tag{16}$$

corresponding to (6). Here

$$\begin{aligned} I_T^{var} &= \int_0^T \sigma_t^2\,dt, \\ I_T^{vol} &= \int_0^T \sigma_t\,dt, \\ I_T^{i\text{-}vol} &= \int_0^T \sigma_t^i\,dt, \end{aligned}$$

so that $F(\sigma_t) = \sigma_t^2$ for I_t^{var}, $F(\sigma_t) = \sigma_t$ for $I_t^{vol\text{-}ave}$, and for $I_t^{i\text{-}vol}$, $F(\sigma_t^i)$ is the implied volatility of an at-the-money call option with the current value of S_t and σ_t.

In some cases, such as an implied volatility swap where the strike of the reference option is fixed (i.e. not floating at-the-money), F may additionally depend on S_t. We do not treat such cases here. We then have $V_t = V(S_t, t, \sigma_t, I_t)$ and the analysis is very similar to that of Asian options. Following a general no-arbitrage approach (see [12], [14]), we derive a partial differential equation for V (we drop the subscript t without ambiguity),

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho S \sigma Q \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2} Q^2 \frac{\partial^2 V}{\partial \sigma^2} + F(\sigma) \frac{\partial V}{\partial I} \\ + rS \frac{\partial V}{\partial S} + (M - \lambda Q) \frac{\partial V}{\partial \sigma} - rV = 0, \end{aligned} \tag{17}$$

where λ is the market price of volatility risk. The terminal condition is $V(S, \sigma_T, T, I) = P(S, I)$, where P is the payoff. We can thus price either by solving equation (17) with the payoff condition, or, equivalently, by taking risk-neutral expectations, as in (10).

2.2 Pricing Volatility and Variance Swaps

The risk-adjusted equation for volatility under the mean-reverting lognormal model is

$$d\sigma_s = \alpha(\bar{\sigma} - \sigma_s)ds + \beta \sigma_s d\hat{W}_s. \tag{18}$$

We need the unconditional ($t = 0$) and conditional ($t > 0$) expectations both for the volatility and the variance. The expectation of the volatility is given by

$$E_t[\sigma_s] = \bar{\sigma}(1 - e^{-\alpha(s-t)}) + \sigma_t e^{-\alpha(s-t)} \qquad s \geq t. \tag{19}$$

This is obtained easily by integrating and taking expectations in (18); note that it tends to $\bar{\sigma}$ as $t \to \infty$. The unconditional expectation is obtained by setting $t = 0$. The stochastic differential equation for the variance is obtained using Ito's formula as

$$d(\sigma_s^2) = 2\sigma_s d\sigma_s + \beta^2 \sigma_s^2 ds.$$

Substituting from (18), we have

$$d(\sigma_s^2) = \Big(\sigma_s^2(\beta^2 - 2\alpha) + 2\alpha\bar{\sigma}\sigma_s\Big)ds + 2\beta\sigma_s^2 d\hat{W}_s. \tag{20}$$

Integrating and taking expectations we get

$$E_t[\sigma_\tau^2] = \sigma_t^2 + 2\alpha\bar{\sigma}\int_t^\tau E_t[\sigma_s]ds + (\beta^2 - 2\alpha)\int_t^\tau E_t[\sigma_s^2]ds \tag{21}$$

where the conditional expectation of the volatility is given by (19). The solution of this integral equation is

$$\begin{aligned} E_t[\sigma_\tau^2] &= \frac{2\alpha\bar{\sigma}^2}{2\alpha - \beta^2}\Big(1 - e^{-(2\alpha-\beta^2)(\tau-t)}\Big) \\ &+ \frac{2\alpha\bar{\sigma}(\sigma_t - \bar{\sigma})}{\alpha - \beta^2}\Big(e^{-\alpha(\tau-t)} - e^{-(2\alpha-\beta^2)(\tau-t)}\Big) \\ &+ \sigma_t^2 e^{-(2\alpha-\beta^2)(\tau-t)}, \qquad \tau \geq t, \end{aligned} \tag{22}$$

where clearly the mean reversion only keeps the expectation finite if $2\alpha > \beta^2$. (The case $\alpha = \beta^2$ has terms $\tau e^{-\alpha(\tau-t)}$ and we do not deal with it here.) Note that

$$E_t[\sigma_\tau^2] \to \overline{\sigma^2} = 2\alpha\bar{\sigma}^2/(2\alpha - \beta^2), \qquad \text{as} \quad \tau \to \infty,$$

the long-term average of σ^2. Using these expectations, we are able to derive formulae for the fair price of the variance swap with payoff (13) and the volatility-average swap with payoff (15); the payoffs (14) and (16) can only be priced numerically. We begin with the easier volatility-average swap (15).

2.2.1 Volatility-Average Swap

Consider the value of the volatility-average swap at any time $0 \leq t \leq T$. In this case the contribution

$$I_t^{\textit{vol-ave}} = \int_0^t \sigma_s\, ds$$

to the final value $I_T^{\textit{vol-ave}}$ has been realized, and the payoff can be decomposed as follows:

$$\begin{aligned} P^{\textit{vol-ave}} &= \frac{1}{T}\Big(\int_0^t \sigma_s\, ds + \int_t^T \sigma_s\, ds\Big) - K^{\textit{vol-ave}} \\ &= \frac{1}{T}\Big(I_t^{\textit{vol-ave}} + \int_t^T \sigma_s\, ds\Big) - K^{\textit{vol-ave}}. \end{aligned} \tag{23}$$

The value of the swap at time t is now

$$V_t \;=\; e^{-r(T-t)}\Big[\frac{1}{T}\Big(\int_0^t \sigma_s\,ds + \int_t^T E_t[\sigma_s]\,ds\Big) - K^{vol\text{-}ave}\Big].$$

Combining with (19) we immediately get

$$V_t = e^{-r(T-t)}\left\{\frac{I_t^{vol\text{-}ave}}{T} + \frac{\bar{\sigma}}{T}(T-t) - \Big(\frac{\sigma_t}{\alpha T} - \frac{\bar{\sigma}}{\alpha T}\Big)(e^{-\alpha(T-t)}-1) - K^{vol\text{-}ave}\right\}. \quad (24)$$

For $t = 0$ we get V_0, which must be 0; from this condition we derive the fair price for $K^{vol\text{-}ave}$,

$$K^{vol\text{-}ave} = \frac{1-e^{-\alpha T}}{\alpha T}(\sigma_0 - \bar{\sigma}) + \bar{\sigma}. \quad (25)$$

We emphasize here the dependence on the model parameters α and $\bar{\sigma}$, and note that we need to know the initial volatility σ_0. Substituting this back into (24), the price V_t takes the final form

$$\begin{aligned} V_t &= \frac{e^{-r(T-t)}}{T}\left\{I_t^{vol\text{-}ave} - t\bar{\sigma} + \frac{1}{\alpha}(e^{-\alpha(T-t)}-1)(\sigma_0-\sigma_t)\right\} \\ &= \frac{e^{-r(T-t)}}{T}\left\{\int_0^t (\sigma_s - \bar{\sigma})ds + \frac{1}{\alpha}(e^{-\alpha(T-t)}-1)(\sigma_0-\sigma_t)\right\}. \end{aligned} \quad (26)$$

We can directly verify that the above expression satisfies equation (17) for $M = \alpha(\bar{\sigma}-\sigma)$ and $Q = \beta\sigma$ with $F(\sigma) = \sigma$. It is now clear that hedging is possible, since the derivative of V with respect to σ (Vega) can be obtained:

$$\frac{\partial V}{\partial \sigma} = \frac{e^{-r(T-t)}}{\alpha T}(1 - e^{-\alpha(T-t)})$$

(note that this is independent of β as indeed is V_t). Here, it should be emphasized that (26) is the value of a par volatility swap, to distinguish from other more complicated contracts.

2.2.2 Variance Swap

We now consider the case of the variance swap. The payoff is

$$P^{var} = \frac{1}{T}\int_0^T \sigma_s^2\,ds - K^{var}.$$

We proceed to derive a formula for the value of the contract at any time $0 \le t \le T$. We have

$$\begin{aligned} V_t &= e^{-r(T-t)}E_t\Big[\frac{1}{T}\int_0^T \sigma_s^2\,ds - K^{var}\Big] \\ &= e^{-r(T-t)}\left\{\frac{I_t^{var}}{T} + \frac{1}{T}\int_t^T E_t[\sigma_s^2]\,ds - K^{var}\right\}, \end{aligned} \quad (27)$$

where

$$I_t^{var} = \int_0^t \sigma_s^2 \, ds.$$

The conditional expectation is given by (22), and we obtain

$$\begin{aligned}
V_t &= e^{-r(T-t)}\Big\{\frac{I_t^{var}}{T} + \frac{2\alpha\bar{\sigma}^2}{(2\alpha-\beta^2)T}\Big((T-t) - \frac{1-e^{-(2\alpha-\beta^2)(T-t)}}{2\alpha-\beta^2}\Big) \\
&+ \frac{2\alpha\bar{\sigma}(\sigma_t-\bar{\sigma})}{(\alpha-\beta^2)T}\Big(\frac{1-e^{-\alpha(T-t)}}{\alpha} - \frac{1-e^{-(2\alpha-\beta^2)(T-t)}}{2\alpha-\beta^2}\Big) \\
&+ \frac{\sigma_t^2}{2\alpha-\beta^2}\Big(1-e^{-(2\alpha-\beta^2)(T-t)}\Big) - K^{var}\Big\} \qquad (28)
\end{aligned}$$

Setting $t=0$ we obtain V_0 which must be 0, and so we derive for the fair variance

$$\begin{aligned}
K^{var} &= \frac{2\alpha\bar{\sigma}^2}{(2\alpha-\beta^2)T}\Big(T - \frac{1-e^{-(2\alpha-\beta^2)T}}{2\alpha-\beta^2}\Big) \\
&+ \frac{2\alpha\bar{\sigma}(\sigma_0-\bar{\sigma})}{(\alpha-\beta^2)T}\Big(\frac{1-e^{-\alpha T}}{\alpha} - \frac{1-e^{-(2\alpha-\beta^2)T}}{2\alpha-\beta^2}\Big) \\
&+ \frac{\sigma_0^2}{(2\alpha-\beta^2)T}\Big(1-e^{-(2\alpha-\beta^2)T}\Big). \qquad (29)
\end{aligned}$$

We substitute this into (28) and the final result is

$$\begin{aligned}
V_t &= e^{-r(T-t)}\Big\{\frac{I_t^{var}}{T} + \frac{2\alpha\bar{\sigma}^2}{(2\alpha-\beta^2)T}\Big(-t + \frac{e^{-(2\alpha-\beta^2)(T-t)} - e^{-(2\alpha-\beta^2)T}}{2\alpha-\beta^2}\Big) \\
&+ \frac{2\alpha\bar{\sigma}}{(\alpha-\beta^2)T}\Big\{\sigma_t\Big(\frac{1-e^{-\alpha(T-t)}}{\alpha} - \frac{1-e^{-(2\alpha-\beta^2)(T-t)}}{2\alpha-\beta^2}\Big) \\
&- \sigma_0\Big(\frac{1-e^{-\alpha T}}{\alpha} - \frac{1-e^{-(2\alpha-\beta^2)T}}{2\alpha-\beta^2}\Big\} \\
&+ \bar{\sigma}\Big(\frac{e^{-\alpha(T-t)}-e^{-\alpha T}}{\alpha} - \frac{e^{-(2\alpha-\beta^2)(T-t)} - e^{-(2\alpha-\beta^2)T}}{2\alpha-\beta^2}\Big)\Big) \\
&+ \frac{1}{2\alpha-\beta^2 T}\Big(\sigma_t^2(1-e^{-(2\alpha-\beta^2)(T-t)}) - \sigma_0^2(1-e^{-(2\alpha-\beta^2)T})\Big)\Big\} \qquad (30)
\end{aligned}$$

Again, this is the value for the par variance swap.

3 Asymptotic analysis

We now present an asymptotic approach similar to that of Fouque et al. [7], who take volatility to be a fast mean-reverting Ornstein–Uhlenbeck process. They derive results for vanilla calls and under this framework they are able to fit the skew, deriving a linear relationship between the implied volatility and the *log-moneyness-to-maturity-ratio*,

$$LMMR = \frac{\log(K/S)}{T-t}.$$

In order to fit the smile, in addition to any skew, one should include higher–order terms in the asymptotic analysis [13].

3.1 General Payoffs

We begin by discussing a larger family of contracts with payoffs of the form $P(S_T, I_T)$, where

$$I_T = \int_0^T F(\sigma_s)\, ds.$$

Having given the general approach, we illustrate the method with some simple examples.

Recall the risk-adjusted process for σ, namely

$$d\sigma_t = \alpha(\bar{\sigma} - \sigma_t)dt + \beta\sigma_t\, d\hat{W}_t, \tag{31}$$

where $\hat{W}_t$ is a Brownian motion with respect to the risk-neutral measure and $\bar{\sigma}$ is the mean. Taking the correlation to be zero for simplicity, the general pricing equation (17) becomes

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\beta^2\sigma^2 \frac{\partial^2 V}{\partial \sigma^2} + rS\frac{\partial V}{\partial S} + F(\sigma)\frac{\partial V}{\partial I} + \alpha(\bar{\sigma} - \sigma)\frac{\partial V}{\partial \sigma} - rV = 0. \tag{32}$$

The stationary density for σ, i.e. the time-independent solution of the forward Kolmogorov equation, is given by

$$p_\infty(\sigma) = Ae^{-2\alpha\bar{\sigma}/\beta^2\sigma}\sigma^{-2-2\alpha/\beta^2}, \tag{33}$$

where

$$A = \frac{(2\alpha\bar{\sigma}/\beta^2)^{1+2\alpha/\beta^2}}{\Gamma(1+2\alpha/\beta^2)},$$

is the normalization constant. We also note here that higher moments of σ may not exist for small values of α/β^2.

We now assume that the characteristic time-scale for the volatility process is small compared with the time to maturity of the volatility swap. We introduce a small parameter by writing $\alpha = a/\epsilon$, where $a = \mathcal{O}(1)$ and, bearing in mind the need for a non-trivial standard deviation for $p_\infty(\sigma)$, we scale the volatility of volatility so that the random walk (18) takes the form

$$d\sigma_t = \frac{a}{\epsilon}(\bar{\sigma} - \sigma_t)dt + \frac{b}{\sqrt{\epsilon}}\sigma_t\, d\hat{W}_t$$

for $0 < \epsilon \ll 1$ (note that the invariant density $p_\infty(\sigma)$ can be written in terms of a and b by replacing α by a and β by b in (33)). We introduce the operators

$$\mathcal{L}_0 = \frac{1}{2}b^2\sigma^2\frac{\partial^2}{\partial\sigma^2} + a(\bar{\sigma} - \sigma)\frac{\partial}{\partial\sigma},$$

$$\mathcal{L}_1 = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2}{\partial S^2} + rS\frac{\partial}{\partial S} + F(\sigma)\frac{\partial}{\partial I} - r.$$

The equation for V then becomes

$$\Big(\mathcal{L}_1 + \frac{1}{\epsilon}\mathcal{L}_0\Big)V = 0.$$

Consider now the expansion

$$V \sim V_0 + \epsilon V_1 + \epsilon^2 V_2 + \cdots \qquad .$$

Substituting, we have

$$\frac{1}{\epsilon}\mathcal{L}_0 V_0 + (\mathcal{L}_1 V_0 + \mathcal{L}_0 V_1) + \epsilon\,(\mathcal{L}_1 V_1 + \mathcal{L}_0 V_2) + \cdots = 0. \tag{34}$$

Equating coefficients, at lowest order we have

$$\mathcal{L}_0 V_0 = 0$$

and so

$$V_0 = V_0(S, t, I)$$

since the operator $\mathcal{L}_0$ consists of derivatives with respect to σ only. For the terms of $\mathcal{O}(1)$ we find

$$\mathcal{L}_0 V_1 + \mathcal{L}_1 V_0 = 0. \tag{35}$$

This equation can be treated as a Poisson equation for V_1, considering V_0 known. The solvability (Fredholm Alternative) condition for this equation can be expressed as

$$< \mathcal{L}_1 V_0, p_\infty >= 0$$

where $< .,. >$ denotes the usual inner product. Thus, $\mathcal{L}_1 V_0$ is orthogonal to p_∞, which, being a solution of the stationary forward Kolmogorov equation for σ, is an eigenfunction of the adjoint of $\mathcal{L}_0$. That is,

$$\frac{\partial V_0}{\partial t} + \frac{1}{2}S^2\frac{\partial^2 V_0}{\partial S^2}\int_0^\infty p_\infty(\sigma)\sigma^2 d\sigma + rS\frac{\partial V_0}{\partial S} + \frac{\partial V_0}{\partial I}\int_0^\infty p_\infty(\sigma)F(\sigma)d\sigma - rV_0 = 0,$$

where we have used the result that V_0 is independent of σ. Noting that the integrals are equal to $\overline{\sigma^2}$ and $\overline{F} = \overline{F(\sigma)}$ respectively, we have

$$\frac{\partial V_0}{\partial t} + \frac{1}{2}\overline{\sigma^2}S^2\frac{\partial^2 V_0}{\partial S^2} + rS\frac{\partial V_0}{\partial S} + \overline{F}\frac{\partial V_0}{\partial I} - rV_0 = 0. \tag{36}$$

Making the transformation

$$\bar{I} = I + \overline{F}(T - t)$$

and writing $V_0(S, t, I) = \overline{V}_0(S, t, \bar{I})$, reduces this further, to

$$\frac{\partial V_0}{\partial t} + \frac{1}{2}\overline{\sigma^2}S^2\frac{\partial^2 V_0}{\partial S^2} + rS\frac{\partial V_0}{\partial S} - rV_0 = 0 \tag{37}$$

which is the Black–Scholes equation with volatility $\left(\overline{\sigma^2}\right)^{1/2}$. The dependence on I is retained parametrically via the payoff, which takes the form

$$P(S, I) = V_0(S, T, I) = \overline{V_0}(S, T, \bar{I}),$$

and so we have $V_0(S,t,I) = \overline{V}_0(S,t,I+(T-t)\overline{F})$.

The next step is to calculate V_1. This is a complex task and here we only give an outline of the procedure; further details will be given elsewhere. First, observe that $\mathcal{L}_1 V_0$ can be written as

$$\mathcal{L}_1 V_0 = \frac{1}{2}(\sigma^2 - \overline{\sigma^2}) S^2 \frac{\partial^2 V_0}{\partial S^2} + (F(\sigma) - \overline{F}) \frac{\partial V_0}{\partial I}. \tag{38}$$

Hence, (35) can be written as

$$\mathcal{L}_0 V_1 = \frac{1}{2}(\overline{\sigma^2} - \sigma^2) S^2 \frac{\partial^2 V_0}{\partial S^2} + (\overline{F} - F(\sigma)) \frac{\partial V_0}{\partial I}. \tag{39}$$

We seek a solution of the form

$$V_1(S,t,\sigma,I) = f_2(\sigma) S^2 \frac{\partial^2 V_0}{\partial S^2} + f_1(\sigma) \frac{\partial V_0}{\partial I} + H(S,t,I), \tag{40}$$

where H is independent of σ. The functions f_1 and f_2 are then solutions of the equations

$$\begin{aligned} \frac{1}{2} b^2 \sigma^2 \frac{d^2 f_2}{d\sigma^2} + a(\bar{\sigma} - \sigma)) \frac{df_2}{d\sigma} &= \frac{1}{2}(\overline{\sigma^2} - \sigma^2), \\ \frac{1}{2} b^2 \sigma^2 \frac{d^2 f_1}{d\sigma^2} + a(\bar{\sigma} - \sigma) \frac{df_1}{d\sigma} &= \overline{F} - F(\sigma). \end{aligned}$$

and can readily be found in integral form. However, $H(S,t,I)$ can only be determined by proceeding to next order and applying the solvability condition to the equation

$$\mathcal{L}_0 V_2 + \mathcal{L}_1 V_1 = 0.$$

We further see that the solution (40), which depends on σ, cannot satisfy the payoff condition $V_1(S,T,\sigma,I) = 0$. This discrepancy is resolved by a boundary layer analysis in which $T - t = \mathcal{O}(\epsilon)$ (if the payoff has discontinuities, as for a volatility option, further local analysis near these points are also necessary). We point out that the lowest order analysis is quite general, and not specific to the random walk (18). Only at higher order do we need to know more about these details, and even then only certain moments need be calculated. As stated above, details of the higher order analysis will be described elsewhere.

3.2 Examples

We illustrate the theory with the four versions of variance/volatility swaps described in §2.1, for two of which we could not easily obtain explicit solutions. In each case we give the lowest-order solution only, leaving the details of the higher-order solution to a later publication.

3.2.1 The variance swap

For this contract, we have $F(\sigma) = \sigma^2$ and so $\overline{F} = \overline{\sigma^2}$, the average variance to be used in the Black-Scholes equation (37). The payoff is $I^{var}/T - K^{var} = \bar{I}^{var}/T - K^{var}$ and the solution to the leading order problem is (reintroducing the subscript t to denote time-t values)

$$\overline{V}_{t0}(S_t, t, \bar{I}_t) = e^{-r(T-t)}\Big(\frac{\bar{I}_t^{var}}{T} - K^{var}\Big), \tag{41}$$

so the leading order approximation to the variance swap value is

$$V_{t0}(S, t, I) = e^{-r(T-t)}\Big(\frac{I_t^{var} + \overline{\sigma^2}(T-t)}{T} - K^{var}\Big). \tag{42}$$

For the random walk (20) for which $\overline{\sigma^2} = 2\alpha\bar{\sigma}^2/(2\alpha - \beta^2) = 2a\bar{\sigma}^2/(2a - b^2)$, it is easily confirmed that we recover the $\mathcal{O}(1)$ terms of the exact result (28).

3.2.2 The volatility swap

For the standard volatility swap payoff (14), we still have $F(\sigma) = \sigma^2$ but now the payoff is $(I^{var}/T)^{1/2} - K^{vol}$. Hence,

$$\overline{V}_{t0}(S_t, t, \bar{I}_t) = e^{-r(T-t)}\Big(\Big(\frac{\bar{I}_t^{var}}{T}\Big)^{1/2} - K^{vol}\Big) \tag{43}$$

and the leading order term in the expansion of the volatility swap price is

$$V_{t0}(S_t, t, \bar{I}_t) = e^{-r(T-t)}\Big(\Big(\frac{I_t^{var} + \overline{\sigma^2}(T-t)}{T}\Big)^{1/2} - K^{vol}\Big). \tag{44}$$

3.2.3 The volatility-average swap

For the payoff (15) we have $F(\sigma) = \sigma$ and we find that

$$\overline{V}_{t0}(S_t, t, \bar{I}_t) = e^{-r(T-t)}(\frac{\bar{I}_t}{T} - K^{vol\text{-}ave}),$$

so that

$$V_{t0}(S_t, t, I_t) = e^{-r(T-t)}\Big(\frac{I_t^{vol-ave} + \bar{\sigma}(T-t)}{T} - K^{vol\text{-}ave}\Big). \tag{45}$$

It is easily confirmed that the $\mathcal{O}(1)$ terms in (24) are consistent with this approximation. We also note that in this case, and for the random walk (18), we readily find that

$$V_{t1}(S_t, t, I_t, \sigma) = (\sigma_t - \bar{\sigma})/a + h(I_t, t) \tag{46}$$

and comparison with (24) reveals that $h(I_t, t)$ is proportional to $e^{-\alpha(T-t)} = e^{-a(T-t)/\epsilon}$; this is the boundary layer correction referred to above, and it decays very rapidly as t decreases from T.

3.2.4 The implied volatility swap

In this case, $F(\sigma)$ is the implied volatility of an at-the-money option. Now we can apply the same asymptotic procedure to standard options, and it is clear that the lowest order solution for a call option under this approximation is given by solving (36) with no I-dependence, i.e. the Black–Scholes equation with volatility $\left(\overline{\sigma^2}\right)^{\frac{1}{2}}$. Hence to leading order, we have $F = \overline{F} = \left(\overline{\sigma^2}\right)^{\frac{1}{2}}$. The leading order value of this swap is consequently

$$V_{t0} = (S_t, t, I_t) = e^{-r(T-t)}\left(\frac{I_t^{i\text{-}vol} + \left(\overline{\sigma^2}\right)^{\frac{1}{2}}(T-t)}{T} - K^{i\text{-}vol}\right). \tag{47}$$

4 Conclusion

In this note, we have presented closed–form solutions for the prices of volatility and variance swaps, under the assumption that volatility is a mean–reverting log–normal process. The work presented here can be extended in several ways. The obvious next step is calibration of the model, which will give empirical proof (or otherwise) of the effectiveness of the approach. Furthermore, the generality of the model permits the pricing of other derivative instruments using a similar approach.

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