A Comparison of q-optimal option prices in a Stochastic Volatility Model with correlation [†]

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This paper investigates option prices in an incomplete stochastic volatility model with correlation. In a general setting, we prove an ordering result which says that prices for European options with convex payoffs are decreasing in the market price of volatility risk.

As an example, and as our main motivation, we investigate option pricing under the class of q-optimal pricing measures. Using the ordering result, we prove comparison theorems between option prices under the minimal martingale, minimal entropy and variance-optimal pricing measures. If the Sharpe ratio is deterministic, the comparison collapses to the well known result that option prices computed under these three pricing measures are the same.

As a concrete example, we specialise to a variant of the Heston model for which the Sharpe ratio is increasing in volatility. For this example we are able to deduce option prices are decreasing in the parameter q. Numerical solution of the pricing pde corroborates the theory and shows the magnitude of the differences in option price due to varying q. Choice of q is shown to influence the level of the implied volatility smile for options of varying maturity.

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1 Introduction

Stochastic volatility models were developed as it became apparent that the Black Scholes option pricing formula exhibits pricing biases across moneyness and maturity. In particular, the Black Scholes formula underprices deep out-of-the-money puts and calls. Further, empirical evidence suggests that stock return distributions are negatively skewed with higher kurtosis than the lognormal distibution, see Bates [2] and Bakshi *et al* [1]. The evidence for negative correlation between asset and volatility is particularly strong in the equity markets, see Nandi [35] and Belledin and Schlag [3]. Stochastic volatility models provide a potential explanation of both the skew and kurtosis effects.

In an incomplete market model, such as a stochastic volatility model, there are no unique preference-independent prices for options. In recent years there has been much research in the area of characterising pricing measures in incomplete markets. Of particular interest is the special case of option prices under stochastic volatility models, since such models exhibit incompleteness without the additional complication of jumps. This paper adds to this literature by presenting a comparison of option prices under various choices of pricing measures, or equivalently, various market prices of volatility risk.

The contribution of this article is threefold. First, in a general setting of an autonomous stochastic volatility model with correlation, we prove an ordering result that the prices of options with convex payoff structures are decreasing in the market price of volatility risk. This result should be compared with, and is an extension of, the results of Bergman *et al* [4], El Karoui *et al* [12], Hobson [25] and Romano and Touzi [40] which show that the option price is increasing in the initial value of volatility.

Second, we apply these results to the class of q-optimal pricing measures which have received much attention recently in the mathematical finance literature. The minimal entropy martingale measure [17], the variance-optimal martingale measure [14] and the minimal reverse entropy martingale measure [43] are all special cases of q-optimal measures. Our goal is to prove comparison theorems between option prices under these various pricing measures and under the minimal martingale measure [13]. The analysis utilises recent results of Hobson [26] on characterising q-optimal measures in stochastic volatility models. For example we find that if the Sharpe ratio is deterministic then in our jump-free setting the q-optimal measures all collapse to the minimal martingale measure. This class of models is often described as being 'almost complete', see Schweizer [42], [43] and Pham *et al* [36]. More importantly, our paper analyses option price orderings outside this special 'almost complete' case.

Third, we undertake a numerical investigation of our results in the Heston model [24]. In this model we can write down explicit forms for many of the quantities of interest (including the market price of volatility risk, and the form of the q-optimal measure). These numerical results support the theory by illustrating the fact that option prices are monotonic in the parameter q, and also provide evidence of the magnitude of the price changes with respect to q.

Our theoretical results can be seen as an extension of the results in Henderson [23]. Henderson studies the special case when there is no correlation between the asset and volatility. In her case, stronger ordering results are obtained, but only under the restrictive assumption of zero correlation. As we remarked above this is an unrealistic model in many markets. Our techniques also differ from Henderson [23]. We use partial differential equation (pde) arguments which generalise more simply to non-zero correlation than the coupling methods of Henderson [23].

Similarly our numerical results can be seen as extensions of the results of Heath *et al* [22]. These authors compare option prices under the variance-optimal ($q = 2$) and minimal martingale measure ($q = 0$) in the Heston model with zero correlation. Our paper extends their results to non-zero correlation and to arbitrary values of q.

The remainder of the paper is organised as follows. Section 2 begins by defining the class of stochastic volatility models under consideration in the paper and describes the form of the equivalent martingale measures. The general option price ordering result is stated and proved in Section 3. In the following section, we specialise to the class of q-optimal measures and summarise their properties. Section 5 employs the general ordering result together with the characterisation of the q-optimal measures to compare option prices in a general stochastic volatility model. We can obtain stronger results by specialising to the Heston [24] stochastic volatility model, and this is the subject of Section 6. Option prices and implied volatility smiles are generated under the Heston model and their dependence on the choice of q explained. The final section concludes the paper.

2 Stochastic Volatility models

Let S be the price of the traded asset (we will assume a zero interest rate so S is actually a discounted price) and let V be the stochastic volatility. In principle, S and V could be vector valued quantities, but in this paper we will only consider the univariate case. Under the real world measure $\mathbb{P}$ let S and V solve:

$$\frac{dS_t}{S_t} = V_t \left(\alpha(t, V_t)dt + dB_t\right) \qquad dV_t = a(t, V_t)dt + b(t, V_t)dW_t, \tag{1}$$

where B, W are correlated Brownian motions with a constant correlation coefficient ρ. We assume ρ takes a value in (-1,1) and we write $\bar{\rho} = \sqrt{1-\rho^2}$ so that W can be represented via $dW_t = \rho dB_t + \bar{\rho} dZ_t$ where Z is a Brownian motion which is independent of B.

We assume V is a non-negative process, this covers the main models in the literature including Heston [24] and Hull and White [27], [28]. By convention, we take $b(t, V_t) > 0$ throughout.

The model (1) is not the most general stochastic volatility model. For example it is possible to let α, a and b be functions of S as well as V (or even to let them be non-Markovian) and to let ρ depend on t, S or V. However the framework (1) does include most of the standard stochastic volatility models in the literature and has the feature that the volatility process is an autonomous diffusion. It is this feature that will allow us to prove many of our results.

The price process S in (1) has drift $\alpha(t, V_t)V_t$ and volatility V_t. Such a parameterisation gives an interpretation of $\alpha(t, V_t)$ as the Sharpe ratio or equity risk premium. The variable $\alpha(t, V_t)$ will play an important role in our comparisons.

Our analysis takes place on a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \le T_+}, \mathbb{P})$ which supports the pair of independent Brownian motions B and Z, and is such that these processes generate the filtration $(\mathcal{F}_t)$. $\mathbb{P}$ is the real world probability measure. (Note that in many option pricing papers the role of $\mathbb{P}$ is merely to determine the set of null events. In our case, a different choice of real world measure will have an impact on pricing.) The time T_+ is a finite horizon time and we are interested in events up to the fixed time $T < T_+$. We need to assume that there exists a unique non-explosive strong solution to (1). Unfortunately, in the parameterisation (S, V) the standard conditions for existence and uniqueness of solutions to SDE's (e.g Rogers and Williams [39, Theorem V 11.1] or Duffie [10, Appendix E]) do not apply. Instead to prove the necessary properties it is convenient to find a reparameterisation $S = e^X$ and $V = \Upsilon(Y)$ for some pair (X, Y) and a suitable function Υ for which the standard conditions apply. Once the existence and uniqueness of (X, Y) has been proved, these properties will carry over to S and V. We continue to work with S and V since these are the economically significant variables.

Since S is the only traded asset in the model, and V is not traded, it is not possible under the model in (1) to perfectly replicate a derivative on the stock price S. The market is incomplete, and there are many probability measures under which the traded asset is a (local) martingale. Denote the set of such measures by $\mathcal{Q}$. Under the assumption of no arbitrage, mild conditions on the coefficients guarantee that $\mathcal{Q}$ is non-empty.

We follow Frey [15] to characterise the family of equivalent martingale measures. A probability measure $\mathbb{Q} \in \mathcal{Q}$ equivalent to $\mathbb{P}$ on $\mathcal{F}_T$ is a local martingale measure for S on $\mathcal{F}_T$ if and only if there is a progressively measurable process $\lambda = (\lambda_t)_{0 \le t \le T}$ and with $\int_0^T \lambda_s^2 ds < \infty$ $\mathbb{P}$ a.s. such that the local martingale $(M_t)_{0 \le t \le T}$ with

$$M_t = \exp\left(\int_0^t \left[-\alpha(u, V_u)dB_u - \frac{1}{2}\alpha(u, V_u)^2 du - \lambda_u dZ_u - \frac{1}{2}\lambda_u^2 du\right]\right) \qquad (2)$$

satisfies $\mathbb{E}M_T = 1$ and $M_T = \frac{d\mathbb{Q}}{d\mathbb{P}}$ on $\mathcal{F}_T$. If M_t is of form (2), S is a $\mathbb{Q}$-local martingale. The condition $\mathbb{E}M_T = 1$ guarantees that M is a true $\mathbb{P}$-martingale and that $\mathbb{Q}$ is a probability measure.

The space of equivalent local martingale measures is parameterised by the process λ_t which governs the change of drift of Z. By Girsanov's theorem, under the change of measure M_T we have two independent $\mathbb{Q}$-Brownian motions $B^{\mathbb{Q}}$ and $Z^{\mathbb{Q}}$ defined by

$$dB_t^{\mathbb{Q}} = dB_t + \alpha(t, V_t)dt \qquad dZ_t^{\mathbb{Q}} = dZ_t + \lambda_t dt$$

and then

$$\begin{aligned} \frac{dS_t}{S_t} &= V_t dB_t^{\mathbb{Q}} \\ dV_t &= [a(t, V_t) - \rho\alpha(t, V_t)b(t, V_t) - \bar{\rho}\lambda_t\, b(t, V_t)]dt + b(t, V_t)dW^{\mathbb{Q}} \qquad (3) \end{aligned}$$

where $dW^{\mathbb{Q}} = \rho dB^{\mathbb{Q}} + \bar{\rho} dZ^{\mathbb{Q}}$.

Under $\mathbb{Q}$ the change of drift on Z is λ_t, the associated change of drift on W is $\rho\alpha(t, V_t) + \bar{\rho}\lambda_t$ and the change of drift on V is $(\rho\alpha(t, V_t) + \bar{\rho}\lambda_t)b(t, V_t)$. The quantity $\rho\alpha(t, V_t) + \bar{\rho}\lambda_t$ is often termed the 'market price of volatility risk' or volatility risk premium. We sometimes call λ_t the market price of Z risk, also known in Lewis [32] as the hedging portfolio risk premium. Note that since $\bar{\rho}$ is positive, then for a given model under $\mathbb{P}$, the market price of volatility risk is positively related to the market price of Z risk.

One simple example of a candidate equivalent local martingale measure is the minimal martingale of Föllmer and Schweizer [13]. This is the measure which corresponds to the choice $\lambda \equiv 0$. (For this candidate to truly be an element of $\mathcal{Q}$ we need to verify that $\mathbb{E}M_T = 1$. A necessary condition is that $\int_0^T \alpha(u, V_u)^2 du < \infty$ almost surely.) Intuitively, under the minimal martingale measure the drifts of Brownian motions which correspond to traded assets are modified to make those assets into martingales, but the drifts of Brownian motions which are orthogonal to the traded assets are left unchanged.

We now turn to the question of option pricing within this model. A European option with payoff $h(S_T)$ can be priced by fixing $\mathbb{Q}$ in $\mathcal{Q}$ and then taking expectation under $\mathbb{Q}$. The time-t option price $C(t, S_t, V_t)$ becomes

$$C(t, S_t, V_t) = \mathbb{E}_t^{\mathbb{Q}} h(S_T)$$

where the superscript $\mathbb{Q}$ refers to the fact that we are taking expectations with respect to $\mathbb{Q}$ and the subscript t refers to the fact that we are conditioning on information at time t.

The advantages of fixing a measure $\mathbb{Q}$ and using it for pricing are that the pricing functional is linear and that it agrees with the arbitrage free price for those options which can be replicated. Given the characterisation above, selecting a particular $\mathbb{Q}$ is equivalent to choosing a market price of volatility risk, and the key to identifying the price is understanding the dynamics in (3).

3 The General Option Price Ordering Result

This section proves our main ordering result which says that convex option prices are decreasing in the market price of Z risk parameter λ_t, or equivalently decreasing in the market price of volatility risk. The intuition is that an increase in either λ_t or the market price of volatility risk corresponds to a decrease in the drift of the volatility under the pricing measure.

The first assumption that we make is to only consider changes of drift λ_t for the Brownian motion Z which are Markov functions of the volatility process. Thus we suppose $\lambda_t = \lambda(t, V_t)$. We show in Section 4 that the market price of Z risk for the q-optimal measure takes this form. Secondly, throughout this section and in subsequent sections we will use the Feynman-Kac theorem (see Karatzas and Shreve [29, Section 5.7 B] or Duffie [10, Appendix E]) to convert the solution of a Cauchy problem expressed via a pde into a stochastic representation. To use this result we need to know that the solution exists, and is unique (at least among the class of solutions satisfying a polynomial growth condition). Again the standard conditions for the existence and uniqueness of a solution (see Friedman [16, p147], Karatzas and Shreve [29] or Duffie [10]) will not be satisfied in our parameterisation. However, if $S = e^X$ and $V = \Upsilon(Y)$, and if the pair (X, Y) satisfy appropriate regularity conditions, then the stochastic representation will hold. The appropriate conditions include the fact that the coefficients of the SDE are differentiable and satisfy appropriate continuity, boundedness and growth conditions (above and below) and that the payoff function h satisfies a growth condition. See for example the discussion in Romano and Touzi [40] and especially the conditions (i) to (iv) on p401. Note that these conditions are not satisfied by the Heston model [24] but in that case it is possible to justify the stochastic representations directly without recourse to the pde approach: see (6) and (7) below. We will also assume that the coefficients of the diffusion processes are sufficiently smooth so that we can differentiate the related infinitesimal generators.

We begin this section by writing down a pair of stochastic volatility models under a pricing measure. (Note that we began the previous section by writing down a single model under the real world dynamics. The difference will be that under the pricing measure S is a local martingale.) Let the price process S and volatility V satisfy

$$\frac{dS_t}{S_t} = V_t dB_t \qquad\qquad dV_t = \eta(t, V_t)dt + b(t, V_t)dW \tag{4}$$

where, as before, $dB\,dW = \rho dt$. Suppose that the drift on the volatility either takes the form $\eta(t, v) = \eta^+(t, v)$ or $\eta(t, v) = \eta^-(t, v)$ and let $\mathbb{E}^+$ (respectively $\mathbb{E}^-$) denote expectation with respect to the model with drift η^+ (respectively η^-.) For a function h define $J^+(t, s, v) = \mathbb{E}^+[h(S_T)|S_t = s, V_t = v]$, and similarly let $J^-(t, s, v) = \mathbb{E}^-[h(S_T)|S_t = s, V_t = v]$.

Theorem 1 *Consider the pair (S, V) as defined in (4) and a convex function h. If $\eta^+(t, v) \geq \eta^-(t, v)$ for all t and v, then*

$$J^+(t, s, v) \geq J^-(t, s, v).$$

Proof: The function J^+ solves $\mathcal{L}^+J^+ = 0$ subject to $J^+(T,s,v) = h(s)$ where

$$\mathcal{L}^+ = \frac{1}{2}s^2v^2\frac{\partial^2}{\partial s^2} + \rho svb(t,v)\frac{\partial^2}{\partial s\partial v} + \frac{b(t,v)^2}{2}\frac{\partial^2}{\partial v^2} + \eta^+(t,v)\frac{\partial}{\partial v} + \frac{\partial}{\partial t}.$$

Similarly, under the alternative dynamics the function J^- solves $\mathcal{L}^-J^- = 0$ subject to $J^-(T,s,v) = h(s)$ where $\mathcal{L}^-$ is obtained from $\mathcal{L}^+$ by replacing η^+ with η^-.

As a consequence, for any function $g(t,s,v)$ we have $\mathcal{L}^+g - \mathcal{L}^-g = (\eta^+ - \eta^-)g_v$. Here the subscript denotes a partial derivative. In general, a subscript t or u denotes a time parameter, other subscripts are partial derivatives, and the partial derivative with respect to time is denoted by a dot.

Consider $\tilde{J} = J^+ - J^-$ so that $\tilde{J}(T,s,v) = 0$. Then

$$\mathcal{L}^-\tilde{J} = \mathcal{L}^-(J^+ - J^-) = \mathcal{L}^+J^+ - \mathcal{L}^-J^- - (\mathcal{L}^+ - \mathcal{L}^-)J^+ = -(\eta^+ - \eta^-)J^+_v.$$

By the Feynman-Kac representation

$$\tilde{J}(t,s,v) = \mathbb{E}^-\left[\int_t^T (\eta^+ - \eta^-)J^+_v(u,S_u,V_u)du \,\middle|\, S_t = s, V_t = v\right].$$

In particular $\tilde{J} = J^+ - J^- \geq 0$ provided that $J^+_v \geq 0$. (Alternatively the conclusion follows from an application of the maximum principle to the pde formulation.)

It remains to show that the prices of options with convex payoffs are increasing in the initial value of the volatility, or equivalently that $J^+_v \geq 0$. In fact, as Roger Lee has pointed out to us this result is already to be found in the literature in an interesting paper by Romano and Touzi [40]. However, for completeness we sketch a proof of this result.

We drop the superscript $+$ from both the operator $\mathcal{L}$ and the function J. Differentiating $\mathcal{L}J = 0$ with respect to volatility gives

$$0 = (\mathcal{L}J)_v = \mathcal{L}_vJ + \mathcal{L}J_v. \tag{5}$$

If we write $g = J_v$ (so that $g(T,s,v) = 0$) we get

$$0 = s^2vJ_{ss} + \rho sb(t,v)g_s + \rho svb_v(t,v)g_s + b(t,v)b_v(t,v)g_v + \eta_v(t,v)g + \mathcal{L}g.$$

Now let

$$\mathcal{L}^\dagger = \mathcal{L} + \rho sb(t,v)\frac{\partial}{\partial s} + \rho svb_v(t,v)\frac{\partial}{\partial s} + b(t,v)b_v(t,v)\frac{\partial}{\partial v}.$$

Then $0 = s^2 v J_{ss} + \eta_v(t,v)g + \mathcal{L}^\dagger g$ and by the Feynman-Kac representation,

$$g(t,S_t,V_t) = \mathbb{E}_t^\dagger \left[\int_t^T S_w^2 V_w J_{ss}(w,S_w,V_w) \exp\left(\int_t^w \eta_v(u,v)du \right) dw \right]$$

where $\mathbb{E}^\dagger$ denotes expectation with respect to the measure under which the pair (S,V) has generator $\mathcal{L}^\dagger$. We conclude that $g = J_v \geq 0$ provided that $J_{ss} \geq 0$.

The final part of the proof is to show that $J_{ss} \geq 0$. The result that the option price J inherits a convexity property from the payoff function is standard in the literature, see Bergman *et al* [4], El Karoui *et al* [12], Hobson [25], Romano and Touzi [40] for a variety of proofs. Again for completeness we sketch a short proof.

Differentiating $\mathcal{L}J = 0$ twice with respect to s we get

$$\begin{aligned} 0 &= (\mathcal{L}J)_{ss} = \mathcal{L}_{ss}J + 2\mathcal{L}_s J_s + \mathcal{L}J_{ss} \\ &= v^2 J_{ss} + \widetilde{\mathcal{L}} J_{ss} \end{aligned}$$

where $\widetilde{\mathcal{L}} = \mathcal{L} + 2sv^2\frac{\partial}{\partial s} + 2\rho v b(t,v)\frac{\partial}{\partial v}$. By assumption, $J_{ss}(T,s,v) = h_{ss}(s) \geq 0$ (in the sense of distributions if necessary, eg. for a call option where $h_{ss}(s) = \delta(s-K)$). Again, by the Feynman-Kac representation,

$$J_{ss}(t,S_t,V_t) = \widetilde{\mathbb{E}}_t \left[h_{ss}(S_T) \exp\left(-\int_t^T V_u^2 du \right) \right] \geq 0$$

where $\widetilde{\mathbb{E}}$ is the expectation operator associated with the generator $\widetilde{\mathcal{L}}$. □

The style of proof given above can easily be adapted to give simple and direct arguments concerning the 'comparative statics of option pricing', namely the ordering of option prices as model parameters change. In the above proof we showed that the convexity of option prices is inherited from the payoff function. It is also easy to show that if the payoff function is increasing then the 'delta' of the option is positive, and that if the payoff function is convex then the 'theta' of the option is positive. Similarly in a model with interest rates and dividends it is possible to derive results on relationships between the option price and the parameters governing these variables, subject to the imposition of suitable conditions on the payoff function.

There are two main ways in which Theorem 1 can be interpreted and applied. In both cases the theorem compares the prices of European-style options with convex payoffs. The first interpretation is to take the theorem literally and to compare two different models (with the same volatility structure but a different drift on the volatility process), where both models are specified under the pricing measure. The second interpretation, which we will use extensively later in the paper, is to consider a single model under a real-world measure $\mathbb{P}$ and to consider this model under two alternative specifications of martingale measure $\mathbb{Q}^+$ and $\mathbb{Q}^-$.

In particular suppose the stochastic volatility model is as given in (1), and there are two candidate pricing measures defined via $(M^+, \lambda^+, \mathbb{Q}^+)$ and $(M^-, \lambda^-, \mathbb{Q}^-)$ respectively. Then if we set $\eta^{\pm} = a - \rho\alpha b - \overline{\rho}\lambda^{\pm} b$ and if $\lambda^+(t,v) \leq \lambda^-(t,v)$ then we have $\eta^+ \geq \eta^-$. It follows from Theorem 1 that convex option prices are higher under the pricing measure $\mathbb{Q}^+$.

4 The Class of q-optimal pricing measures

For the remainder of this paper, we focus on a class of equivalent martingale measures, namely the q-optimal measures. For a given q the q-optimal measure is the martingale measure which is closest to the original real world measure $\mathbb{P}$ in the sense of the q^{th} moment; see Grandits and Krawczyk [19], Delbaen *et al* [8], Grandits and Rheinländer [20] and Hobson [26].

In later sections we will compare option prices under stochastic volatility models of the form (1) where option prices are computed under various q-optimal measures. Note that in order to calculate the q-optimal measure it is necessary to know the real world dynamics and probability measure $\mathbb{P}$.

The fundamental idea is to choose a martingale measure $\mathbb{Q}$ as close as possible to $\mathbb{P}$. For $q \in \mathbb{R} \setminus \{0,1\}$ define

$$H_q(\mathbb{P},\mathbb{Q}) = \begin{cases} \mathbb{E}[\frac{q}{q-1}(M_T)^q] & \text{if } \mathbb{Q} \ll \mathbb{P} \\ \infty & \text{otherwise,} \end{cases}$$

and for $q \in \{0,1\}$ define

$$H_q(\mathbb{P},\mathbb{Q}) = \begin{cases} \mathbb{E}[(-1)^{1+q} M_T^q \ln(M_T)] & \text{if } \mathbb{Q} \ll \mathbb{P} \\ \infty & \text{otherwise.} \end{cases}$$

For $q \in \mathbb{R}$ the q-optimal measure is the measure $\mathbb{Q}^{(q)}$ which minimises $H_q(\mathbb{P},\mathbb{Q})$.

A number of well studied martingale measures are special cases in the q-optimal class and, as such, this class provides a unifying framework. When $q = 1$, $\mathbb{Q}^{(1)}$ is the minimal relative entropy measure, see Frittelli [17]. There are strong links between this pricing measure and pricing under exponential utility, see Delbaen *et al* [7]. Taking $q = 0$ gives the minimal reverse entropy measure $\mathbb{Q}^{(0)}$. (The extra log terms when $q = 0$ and $q = 1$ can be related to log terms which arise when we integrate x^{q-2} twice.) The case $q = 2$ gives the variance-optimal measure, which is related to mean-variance hedging, see Föllmer and Sondermann [14] and Duffie and Richardson [11]. In a stochastic volatility context, the variance optimal measure has been studied extensively by Laurent and Pham [31], Biagini *et al* [5] and Heath *et al* [22].

The case $q = 0$ has some special features. In a continuous setting such as ours (and under certain regularity assumptions) Schweizer [43] has shown that the measure which minimises the reverse relative entropy is precisely the (Föllmer-Schweizer) minimal martingale measure. This corresponds to the choice $\lambda \equiv 0$ and implies that the non-hedgeable risk is not priced. We will recover this result in Corollary 4 below.

Although the cases $q = 0, 1, 2$ have special properties, the criterion of choosing $\mathbb{Q}$ to minimise H_q makes sense for all q in $\mathbb{R}$. Further, for $q \neq 0, 1$ there are links between pricing under the q-optimal measure and utility indifference pricing under a power-law utility. The cases $q < 1$ correspond to strictly increasing, concave utility functions defined on $\mathbb{R}^+$.

Existence of the q-optimal measure has been dealt with by a number of authors. When $q = 1$, Rheinländer [37] gives a necessary and sufficient condition for the existence of an equivalent measure with finite relative entropy. A result of Frittelli [17] then guarantees the minimal entropy measure exists. When $q = 2$, existence is dealt with in Delbaen and Schachermayer [9], Gourieroux *et al* [18] and Rheinländer and Schweizer [38], and more generally for $q > 1$ related results can be found in Grandits and Rheinländer [20]. In the particular setting of our stochastic volatility model, and given certain smoothness and boundedness conditions on the parameters, Hobson [26], shows that if $q(1 - q\rho^2) > 0$ then the q-optimal measure always exists. If, on the other hand $q(1 - q\rho^2) \leq 0$ then it may be the case that the q-optimal measure ceases to exist beyond a certain q-dependent time horizon.

To investigate the dependence of option prices on the choice of q-optimal measure, we need to be able to characterise such measures. Motivated by results in Rheinländer [37] for the minimal entropy case, Hobson [26] derives a representation equation which characterises the q-optimal measure. See also Laurent and Pham [31] and Mania *et al* [33] for related approaches. It remains to solve the representation equation and to find the form of the market price of risk associated with the q-optimal measure. In the variance-optimal case with zero correlation, this was achieved by Laurent and Pham [31] and Biagini *et al* [5], and in the minimal entropy case within the Stein-Stein volatility model (with non-zero correlation) by Rheinländer [37]. Instead we follow Hobson [26] who shows how to solve the q-optimal representation equation for models with correlation, at least for models of the form (1). In fact, he only considers $q \geq 1$ but his representation holds equally for $q < 1$ so we are able to treat all the special cases of the general theory within the q-optimal setting. Defining $R = 1 - q\rho^2$, the identification of the q-optimal measure $\mathbb{Q}^{(q)}$ is given via the market price of Z risk,

$$\lambda^{(q)}(t, V_t) = \bar{\rho} b(t, V_t) f_v(t, V_t) \tag{6}$$

where $f = 0$ if $q = 0$ and otherwise

$$f(t, v) = -\frac{1}{R} \log \hat{\mathbb{E}} \left[\exp \left(-\frac{q}{2} R \int_t^T \alpha(u, V_u)^2 du \right) \middle| V_t = v \right] \tag{7}$$

or

$$f(t, v) = \hat{\mathbb{E}} \left[\frac{q}{2} \int_t^T \alpha(u, V_u)^2 du \middle| V_t = v \right] \tag{8}$$

in the special case of $R = 0$. Under $\hat{\mathbb{P}}$, V has dynamics

$$dV_t = (a(t, V_t) - q\rho\alpha(t, V_t)b(t, V_t))dt + b(t, V_t)d\hat{W}$$

with $\hat{\mathbb{P}}$-Brownian motion $\hat{W}$. If $\rho = 0$ or $q = 0$ then $\hat{\mathbb{P}}$ is just the real world measure $\mathbb{P}$.

By the Feynman-Kac theorem, f solves the representation equation

$$\frac{q}{2}\alpha(t, v)^2 - q\rho b(t, v)\alpha(t, v) f_v - \frac{R}{2} b(t, v)^2 (f_v)^2 + a(t, v) f_v + \frac{1}{2} b(t, v)^2 f_{vv} + \dot{f} = 0 \tag{9}$$

with $f(T, v) = 0$.

Consider the representation (7). If $qR > 0$ (or equivalently, $q > 0$ and $\rho^2 < \frac{1}{q}$) then f is positive and finite, and the q-optimal measure is defined over all time horizons. However, if $qR \leq 0$, additional conditions are necessary to ensure the change of drift $\lambda^{(q)}$ is finite. Typically the function f explodes at a finite time-horizon, beyond which the q-optimal measure ceases to be defined.

Under $\mathbb{Q}^{(q)}$, the dynamics in (3) become

$$\begin{aligned} \frac{dS_t}{S_t} &= V_t dB_t^{\mathbb{Q}^{(q)}} \\ dV_t &= [a(t, V_t) - \rho\alpha(t, V_t)b(t, V_t) - \bar{\rho}^2 b(t, V_t)^2 \, f_v(t, V_t)]dt + b(t, V_t)dW_t^{\mathbb{Q}^{(q)}} \end{aligned}$$

with $B_t^{\mathbb{Q}^{(q)}} = B_t + \int_0^t \alpha(u,V_u)du$, $Z_t^{\mathbb{Q}^{(q)}} = Z_t + \int_0^t \bar{\rho} b(u,V_u) f_v(u,V_u)du$ and $dW^{\mathbb{Q}^{(q)}} = \rho dB^{\mathbb{Q}^{(q)}} + \bar{\rho} dZ^{\mathbb{Q}^{(q)}}$. These are the general model dynamics for the class of q-optimal measures.

5 Comparisons between the q-optimal measures

The general ordering theorem says option prices are decreasing in λ, or equivalently decreasing in the market price of volatility risk. Our task in this section is to analyse $\lambda^{(q)}$ for the q-optimal class of measures and hence to compare option prices. As we saw earlier, the q-optimal market price of Z risk may be written as

$$\lambda^{(q)}(t,V_t) = \bar{\rho} b(t,V_t) f_v(t,V_t)$$

where f is given in (7) or (8) and solves the pde (9).

We first investigate the sign of $\lambda^{(q)}$. If $q=0$ then $f \equiv 0$ and $\lambda^{(0)} \equiv 0$. Otherwise, consider first the case $R \neq 0$. Under the transformation $g = e^{-Rf}$, g is given by

$$g(t,v) = \hat{\mathbb{E}} \exp\left(-\frac{q}{2} R \int_t^T \alpha(u,V_u)^2 du \right) > 0,$$

see (7), and (9) becomes

$$-\frac{q}{2} R\alpha(t,v)^2 g + \left(a(t,v) - q\rho b(t,v)\alpha(t,v)\right) g_v + \frac{1}{2} g_{vv} b(t,v)^2 + \dot{g} = 0 \qquad (10)$$

subject to $g(T,v) = 1$. Now $g_v = -Rgf_v$ and we can examine the sign of f_v via an analysis of g_v. The above pde can be written as

$$0 = \hat{\mathcal{L}} g - \frac{q}{2} R\alpha(t,v)^2 g \qquad (11)$$

where

$$\hat{\mathcal{L}} = \left(a(t,v) - q\rho\alpha(t,v)b(t,v)\right)\frac{\partial}{\partial v} + \frac{1}{2} b(t,v)^2 \frac{\partial^2}{\partial v^2} + \frac{\partial}{\partial t}.$$

Differentiating (11) with respect to v yields

$$\begin{aligned} 0 &= \left[\hat{\mathcal{L}} g - \left(\frac{q}{2} R\alpha(t,v)^2 \right) g \right]_v \\ &= \hat{\mathcal{L}}_v g + \hat{\mathcal{L}} g_v - \frac{q}{2} R(2g\alpha\alpha_v + \alpha^2 g_v) \\ &= \left((a_v - q\rho b\alpha_v - q\rho\alpha b_v)\frac{\partial}{\partial v} + bb_v \frac{\partial^2}{\partial v^2} \right) g + \hat{\mathcal{L}} g_v - \frac{q}{2} R \left(2g\alpha\alpha_v + \alpha^2 g_v \right). \end{aligned}$$

Define

$$\mathcal{L}^{\ddagger} = \hat{\mathcal{L}} + b(t,v)b_v(t,v)\frac{\partial}{\partial v}.$$

Then $g_v(T,v)=0$ and

$$0=\left(a_v-q\rho b\alpha_v-q\rho\alpha b_v-\frac{q}{2}R\alpha^2\right)g_v+\mathcal{L}^{\ddagger}g_v-qR\alpha\alpha_v g.$$

By the Feynman-Kac representation,

$$g_v(t,v)=-qR\mathbb{E}^{\ddagger}\left[\int_t^T\alpha\alpha_v g\exp\left(\int_t^s(a_v-q\rho b\alpha_v-q\rho\alpha b_v-\frac{q}{2}R\alpha^2)du\right)ds\right]$$

where under $\mathbb{P}^{\ddagger}$, V has drift $(a-q\rho\alpha b+bb_v)$. Now recall $g_v=-Rgf_v$ so

$$f_v=\frac{q}{g}\mathbb{E}^{\ddagger}\left[\int_t^T\alpha\alpha_v g\exp\left(\int_t^s(a_v-q\rho b\alpha_v-q\rho\alpha b_v-\frac{q}{2}R\alpha^2)du\right)ds\right].$$

Since $g>0$ we have that

$$\begin{aligned}\text{if }q\alpha(t,v)\alpha_v(t,v)&>0\text{ then }f_v>0;\\ \text{if }q\alpha(t,v)\alpha_v(t,v)&<0\text{ then }f_v<0;\\ \text{if }q\alpha(t,v)\alpha_v(t,v)&=0\text{ then }f_v=0.\end{aligned}$$

It remains to consider the case $R=0$. In that case (9) becomes

$$\frac{q}{2}\alpha^2-q\rho f_v b\alpha+af_v+\frac{1}{2}b^2f_{vv}+\dot{f}=0,$$

subject to $f(T,v)=0$. Differentiating this equation with respect to v and applying the Feynman-Kac formula gives

$$f_v=q\mathbb{E}^{\ddagger}\left[\int_t^T\alpha(t,v)\alpha_v(t,v)\exp\left(\int_t^s(a_v-q\rho b\alpha_v-q\rho\alpha b_v)du\right)ds\right]$$

and the same conclusions. We have proved the following theorem.

Theorem 2 *Under stochastic volatility dynamics in (1), and for $q\in\mathbb{R}$*
(i) if $q\alpha(t,v)^2$ is non-decreasing in v, then $\lambda^{(q)}\geq 0$;
(ii) if $q\alpha(t,v)^2$ is non-increasing in v, then $\lambda^{(q)}\leq 0$;
(iii) the inequalities on $\lambda^{(q)}$ are strict if $q\alpha(t,v)^2$ is strictly increasing or decreasing in v.

Combining this result with the general ordering of Theorem 1, and the fact that $\lambda^{(0)}=0$ we can make the following conclusion on the ordering of q-optimal option prices.

Corollary 3 *If, for each t, $q\alpha(t,v)^2$ is increasing in v, then option prices under the q-optimal measure are less than those under the minimal martingale measure. Conversely, if $q\alpha(t,v)^2$ is decreasing in v, then option prices under the q-optimal measure exceed those under the minimal martingale measure.*

Corollary 4 *If α is deterministic, then option prices under the q-optimal measure are all equal to the minimal martingale measure option price.*

Remark 5 The restricted set of models where the Sharpe ratio $\alpha(t, V_t)$ is deterministic are 'almost complete' in the terminology of Pham *et al* [36] and Laurent and Pham [31]. Corollary 4 generalises results of Pham *et al* [36] and Schweizer [42], [43] to q-optimal measures.

In the light of Theorem 2 and Corollary 3 it is natural to make the following conjecture:

Conjecture 6 *Option prices under the q-optimal measure are decreasing (respectively increasing) in q if α^2 is increasing (respectively decreasing) in v.*

This conjectures that whether option prices increase or decrease in q depends on whether the (square of the) Sharpe ratio is increasing or decreasing in volatility.

The remainder of this section is devoted to an investigation of this conjecture. As Henderson [23] shows (and as we show below) the conjecture is true for models with zero correlation. (The proof in [23] is based on couplings of stochastic processes.) The conjecture also holds for the special case of the Heston model we consider in the next section.

We are interested in the effect of q on the market price of Z risk given in (6) earlier as

$$\lambda^{(q)}(t, V_t) = \bar{\rho} f_v(t, V_t) b(t, V_t).$$

The variable of interest is therefore f_{vq} which we denote by $k(t, V_t)$. Note that $k(T, v) = 0$. It is sufficient to determine the sign of k. Differentiating the pde for f given in (9) with respect to v and then q gives

$$\begin{aligned} \dot{k} &+ \frac{1}{2} b^2 k_{vv} + k_v(a + bb_v - q\rho b\alpha - Rb^2 f_v) \\ &+ k(a_v - Rb^2 f_{vv} - q\rho(b\alpha)_v - 2Rbb_v f_v) = -\frac{1}{2}[(\alpha - \rho b f_v)^2]_v. \end{aligned}$$

If we set $\rho = 0$ then the Feynman-Kac representation tells us that if $(\alpha^2)_v$ is everywhere positive then $k > 0$. Conversely, if $(\alpha^2)_v < 0$ then $k < 0$. However, if $\rho \neq 0$, Feynman-Kac shows the sign of k depends on the sign of $[(\alpha - \rho b f_v)^2]_v$ which is not so easy to determine, since it depends on the unknown f_v.

Although we cannot obtain a general result for the effect of q on $\lambda^{(q)}$ when correlation is present, we can if we specialise to a particular model. In the next section we consider the Heston [24] model.

6 Option Price Comparisons under the Heston model

In this section, we are interested in comparing various q-optimal option prices in the Heston [24] model. In this special case we can give explicit formulæ for the market price of Z risk (and hence market price of volatility risk) associated with the q-optimal measure. This allows us to analyse the effect of choice of q on option prices.

Later in this section we illustrate these comparisons by solving the q-optimal pricing pde for the Heston model numerically. The graphs reinforce the theoretical results and allow for analysis of implied volatility smiles generated by q-optimal pricing for various q.

The Heston model [24] gives the squared volatility U as a squared Bessel process under the real world probability measure $\mathbb{P}$

$$\begin{aligned} \frac{dS_t}{S_t} &= \alpha(t, \sqrt{U_t})\sqrt{U_t}dt + \sqrt{U_t}dB_t, \\ dU_t &= 2\kappa(m - U_t)dt + 2\beta\sqrt{U_t}dW_t \end{aligned}$$

with B and W correlated Brownian motions. Writing $U = V^2$ and $\bar{m} = m - \beta^2/2\kappa$ we get

$$\frac{dS_t}{S_t} = V_t\left(\alpha(t, V_t)dt + dB_t\right) \qquad dV_t = \kappa(\bar{m}/V_t - V_t)dt + \beta dW_t. \tag{12}$$

This model is equivalent to taking $a(t, V_t) = \kappa(\bar{m}/V_t - V_t)$ and $b(t, V_t) = \beta$ in our original model. We assume $\beta, \kappa > 0$ and $m \geq \beta^2/\kappa$, the latter to guarantee that V does not hit zero. The existence and uniqueness of solutions to (12) is covered in Sin [44]. Sin also provides a discussion of sufficient conditions for S to be a true martingale under a local martingale measure element of $\mathcal{Q}$.

The choice of the Sharpe ratio $\alpha(t, V_t)$ in the model will have a direct impact on the form of the q-optimal measure and hence the ordering of q-optimal option prices, via the function f in (7) or (8). Before discussing the appropriate choice of $\alpha(t, V_t)$, we will describe the martingale measure used by Heston for pricing. In the original Heston model [24] the Sharpe ratio term $\alpha(t, V_t)$ is of the form $\alpha(t, V_t) = \alpha_{-1}/V_t$. To obtain a model under the risk neutral measure Heston adjusts the drift on the traded Brownian motion B so that S is a $\mathbb{Q}$-martingale and then proposes that the effect of the change of measure on the volatility V is to adjust the drift by a term $\lambda^{\mathrm{H}}V$. Thus under the pricing measure the dynamics become

$$\frac{dS_t}{S_t} = V_t dB_t^{\mathbb{Q}} \qquad dV_t = \{\kappa(\bar{m}/V_t - V_t) - \lambda^{\mathrm{H}}V_t\}dt + \beta dW_t^{\mathbb{Q}}. \tag{13}$$

In particular, although Heston makes the choice $\alpha(t, V_t) = \alpha_{-1}/V_t$, his choice of pricing measure makes this term disappear. Note that the market price of volatility risk $\lambda^{\mathrm{H}}V/\beta$ is equivalent to a change of measure for which $dW^{\mathbb{Q}} = dW + (\lambda^{\mathrm{H}}V/\beta)dt$ is a $\mathbb{Q}$-martingale. In turn this corresponds to a change in drift on the orthogonal martingale Z to make $Z^{\mathbb{Q}}$ given by $dZ^{\mathbb{Q}} = dZ - \frac{1}{\bar{\rho}}(\frac{\rho\alpha_{-1}}{V} - \frac{\lambda^{\mathrm{H}}V}{\beta})dt$ into a $\mathbb{Q}$-martingale. (The Brownian motions B and W are correlated, so although the switch to the pair (B, Z) is merely a reparameterisation, we contend that it is more natural to consider the effect of a change of measure in terms of the impact on the independent driving Brownian motions.)

Under the Heston change of drift,

$$dV_t = \kappa^* \left(\frac{\bar{m}^*}{V_t} - V_t \right) dt + \beta dW_t{}^{\mathbb{Q}}$$

where $\kappa^* = \kappa + \lambda^{\mathrm{H}}$, $\bar{m}^* = \frac{\kappa m - \frac{1}{2}\beta^2}{\kappa + \lambda^{\mathrm{H}}}$ are risk adjusted parameters and $W^{\mathbb{Q}}$ is a $\mathbb{Q}$-Brownian motion. Under this framework, Heston was able to obtain a pricing formula via Fourier inversion techniques involving numerical integration of the real part of a complex function. However, Heston's choice of pricing measure is essentially arbitrary and, in terms of its effect on the Brownian motion Z, not very natural. Certainly it is not one of the q-optimal measures. The choice does, however allow him to obtain a tractable method for pricing options.

Returning to the question of the specification of Sharpe ratio $\alpha(t, V_t)$ in the real world model (12), there are several possibilities. We disregard a constant Sharpe ratio ($\alpha(t, V_t) = \alpha_0$) case since Corollary 4 implies option prices under all q-optimal measure prices are equal in this case. Heston [24] takes $\alpha(t, V_t) = \alpha_{-1}/V_t$, although as described, this has no impact in his model on option prices. The third choice of Sharpe ratio $\alpha(t, V_t) = \alpha_1 V_t$ is a popular one in the literature, and appears as (H2) in Heath et al [22], and as an example in both Hobson [26] and Henderson [23]. If $\alpha_1 > 0$, this specification implies a Sharpe ratio which is increasing with volatility, a feature which is backed up empirically by Campbell and Cochrane [6]. In this section we will focus on the specification of the Heston model with $\alpha(t, V_t) = \alpha_1 V_t$.

Hobson [26, Proposition 5.1] gives an explicit formula for the solution of (7) and (8) (or equivalently (9)). In this case the solution to (9) is of the form

$$f(t, v) = v^2 F(T-t)/2 + G(T-t)$$

where $F(0) = 0 = G(0)$ and F and G solve

$$\dot{F} = q\alpha_1^2 - 2(\kappa + q\rho\beta\alpha_1)F - R\beta^2 F^2, \qquad \dot{G} = \kappa m F. \tag{14}$$

The differential equation for F can be solved explicitly on a case by case basis depending on the signs of q and R. For example if q and R are both positive,

$$F(\tau) = \frac{C}{A}\tanh\left(AC\tau + \tanh^{-1}\left(\frac{AB}{C}\right)\right) - B$$

where the constants A, B and C are given by

$$A^2 = R\beta^2 \qquad B = \frac{\kappa + q\rho\beta\alpha_1}{\beta^2 R} \qquad C^2 = q\alpha_1^2 + \frac{(\kappa + q\rho\beta\alpha_1)^2}{\beta^2 R}.$$

This solution is finite for all time. For other parameter values the function F may explode.

We are interested in the market price of Z risk

$$\lambda^{(q)}(t, V) = \bar{\rho}\beta V_t F(T-t)$$

for the change of drift of V under the q-optimal measure. The dynamics of V under $\mathbb{Q}^{(q)}$ are

$$dV_t = (\kappa(\bar{m}/V_t - V_t) - \rho\beta\alpha_1 V_t - \bar{\rho}^2\beta^2 V_t F(T-t))dt + \beta(\rho dB_t^{\mathbb{Q}^{(q)}} + \bar{\rho} dZ_t^{\mathbb{Q}^{(q)}})$$

with $dZ_t^{\mathbb{Q}^{(q)}} = dZ_t + \bar{\rho}\beta V_t F(T-t)dt$. Note the dynamics under the q-optimal measure are time-inhomogeneous.

We can now investigate the dependence of $\lambda^{(q)}$ on q under the Heston model.

Proposition 7 *For the Heston model (12), with $\alpha(t, V_t) = \alpha_1 V_t$, the market price of risk under the q-optimal class of pricing measures $q \in \mathbb{R}$ is increasing in q.*

Proof: To decide the sign of $(\lambda^{(q)})_q$ it is enough to know the sign of f_{vq}. Further, since $f(t, v) = v^2 F(T-t)/2 + G(T-t)$ it is sufficient to know the sign of $H = F_q$ where $F(0) = 0$ and F solves (14). Differentiating (14) with respect to q gives

$$\dot{H} = (\alpha_1^2 - 2\rho\beta\alpha_1 F + \rho^2\beta^2 F^2) - 2(\kappa + q\rho\beta\alpha_1)H - 2R\beta^2 FH = \theta + \Theta H$$

where $\theta = (\alpha_1^2 - 2\rho\beta\alpha_1 F + \rho^2\beta^2 F^2) \geq 0$ and $\Theta = -2(R\beta^2 F + \kappa + q\rho\beta\alpha_1)$. Since $H(0) = 0$ and $\dot{H}(0) = \alpha_1^2 > 0$ we conclude $H(t, V) > 0$. □

Combining Proposition 7 and the general comparison result in Theorem 1, we have

Corollary 8 *For the Heston model (12) with $\alpha(t, V_t) = \alpha_1 V_t$, q-optimal prices for European options with convex payoffs are decreasing in q.*

This implies the highest option prices (within the class of q-optimal measure prices) are attained with the measures which correspond to the smallest values of q. In particular, the minimal martingale measure price is larger than the minimal entropy price, which in turn is greater than the variance-optimal price.

Having established the general relationship between option prices and the choice of q in the q-optimal measure in the Heston model, we now investigate numerically the magnitude of the differences in option prices. We also examine the impact of the choice of q on the implied volatilities resulting from this model.

The numerical solutions are generated by solving the Heston pricing pde

$$\begin{aligned} &\dot{C} + C_v \left(\kappa(\bar{m}/v - v) - \rho\beta\alpha_1 v - \bar{\rho}^2\beta^2 v F(T-t)\right) \\ &+\frac{1}{2}(C_{ss}s^2v^2 + C_{vv}\beta^2 + 2C_{sv}sv\beta\rho) = 0 \quad C(T, s, v) = h(s) \end{aligned} \tag{15}$$

using a Crank-Nicolson type finite difference method. We follow Kluge [30] and apply this scheme to the log spot transformed pde ($x := \log s$) which is of convection diffusion type. At the zero volatility boundary, $v = 0$, the diffusion term disappears and only the convection remains. With the restriction $m \geq \beta^2/\kappa$, which implies that the stochastic volatility process does not hit zero, it turns out that this is an outflow boundary. That is why no boundary conditions can be imposed and we use an upwind scheme at this boundary. All other boundaries are artificial due to the fact that the log spot transformed pde lives in $(x, v) \in \mathbb{R} \times \mathbb{R}_+$ and have to be set sufficiently far away from the area of interest. To further reduce the numerical error a non-uniform grid in space and time dimension is used. Numerical accuracy of this scheme has been examined in [30] for $q = 0$ and $\alpha_1 = 0$ and verified against the Heston closed form solution.

Our model parameters are appropriate for the foreign exchange market, although our qualitative conclusions do not depend on such a choice. Melino and Turnbull [34] explore pricing in this market with stochastic volatility, as do Hakala and Wystup [21]. In the foreign exchange market, correlations tend to be small and positive. This differs from the equity market where strong negative correlations are the norm. The leverage effect is often cited to be the cause of this. As the stock price falls, debt to equity ratios rise making the firm riskier and resulting in greater volatility, see Nandi [35].

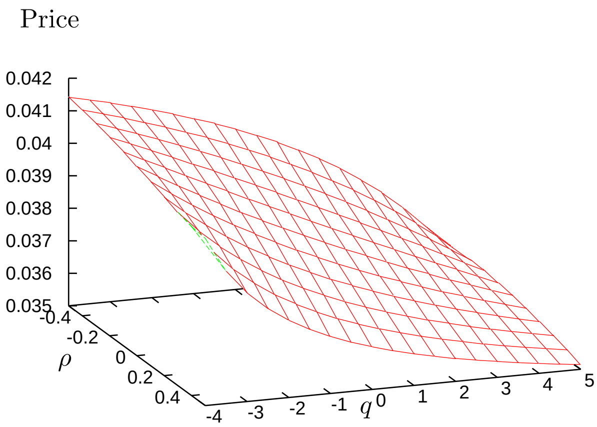

Figure 1: *Price of a 1 year at-the-money put option under the Heston model with the parameters given in the table.*

Parameter		Value
m	long term variance	0.01
κ	rate of mean reversion	1
β	volatility of volatility	0.1
V_0	initial volatility	0.1
α_1	absolute drift on asset $\alpha_1 V^2 S$	4
S_0	initial asset price	1

The parameters used to generate each of the following graphs are given in the above table. We consider a put option with varying strike and maturity. Puts have convex, bounded payoffs, which helps for the calculation of numerical solutions. Put-call parity allows us to infer the prices of calls and since interest rates are zero, the prices of at-the-money puts and calls will be the same.

Figure 1 uses the above parameters values together with strike $K = 1$ and $T = 1$. We plot the put price for $\rho \in [-0.5, 0.5]$ and $q \in [-4, 5]$. Over most of this range $qR > 0$ and the q-optimal measure exists for all time, and even for $q = 5$ and $|\rho| = 0.5$ the q-optimal measure exists up to $T = 1$. As anticipated by Corollary 8, we observe the put price is decreasing in q. The range of the graph represents about a 16% difference in prices between the extreme points.

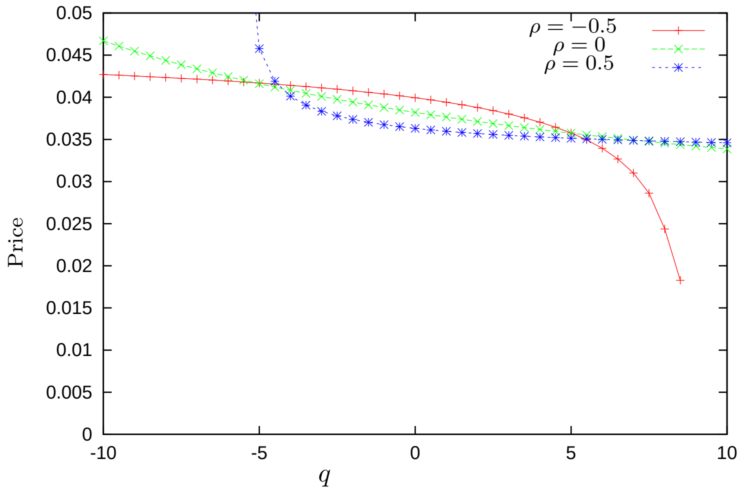

Figure 2: *Price of a 1 year at-the-money put option under the Heston model for various values of q and ρ. Model parameters are given in the table.*

If we examine special cases of moving from say $q = 0$ to $q = 2$, the price change is of the order of a couple of percent, depending on the correlation. This is a non-trivial difference, and highlights the fact that choice of martingale measure or adjustment for risk has an impact on option prices.

In the figure, put option prices are also observed to decrease with correlation. It turns out that this is the case for 'small' q. Note that in the pricing pde (15), there are two drift terms arising from the incompleteness, $\rho\beta\alpha_1 v$ and $\bar{\rho}^2\beta^2 vF(T-t)$. In the small q case, the first of these is dominant. If correlation is negative, this term has a positive effect on the option price, whilst the reverse is true for positive correlation.

As Figure 2 shows, once q is no longer small, the ordering reverses. This is the case as the drift term involving $\bar{\rho}^2 F(T-t)$ becomes dominant. As mentioned earlier, outside the range $qR > 0$, the function F may explode, and the graph shows these effects. For $q = 5$ and $\rho = -0.5$, F explodes to infinity (for sufficiently large T) and prices are small as a result. Similarly, for $q = -5$ and $\rho = 0.5$, F explodes to $-\infty$ and prices are large.

One of the best ways of capturing the effects of a stochastic volatility model is by considering the implied volatilities and the true test of a model is whether

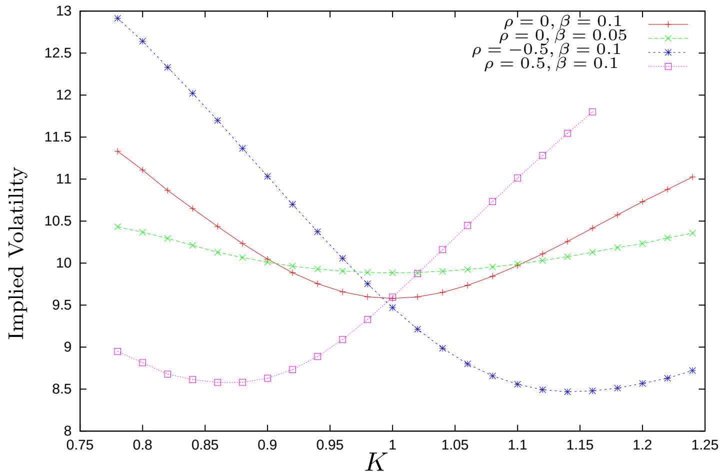

Figure 3: *The effect of correlation and the volatility of volatility on implied volatilities for a 1 year put option priced under the Heston model,* $q = 0, \alpha_1 = 0$.

it can be calibrated well to market data. The Heston [24] model, and stochastic volatility models in general, fit market options prices reasonably well, although they do not perform as well for very short or very long maturities. According to Rubinstein [41], volatility smile patterns and pricing biases are time dependent. Bakshi *et al* [1], Belledin and Schlag [3], Hakala and Wystup [21] and many others observe that smiles are strongest for short term puts and calls, and this is where the fit is least impressive.

By including the adjustment for volatility risk via the q measures, we have a richer class of models which may provide a better fit. We will not focus on calibration here, but rather on the shape of implied volatility smiles from the Heston model under q-optimal measures. This is equivalent to looking at the effect of q on options of differing moneyness.

In Figure 3, we plot the implied volatility from the Heston model against the strike of the option K. The parameters used are those in the table, with the exception that $\alpha_1 = 0, q = 0$ and the option maturity is 1 year. The four smiles correspond to different choices of correlation and volatility of volatility β. When $\rho = 0$, the smile is symmetric about the at-the-money volatility. Increasing β appears to increase the convexity of the smile. Non-zero correlation controls the smile's asymmetry, important in equity markets. Hakala and Wystup [21] document the qualitative effects of changing parameters in the Heston model and note these two effects.

In the final figure we consider the effect of changing the Sharpe ratio (governed by the parameter α_1) and the measure of distance between the real world and candidate pricing measures parameterised by q. We also consider the effects of varying maturity. In each case the implied volatilities are calculated from the Heston model with correlation $\rho = -0.2$. There are two graphs (one for changing the notion of distance between the pricing and real world measures and one for changing the Sharpe ratio) for each of three maturities. The parameter values have been chosen to roughly match the magnitude of these effects, for $T = 1/4$.

The key feature that will aid our understanding is that the market price of volatility risk $\lambda^{(q)}(t, V_t) = \bar{\rho}\beta V_t F(T-t)$ is time-inhomogeneous. Since, for each q, $|F(\tau)|$ is increasing in τ, the effects of changing q will become more pronounced as the maturity increases.

We begin with some generic observations which are typical features of stochastic volatility models. The correlation is negative so the smiles are skewed to the left. As maturity increases, the smile becomes flatter or less convex - beware the change in horizontal scale. By considering the left-hand column we see that as q increases, option prices decrease. This is consistent with Corollary 8. Conversely, in the right-hand column, we see that as α_1 increases, the option price increases. This is a consequence of the drift term appearing in the pde (15). Since $\rho < 0$, the term $-\rho\alpha_1\beta v$ is positive and prices increase as the drift term under the pricing measure increases. (Although this is not shown on the graph, it is possible to find alternative parameter values such that option prices decrease as α_1 increases.) This is because under the pricing measure, the second drift term $-\bar{\rho}^2\beta^2 vF(T-t)$ is negative (for $\rho < 0$ and $q > 0$). The two terms will have competing effects, and the overall effect of a change in α_1 will depend on which term dominates. If F is small (q is small or T small), then the first term dominates and increasing α_1 shifts the smile upward, as we saw in Figure 4. But if F is large then the second term dominates and increasing α_1 will cause the smile to shift down. This depends on correlation, since if correlation is zero, the first term disappears and the smile always shifts down with increased α_1. If correlation is positive, both terms work together to have the same effect.

A final feature of the graphs that we wish to remark on is the relative

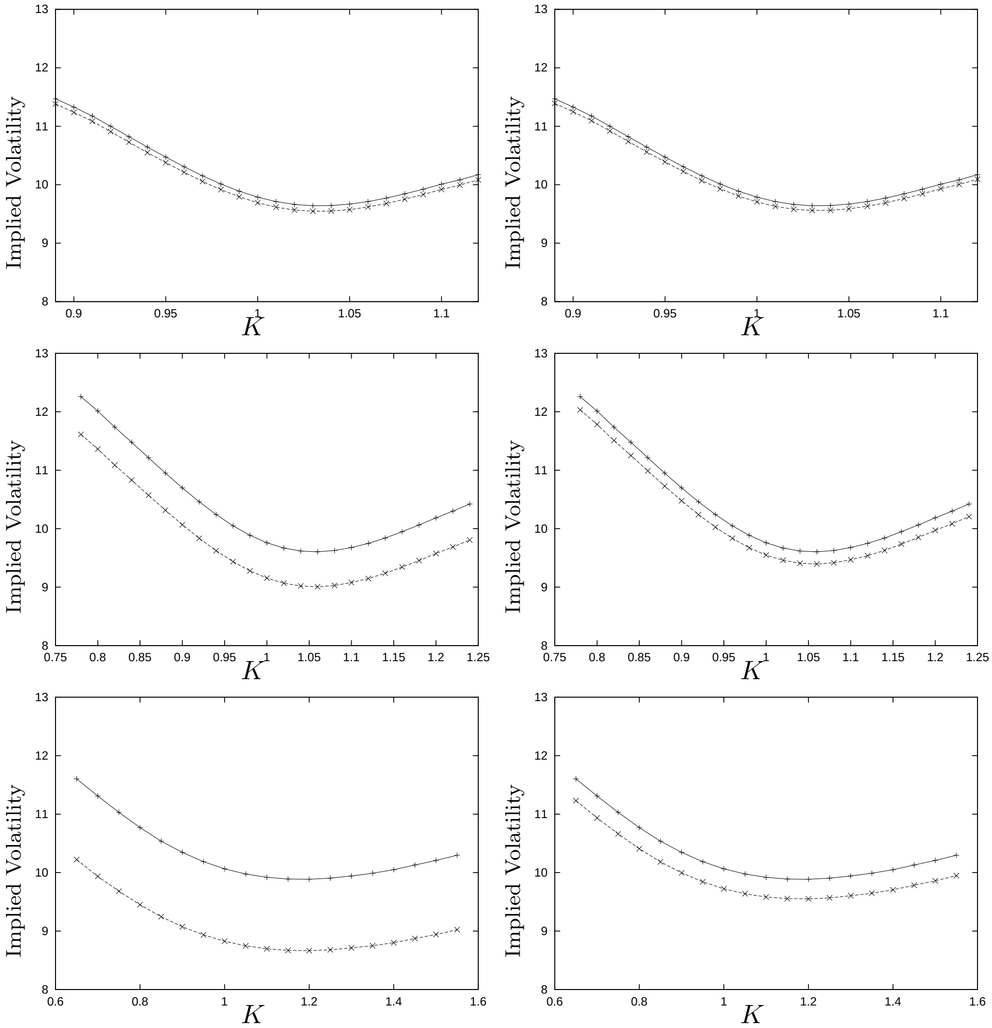

Figure 4: *Implied volatility smiles for the Heston model with $\rho = -0.2$ and $T = 0.25$ (top row), $T = 1$ (middle row) and $T = 4$ (bottom row). The solid line in each graph corresponds to $q = 0, \alpha_1 = 4$. The lower line in the graphs in the left column correspond to a higher value of q ($q = 4, \alpha_1 = 4$) and the lower lines in the graphs on the right correspond to a lower value of α_1 ($q = 0, \alpha_1 = 0$). Note that the horizontal scale changes with maturity (by a factor of $\sqrt{T}$).*

magnitude of the implied volatility shifts as maturity changes. For the graphs in the right column, ($q = 0$ and $\alpha_1 = 0$ or 4) the change in drift induced by the pricing measure is $-\rho\alpha_1\beta v$. The magnitude of the implied volatility changes seems to approximately double each time T increases by a factor of 4. Conversely, on the left hand side, modification to the drift under $\mathbb{Q}^{(q)}$ consists of two terms $-\rho\alpha_1\beta v$ and $-\bar{\rho}^2\beta^2 vF(T-t)$. For $q > 0$, $F(\tau)$ is positive and increasing in τ, so the effect of changing q is comparatively larger when the maturity is large (note that the parameter values have been chosen so when $T = 1/4$ the effect of changing q from 0 to 4 is approximately the same as changing α_1 from 4 to 0).

7 Conclusion

In this paper we have investigated the role of the market price of volatility risk and the choice of pricing measure on the prices of options. In a final section we compared these theoretical results for a general stochastic volatility model to numerical results for the Heston model.

Our first result is that as the market price of risk on the unhedgeable source of randomness Z increases, or equivalently as the market price of volatility risk increases, the prices of European style options with convex payoffs decrease. At least when $\rho = 0$, this result can be seen as an extension to an incomplete market of the standard result that in a complete market (such as the Black Scholes model) option prices are increasing in volatility.

As an application of this result we investigated how the market price of risk changes with q when the pricing measure is chosen to be the q-optimal measure. If the innovations process driving the volatility is independent of that driving the traded asset then we recover the result of Henderson [23] that if (the square of) the Sharpe ratio is increasing in volatility then (European) options with convex payoffs have prices decreasing in q. In the correlated case, the results are less clear, but we are able to show that under the same conditions, the q-optimal price for any positive q is smaller than that for any negative q.

We conjecture that in the correlated case, if $\alpha(t,v)^2$ is increasing in v then option prices are decreasing in q (so that the result for the correlated case is the same as when $\rho = 0$). The evidence for the conjecture is based on the $\rho = 0$ case and also on the analysis of the Heston model. However, this evidence is fairly slim, and it would be extremely interesting if it were possible either to prove the result or to find a counterexample. We hope this motivates the study of other examples, particularly if they lead to further explicit solutions, as this should help aid intuition.

Even when the correlation is zero we can have the following surprising result: namely that as an agent becomes more risk averse (which corresponds to q increasing) q-optimal option prices may go down or up. The precise effect of changing q depends on the Sharpe ratio.

An important observation is that the market price of volatility risk corresponding to the q-optimal measure is time-inhomogeneous. As a consequence, changing q may have different effects for short and long maturity options. This additional feature of the model may provide extra ability for the model to fit market data and extra explanatory power. The issue of calibrating the model and inferring the parameter q or risk aversion of the market is an extremely interesting one, and will be addressed elsewhere.

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