Bounds for Floating-Strike Asian Options using Symmetry [†]

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This paper studies symmetries between fixed and floating-strike Asian options and exploits this symmetry to derive an upper bound for the price of a floating-strike Asian. This bound only involves fixed-strike Asians and vanillas, and can be computed simply given one of the many efficient methods for pricing fixed-strike Asian options. The bound is exact until after the averaging has begun and again at maturity. The bound is compared to benchmark prices obtained via Monte Carlo simulation in numerical examples.

Keywords: Asian options, floating strike Asian options, put call symmetry, bounds, change of numéraire

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1 Introduction

Asian options have a payoff which depends on the average price of the underlying asset during some part of its life. The average is usually arithmetic, and if the asset price is assumed to follow exponential Brownian motion, an explicit option price is not available as the arithmetic average of a set of lognormal distributions is not known explicitly. Instead, pricing of Asian options is usually done numerically.

There are two types of Asian options: the fixed-strike option, where the average relates to the underlying asset and the strike is fixed; and floating strike options where the average relates to the strike price. The fixed-strike option is in some sense easier, and has received most attention in the literature.

The starting point for this paper is a 'symmetry' result in Henderson and Wojakowski [11] (see also Hoogland and Neumann [12]) which proves an equivalence between the price of a floating-strike Asian and the price of a related fixed-strike Asian. In [11] this equivalence is shown to be valid at the start of the averaging period.

In this paper we extend this symmetry result to 'forward starting' Asian options. In particular, for a forward starting floating Asian options, we provide a symmetry with a 'starting' fixed-strike Asian option. In the case where the floating option is 'starting', we recover the special case given in Henderson and Wojakowski [11]. If the option is 'in progress', we show that a floating-strike option can be re-expressed as a generalised 'starting' option but not as any type of fixed-strike option. Instead, we derive an approximate method to price 'in progress' floating-strike options. This approximation is actually an upper bound, and relates the price of a floating-strike Asian option to the sum of the price of a fixed-strike Asian and the price of a vanilla option.

Pricing of the fixed-strike Asian has been the subject of much research over the last ten years and academic interest in these options has experienced a revival recently, see Carr and Schröder [2] and Donati-Martin *et al* [7] continuing the earlier work of Geman and Yor [9]. The current state of the art methods for fixed-strike Asians are numerical inversion of the Laplace transform (Shaw [18], [19]), eigenfunction expansions of Linetsky [13] and the stable pde method of Vecer [22].

The floating-strike Asian option has received far less attention in the literature, perhaps because the problem is more difficult in that the joint law of $\{S_t, A_t\}$ is needed. Chung *et al* [3] and Ritchken *et al* [16] generalise earlier efforts which derive approximations using joint lognormality. A pde approach can also be taken, see Rogers and Shi [17], and Vecer [22] for an excellent new method. Even so, pricing methods for floating strike options are underdeveloped compared with the more established methods for the fixed-strike option. Hence, a method which capitalises on existing algorithms for fixed-strike options to achieve bounds on floating-strike options is useful.

Our upper bound for the price of the floating-strike Asian call option is exact at times up to and including the time the averaging begins and at maturity. Via symmetry, the bound may be expressed as a combination of fixed-strike puts and vanilla call options, optimised over a weighting parameter. One of the main advantages of the bound is that one can capitalise on existing methods to price the fixed-strike option. As such, the speed and accuracy of the method depend on the chosen algorithm to price the fixed-strike component of the bound. We introduce an approximation to choose the weighting parameter optimally, and demonstrate whilst this has little effect on the accuracy, it reduces the computation time dramatically.

Other related work on approximations for Asian option prices includes bounds obtained by conditioning methods by Curran [6] and Rogers and Shi [17], enhancements of the conditioning method by Thompson [20] and Nielsen and Sandmann [15] and adaptation for discrete monitoring by Vanmaele *et al* [21]. The difference between these works and ours is that they obtain an approximate price for the option at the start of the averaging, whilst our method is exact at this time. It is difficult to compare the methods directly, as these papers do not report results during the averaging period.

The main contribution of the paper is an approximation to the price of a floating-strike Asian option which has some desirable properties. Further, the paper builds on symmetry relationships for exotic options and provides an interesting application of such symmetries to pricing.

The paper is structured as follows. The next section outlines the model and defines the floating and fixed-strike Asian option. Section 3 gives some general symmetry results for Asian options and recovers the symmetry found in Henderson and Wojakowski [11] as a special case. The following section derives the upper bound for the price of the floating-strike Asian call. We concentrate in this paper on a bound for the call. Of course, the same method gives a bound for the floating-strike Asian put, this is left to the interested reader. In Section 5 we give an approximate method to reduce the calculation time and report the results of our numerical investigation in Section 6. The final section concludes the paper.

2 The Model

We consider the standard Black Scholes economy with a risky asset (stock) and a money market account. We take as given a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{0\le t\le T_\infty}$, which is right-continuous and such that $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets of $\mathcal{F}$. Here T_∞ is the termination date of our economy, which is certainly greater than the maturity date of any option we might consider. We also assume the existence of a risk-neutral probability measure $\mathbb{Q}$ (equivalent to

$\mathbb{P}$) under which discounted asset prices are martingales, implying no arbitrage. We denote expectation under measure $\mathbb{Q}$ by $\mathbb{E}$, and under $\mathbb{Q}$, the stock price follows

$$\frac{dS_t}{S_t} = (r - \delta)dt + \sigma dW_t \tag{1}$$

where r is the constant continuously compounded interest rate, δ is a continuous dividend yield, σ is the instantaneous volatility of asset return and W is a $\mathbb{Q}$-Brownian motion.

We consider an Asian contract which is based on the value A_T where $(A_t)_{t \geq t_0}$ is the arithmetic average

$$A_t = \frac{1}{t - t_0} \int_{t_0}^{t} S_u du \qquad\qquad t > t_0,$$

and by continuity, we define $A_{t_0} = S_{t_0}$. The contract is written at time 0 (with $0 \leq t_0$) and expires at $T > t_0$. Of interest is to calculate the price of the option at the current time t, where $0 \leq t \leq T$. The position of t compared to the start of the averaging, t_0 may vary. If $t \leq t_0$ the option is 'forward starting'. We will call the special case $t = t_0$ a 'starting' option. If $t > t_0$, the option is termed 'in progress' as the averaging has begun. In this paper we are mainly concerned with in progress options.

We consider a *generalised Asian option* with payoff $(aS_T + bA_T + c)^+$ at time T. The important cases in financial options are

- $(a, b, c) = (0, 1, -K)$ — the fixed-strike Asian call option,
- $(a, b, c) = (0, -1, K)$ — the fixed-strike Asian put,
- $(a, b, c) = (1, -1, 0)$ — the floating-strike Asian call, and
- $(a, b, c) = (-1, 1, 0)$ — the floating-strike Asian put.

Note that vanilla European puts and calls correspond to the choice $b = 0$, $a = \pm 1$ and $c = \pm K$.

By standard arbitrage arguments the time-t price of this generalised option is the discounted expected payoff under $\mathbb{Q}$, and we write

$$V_t(a, b, c; r, \delta; S_t, A_t; t_0, T) = e^{-r(T-t)}\mathbb{E}[(aS_T + bA_T + c)^+ | \mathcal{F}_t].$$

Note that for forward starting options A_t is not well defined and so we write $V_t(a, b, c; r, \delta; S_t, \star; t_0, T)$.

3 Symmetry Results for Asian Options

In this section, we show that the pricing function for the generalised option satisfies certain scaling and symmetry results.

Theorem 1 *(i) V is homogeneous of degree 1 in the parameters a, b, c, so that for $\lambda > 0$,*

$$V_t(\lambda a, \lambda b, \lambda c; r, \delta; S_t, A_t; t_0, T) = \lambda V_t(a, b, c; r, \delta; S_t, A_t; t_0, T).$$

(ii) For an in-progress option ($t > t_0$) we have the identity

$$V_t(a, b, c; r, \delta; S_t, A_t; t_0, T) = V_t\left(a, b\frac{T-t}{T-t_0}, c + b\frac{t-t_0}{T-t_0}A_t; r, \delta; S_t, \star; t, T\right) \tag{2}$$

which allows us to write any generalised in-progress Asian option as a starting Asian option.

(iii) For a starting option we have the symmetry

$$V_{t_0}(a, b, c; r, \delta; S_{t_0}, \star; t_0, T) = V_{t_0}\left(\frac{c}{S_{t_0}}, b, aS_{t_0}; \delta, r; S_{t_0}, \star; t_0, T\right) \tag{3}$$

This is an extension of the result of Henderson and Wojakowski [11]. Note that the roles of r and δ are reversed as well roles of a and c.

(iv) Combining (ii) and (iii) we get for in-progress options

$$V_t(a, b, c; r, \delta; S_t, A_t; t_0, T) = V_t\left(\frac{c}{S_t} + b\frac{t-t_0}{T-t_0}\frac{A_t}{S_t}, b\frac{T-t}{T-t_0}, aS_t; \delta, r; S_t, \star; t, T\right) \tag{4}$$

Proof:
The linearity of the the option pricing function is inherited from the homogeneity of the payoff function: $(\lambda x)^+ = \lambda x^+$ at least for positive λ. The second part is equally trivial and is based on the identity

$$A_T = \frac{t-t_0}{T-t_0}A_t + \frac{T-t}{T-t_0}\frac{1}{T-t}\int_t^T S_u\, du$$

where the first term is $\mathcal{F}_t$ measurable, and the second term is a constant multiplied by the average stock price over the interval $[t, T]$. The final part does indeed follow from earlier parts as indicated, so the main result of this theorem is contained in (*iii*), the proof of which is relegated to the appendix. This proof is an extension of an argument in Henderson and Wojakowski [11] and involves a change of measure and an identification of a time-reversal of a Brownian motion. $\square$

Note that it follows from the first part of the theorem that it is sufficient to consider the cases $b = \pm 1$, together with $b = 0$ which corresponds to vanilla European options. If we take a special case of (*iii*), namely a floating-strike option, we can derive a symmetry which holds whilst the option is forward-starting.

Theorem 2 *For a forward-starting Asian option, $t \leq t_0$ we have*

$$V_t(a, -1, 0; r, \delta; S_t, \star; t_0, T) = V_t(0, -1, aS_t e^{-\delta(t_0 - t)}; \delta, r; S_t e^{-\delta(t_0 - t)}, \star; t, T + t - t_0)$$

In particular a forward starting floating-strike call has the same price as a starting fixed-strike put with r and δ reversing roles and modified maturity. An analogous result holds for $b = 1$, which converts a floating-strike put into a fixed-strike Asian call.

Remark 3 *The special case of this result for a 'starting' option was proved in Henderson and Wojakowski [11] and is given as*

$$V_{t_0}(a, -1, 0; r, \delta; S_{t_0}, \star; t_0, T) = V_{t_0}(0, -1, aS_{t_0}; \delta, r; S_{t_0}, \star; t_0, T)$$

Vanmaele et al [21] also obtain this symmetry result when the average is sampled discretely.

Proof of Theorem 2:

$$V_t(a,-1,0,r,\delta,S_t,\star,t_0,T) = e^{-r(T-t)}\mathbb{E}_t\left[S_{t_0}\left(\frac{aS_T - A_T}{S_{t_0}}\right)^+\right]$$

$$= \quad e^{-r(T-t)}(\mathbb{E}_t S_{t_0})\mathbb{E}_t\left(\frac{aS_T}{S_{t_0}} - \frac{1}{T-t_0}\int_{t_0}^{T}\frac{S_u}{S_{t_0}}du\right)^+$$

where we use the independence of S_{t_0} and increments after t_0. Using a time translation $u \to u-(t_0-t)$ in the second expectation this becomes

$$e^{-r(T-t_0)}S_t e^{-\delta(t_0-t)}\mathbb{E}_t\left(\frac{aS_{T+t-t_0}}{S_t} - \frac{1}{T-t_0}\int_t^{T+t-t_0}\frac{S_u}{S_t}du\right)^+$$

which is

$$V_t(a,-1,0;r,\delta;S_t e^{-\delta(t_0-t)},\star;t,T+t-t_0)$$

a starting floating option (but at time t) with modified maturity. Now applying the symmetry result of Theorem 1[(iii)] for a starting option, we can write this as

$$V_t(0,-1,aS_t e^{-\delta(t_0-t)};\delta,r;S_t e^{-\delta(t_0-t)},\star;t,T+t-t_0)$$

$\square$

Theorems 1 and 2 are useful as they give relationships between various Asian options. The generalised symmetry of Theorem 1[(iii)] can be used to transform starting floating-strike Asians into starting fixed-strike Asians. In addition, a forward starting floating-strike Asian is equivalent to a starting fixed-strike Asian with modified maturity and other parameters, as given in Theorem 2. Any Asian which is in progress may be written as a generalised starting option, as described in (ii).

However, (iv) clarfies that although we can write an in progress Asian (take $a=1, b=-1, c=0$ for call) as a generalised starting Asian, it cannot be simplified further.

Thus, to price a forward starting (and starting) floating-strike call (or put), we can use symmetry and price the equivalent fixed-strike put (or call). If the floating Asian call is in progress however, there is no such symmetry. Instead we derive an upper bound which involves fixed-strike Asian puts and vanilla call options. Similarly we could derive an upper bound for the floating-strike Asian put.

4 An Upper Bound for the Floating-Strike Asian Option

Since the symmetry in Theorem 2 holds only up to and at the moment the averaging begins, we develop an upper bound for the case when the option is in progress. The payoff of a floating-strike Asian call option

$$(S_T - A_T)^+ = \left(S_T - \frac{1}{T-t_0}\int_{t_0}^{T} S_u\, du\right)^+$$

can be rewritten in terms of pre and post-t parts, for $t_0 < t$

$$\left(S_T - \frac{1}{T-t_0}\int_{t_0}^{t} S_u\, du - \frac{1}{T-t_0}\int_{t}^{T} S_u\, du\right)^+. \tag{5}$$

We can use this representation to obtain the following result.

Theorem 4 *For $t \geq t_0$, an upper bound on the price $V_t(1,-1,0;r,\delta;S_t,A_t;t_0,T)$ of an in-progress floating-strike call is given by*

$$\inf_{\alpha}\left\{V_t\left((1-\alpha),0,-\frac{t-t_0}{T-t_0}A_t;r,\delta;S_t,\star;t,T\right) + V_t\left(0,-\frac{T-t}{T-t_0},\alpha S_t;\delta,r;S_t,\star;t,T\right)\right\} \tag{6}$$

Note that we have bounded the floating in-progress option with the notionally simpler fixed-strike Asian put with modified dynamics together with an ordinary European call option.

Corollary 5 *If $t = t_0$ the infimum is attained at $\alpha = 1$. Conversely, if $t = T$ then the infimum is attained at $\alpha = 0$. Further, the bound in (6) gives the exact price for the floating-strike option for times t at both ends of the averaging interval.*

Proof: The upper bound from Theorem 4 gives

$$\begin{aligned}
&V_{t_0}(1,-1,0;r,\delta;S_{t_0},\star;t_0,T)\\
&\quad\leq \inf_{\alpha}\left\{V_{t_0}\left((1-\alpha),0,0;r,\delta;S_{t_0},\star;t_0,T\right) + V_{t_0}(0,-1,\alpha S_{t_0};\delta,r;S_{t_0},\star;t_0,T)\right\}\\
&\quad\leq V_{t_0}\left(0,0,0;r,\delta;S_{t_0},\star;t_0,T\right) + V_{t_0}(0,-1,S_{t_0};\delta,r;S_{t_0},A_{t_0};t_0,T)
\end{aligned} \tag{7}$$

where this last inequality is obtained by replacing the infimum over α with the value when α equals 1. But the first of the two terms in (7) is zero, and by Corollary 2 the second is exactly $V_{t_0}(1,-1,0;r,\delta;S_{t_0},\star;t_0,T)$. Hence both the inequalities are equalities, and the bound is attained when $\alpha = 1$.

Similarly, when $t = T$,

$$\begin{aligned}
&V_T(1,-1,0;r,\delta;S_T,A_T;t_0,T) \\
&\quad\le \inf_{\alpha}\left\{V_T\left((1-\alpha),0,-A_T;r,\delta;S_T,\star;T,T\right) + V_T(0,0,\alpha S_T;\delta,r;S_T,\star;T,T)\right\} \\
&\quad\le V_T\left(1,0,-A_T;r,\delta;S_T,\star;T,T\right) + V_T(0,0,0;\delta,r;S_T,\star;T,T) \\
&\quad= (S_T - A_T)^+
\end{aligned}$$

where again the second inequality is obtained by substituting in a particular value ($\alpha = 0$). But this last expression is exactly the payoff of the floating strike call, so that both the inequalities are equalities, and the bound is attained when $\alpha = 0$. $\square$

Proof of Theorem 4 : Note $(a+b+c) = [(1-\alpha)a+b] + (\alpha a + c)$ and $(x+y)^+ \le x^+ + y^+$. Hence for any α, $(a+b+c)^+ \le ((1-\alpha)a+b)^+ + (\alpha a+c)^+$. Applying these to (5) gives

$$\begin{aligned}
(S_T - A_T)^+ \le \inf_{\alpha}\Bigg\{&\left(S_T(1-\alpha) - \frac{1}{T-t_0}\int_{t_0}^{t} S_u\,du\right)^+ \\
&+ \left(\alpha S_T - \frac{1}{T-t_0}\int_{t}^{T} S_u\,du\right)^+\Bigg\}
\end{aligned}$$

Taking discounted expectations will give an upper bound on the price of a floating-strike Asian call

$$\begin{aligned}
V_t(1,-1,0;r,\delta;S_t,A_t;t_0,T) &= e^{-r(T-t)}\mathbb{E}\left[(S_T-A_T)^+ \middle| \mathcal{F}_t\right] \\
\le \inf_{\alpha}\Bigg\{& e^{-r(T-t)}\mathbb{E}\left[\left(S_T(1-\alpha) - \frac{1}{T-t_0}\int_{t_0}^{t} S_u\,du\right)^+ \middle| \mathcal{F}_t\right] \\
&+ e^{-r(T-t)}\mathbb{E}\left[\left(\alpha S_T - \frac{1}{T-t_0}\int_{t}^{T} S_u\,du\right)^+ \middle| \mathcal{F}_t\right]\Bigg\}. \qquad (8)
\end{aligned}$$

The first term is a call option and can be rewritten as

$$V_t\left(1-\alpha, 0, -\frac{t-t_0}{T-t_0}A_t; r,\delta;S_t,\star;t,T\right)$$

Further, by Theorem 1[(iv)] the second term can be re-expressed as a fixed-strike Asian put:

$$V_t\left(\alpha, -\frac{T-t}{T-t_0}, 0; r,\delta;S_t,\star;t,T\right) = V_t\left(0, -\frac{T-t}{T-t_0}, \alpha S_t; \delta, r; S_t,\star;t,T\right).$$

□

We have managed to construct a bound for floating-strike Asians which depends only on vanilla options and fixed-strike Asians. The fixed-strike Asian option has been well studied. Competing methods include integral formulas of Linetsky [13], inversion of Laplace transform of Geman and Yor [9] (implemented by Shaw [18], [19]), and the stable pde method of Vecer [22]. Each of these methods was shown to give six digit precision by Vecer [23] and Shaw [19]. Thus, given any of these (or another) method for pricing fixed-strike Asian options, the bound can be calculated without any new algorithms. The bound requires optimizing over the parameter α. This potentially means many calls to a routine to price the fixed-strike put need to be made. It is therefore vital to choose a fast (and accurate) method for pricing the fixed-strike put, as the speed and accuracy of the bound depend on this.

However, if we could avoid this optimization over α by choosing an approximate value which achieved very similar accuracy, this would speed up the computation, as only a single call to the fixed-strike pricing routine would be needed. In the next section we give a method for speeding up the calculation by deriving an approximation to the optimal choice of α for the upper bound.

Note that in this section we have concentrated on the pricing of the 'in progress' floating-strike Asian and we have not mentioned hedging at all. However, implicit in the analysis is a simple super-replicating hedge. At time $t \in [t_0, T]$, choose the optimal $\alpha = \alpha(t)$ (or indeed any α) and decompose the floating-strike option into a combination of a vanilla and a fixed-strike Asian option. The vanilla can be replicated in the standard fashion. If the fixed-strike Asian is hedged approximately, then we have a super-hedge for the floating-strike Asian option.

5 Optimal Choice of the parameter α for the upper bound

The purpose of this section is to find an efficient method to choose a value of the parameter α to give a good approximate upper bound in Theorem 4. Recall that if $t = t_0$, so we are in the case of a starting option, then the optimal α is given by $\alpha = 1$. We consider 'in-progress' options, so that $t_0 < t < T$, and we will make a series of approximations and assumptions to derive a suitable choice of α. Our linearizing of exponential terms is similar to that used in the pricing approximation in Chung *et al* [3] and Bouaziz *et al* [1].

We begin by recalling

$$(S_T - A_T)^+ \le \left((1-\alpha)S_T - \frac{t-t_0}{T-t_0}A_t\right)^+ + \left(\alpha S_T - \frac{1}{T-t_0}\int_t^T S_u du\right)^+$$

Note that, for $u \ge v$, $S_u = S_v \exp\{\sigma(W_u - W_v) + (r - \delta - \sigma^2/2)(u-v)\}$.

Assumption 6 *$r-\delta$ and σ^2 are small, or more precisely $(r-\delta)(T-t_0)$ and $\sigma^2(T-t_0)$ are small.*

Under this assumption, for $u \ge t$, we can approximate S_u by $S_t(1+\sigma(W_u - W_t)) = S_t(1+\sigma\int_t^u dW_v)$, and then, with $\doteq$ denoting approximately equal

$$\begin{aligned}
&(1-\alpha)S_T - \frac{t-t_0}{T-t_0}A_t \doteq (1-\alpha)S_t - \frac{t-t_0}{T-t_0}A_t + S_t(1-\alpha)\sigma\int_t^T dW_v \\
&\sim N\left((1-\alpha)S_t - \frac{t-t_0}{T-t_0}A_t;\ S_t^2(1-\alpha)^2\sigma^2(T-t)\right).
\end{aligned} \tag{9}$$

Also

$$\frac{1}{T-t_0}\int_t^T S_u du \doteq \frac{(T-t)}{T-t_0}S_t + \frac{\sigma S_t}{T-t_0}\int_t^T (W_u - W_t)du.$$

and hence

$$\begin{aligned}
&\alpha S_T - \frac{1}{T-t_0}\int_t^T S_u du \\
&\doteq \alpha S_t + \alpha S_t\sigma\int_t^T dW_t - \frac{(T-t)}{T-t_0}S_t - \frac{\sigma S_t}{T-t_0}\int_t^T (T-u)dW_u \\
&= \left(\alpha - \frac{T-t}{T-t_0}\right)S_t + S_t\sigma\int_t^T\left(\alpha - \frac{T-u}{T-t_0}\right)dW_u \\
&\sim N\left(\left(\alpha - \frac{T-t}{T-t_0}\right)S_t\ ;\ \frac{T-t_0}{3}S_t^2\sigma^2\left[\alpha^3 - \left(\alpha - \frac{T-t}{T-t_0}\right)^3\right]\right).
\end{aligned} \tag{10}$$

Note that the covariance of the terms in (9) and (10) is

$$\begin{aligned}
&S_t^2\sigma^2(1-\alpha)\int_t^T\left(\alpha - \frac{T-u}{T-t_0}\right)du \\
&= (T-t_0)S_t^2\sigma^2\frac{(1-\alpha)}{2}\left[\alpha^2 - \left(\alpha - \frac{T-t}{T-t_0}\right)^2\right]
\end{aligned} \tag{11}$$

Let G_1 and G_2 be normal random variables with distributions

$$\begin{aligned} G_1 &\sim N\left((1-\alpha)S_t - \frac{t-t_0}{T-t_0}A_t;\ S_t^2(1-\alpha)^2\sigma^2(T-t)\right) \\ G_2 &\sim N\left(\left(\alpha - \frac{T-t}{T-t_0}\right)S_t\ ;\ \frac{T-t_0}{3}S_t^2\sigma^2\left[\alpha^3 - \left(\alpha - \frac{T-t}{T-t_0}\right)^3\right]\right) \end{aligned}$$

and covariance as given in (11).

We are assuming that σ is small. If in fact if σ is zero then $A_t \doteq S_t$ and the variance of both G_1 and G_2 is zero. In that case it is optimal to take $\alpha \equiv (T-t)/(T-t_0)$ as then both G_1 and G_2 are identically zero.

More generally, our goal is to minimise $G_1^+ + G_2^+$. Note that G_1+G_2 has mean $(S_t - A_t)(t-t_0)/(T-t_0)$ which is independent of α. We can imagine choosing α to distribute this mean between the two variables G_i. The proportion of this quantity that we assign to each normally distributed random variable should depend on their respective variances.

In particular we consider α of the form

$$\alpha = \frac{T-t}{T-t_0} + \gamma\left\{\frac{t-t_0}{T-t_0}\left(\frac{S_t - A_t}{S_t}\right)\right\}.$$

Assumption 7 *To leading order $\alpha = (T-t)/(T-t_0)$. Further, when we substitute α in to the expressions for the variances of G_1 and G_2 we can neglect higher order terms.*

Note that under the approximations σ is small and $S_t \doteq S_{t_0}(1+\sigma(W_t - W_{t_0}))$ we find that $(S_t - A_t)/S_t$ is approximately mean zero with variance $\sigma^2(t-t_0)/3$ so that the first part of this latest assumption follows from Assumption 6.

For this α, and using the leading order expression for the variances we have

$$\begin{aligned} G_1 &\sim N\left((1-\gamma)\frac{t-t_0}{T-t_0}(S_t - A_t);\ S_t^2\sigma^2(T-t)\frac{(t-t_0)^2}{(T-t_0)^2}\right) \\ G_2 &\sim N\left(\gamma\frac{t-t_0}{T-t_0}(S_t - A_t)\ ;\ S_t^2\sigma^2\frac{1}{3}\frac{(T-t)^3}{(T-t_0)^2}\right) \end{aligned}$$

Note also that the ratio of the standard deviations is given by:

$$\sqrt{\mathrm{Var}(G_1)} : \sqrt{\mathrm{Var}(G_2)} = \sqrt{3}(t-t_0) : (T-t)$$

We choose γ such that the ratio of the means is equal to the ratio of the standard deviations, so

$$(1-\gamma) : \gamma = \sqrt{3}(t-t_0) : (T-t)$$

and hence $\gamma = (T-t)/(T-t+\sqrt{3}(t-t_0))$. This choice can be justified rigorously if G_1 and G_2 are uncorrelated (whereas in fact they have correlation $\sqrt{3}/2$), and if the means are large in comparison with the standard deviations, but remains a plausible choice in many circumstances.

In conclusion, the proposed choice of α is

$$\hat{\alpha} = \frac{T-t}{T-t_0} + \frac{t-t_0}{T-t_0}\left(\frac{T-t}{T-t+\sqrt{3}(t-t_0)}\right)\left(\frac{S_t - A_t}{S_t}\right).$$

To summarize, this approximation will be better when $(r-\delta)(T-t_0)$ and $\sigma^2(T-t_0)$ are small. We test this approximation in the next section, where both the full α minimisation in (6) and this approximate $\hat{\alpha}$ are used in some examples.

6 Implementation and Results

We implement the upper bound using Laplace transform inversion methods for the fixed-strike option. Shaw [18] performs the Laplace transform inversion by direct numerical integration along the truncated Bromwich contour. This contour is a vertical line to the right of any finite singularities, and the truncation can be adjusted to obtain higher accuracy. He has improved upon this implementation in Shaw [19] by transforming the hypergeometric function into a collection of geometric series using Mellin transforms. This improved the computation time dramatically, especially for low volatility examples. We have employed this improved method in our calculation of the fixed-strike options.

The advantages of this choice are that it is accurate, reasonably fast, and it is relatively easy to code (approximately fourteen lines in Mathematica). A further advantage from a coding perspective is that we can then calculate the bound in Mathematica using its inbuilt optimisation routines. In this implementation, we also utilised put-call parity for floating-strike Asians.

We will test our bound against a benchmark price for the floating-strike Asian option, which we calculate by Monte Carlo simulation, with variance reduction techniques. Monte Carlo and Quasi Monte Carlo simulation are used extensively in finance to obtain benchmark prices, see Corwin *et al* [5] and Fu *et al* [8]. Our random numbers were generated using a twisted GFSR (see Matsumoto and Nishimura [14]). Fu *et al* [8] find that for fixed-strike Asians, the continuous geometric average Asian served as a high quality control variate. We take the same approach and follow Conze and Visvanathan [4] to derive a formula for the floating-strike geometric average call option and use this as a control variate. For each example following, our calculated simulation prices at time 0 agreed (up to reported accuracy) with those in Chung *et al* [3].

Our first example uses parameters $t = t_0 = 0$, $T = 1$, $r = 0.1, q = 0, \sigma = 0.5, S_t = 100$ and A_t is either 90, 100 or 110. The second example uses the same parameters but reduces volatility to 0.3. Both are ‘starting’ options. Figures 1 and 2 plot the upper bound (C^u calculated using Theorem 4 and optimising over α) and the benchmark prices (C) over the 1 year life of the option, for each value of A_t. It can be seen the bound is exact at time 0 and at maturity. If we were to compute the value for a forward starting option, the bound would simply be exact up to and including the time the averaging began. More detail is given in Tables 1 and 2. Simulation benchmark values C with standard deviation δ and the upper bound C^u with optimised value of α are reported. The remaining two columns contain the price bound $\hat{C}^u$ calculated using the approximation $\hat{\alpha}$ given in Section 5.

Our first comparison is between C, the benchmark and C^u the upper bound. For $t = 0$, the option is ‘starting’ and the upper bound should be exact, at least theoretically. We see that 3 or 4 digits of accuracy are obtained between the simulation and the bound, due to errors inherent in the numerical estimation of each. This represents around a 0.02 % error. We can use this as a guide for evaluating the errors over the life of the option, since this is a ‘base‘ error we are starting with. For volatility 0.5 (Table 1), the accuracy is reduced to 2 digits when $t > 0$ and 1 or 2 if the volatility is 0.3 (Table 2). As a percentage (when volatility is 0.5), errors range from about 0.1 % to 3.5 %, for the worst out-of-the-money option with $t = 0.4$. These are slightly better for the lower volatility case. If we compare these errors to a 1 % misspecification in volatility, we find the bound in Table 2 is less than the simulated price C with volatility 0.31.

These calculations are time consuming, however. For example, when volatility is 0.5, and $A_t = 90$, it took between 63 and 90 seconds to compute C^u for various points in time, t, for a truncation of 10000 in Shaw’s implementation. These times can be reduced dramatically with little loss of accuracy by using $\hat{C}^u$ and $\hat{\alpha}$, the approximation to α. Under both volatilities, the difference in accuracy between $\hat{C}^u$ and C^u is insignificant, although $\hat{\alpha}$ is closer to α when volatility is 0.3, as expected. Recall $\hat{\alpha}$ does not depend on volatility. Using the approximation for α reduced the computation time for the $A_t = 90, \sigma = 0.5$ example to around a twelfth of the times reported earlier. For example, when $t = 0.2$, the time reduces from 75.6 to 6.3 seconds. Thus the approximation method retains virtually the same accuracy as the full optimized upper bound, but for a fraction of the computation time.

7 Conclusion

This paper has explored symmetries in Asian option pricing and exploited such relationships to derive a new approximation to the price of a floating-strike Asian. This approximation takes the form of a one-sided bound on the true price. The

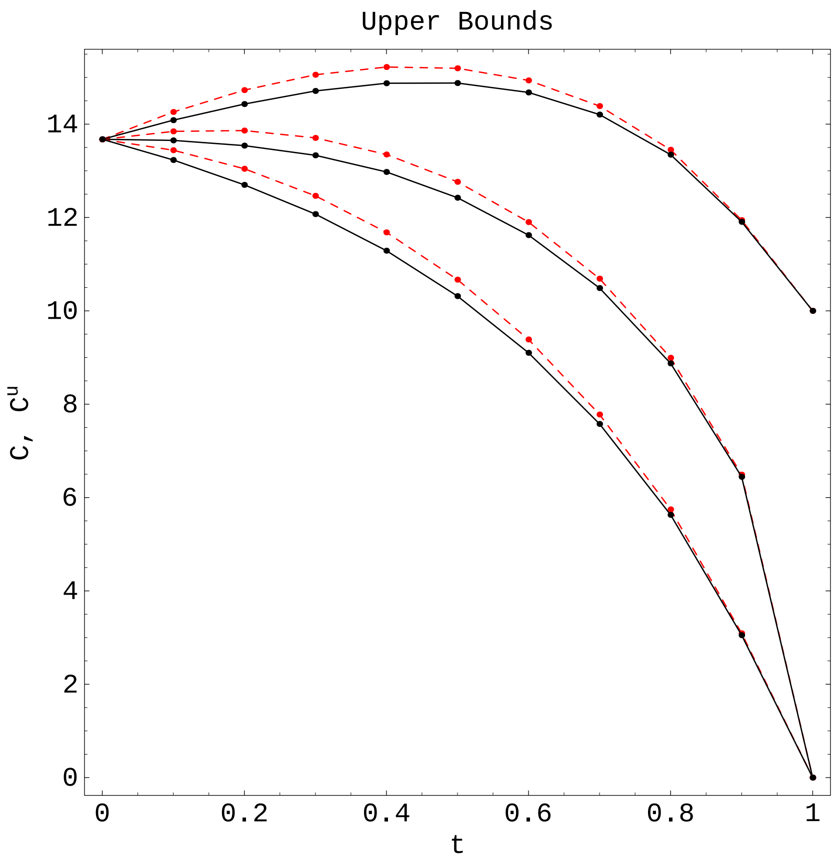

Figure 1: *Upper Bounds* C^u (dashed lines with dots) vs optimized control variate Monte-Carlo estimates of the arithmetic Asian option price C (solid lines with dots). Parameters are $S_t = 100$, $\sigma = 0.5$, $r = 0.1$, $q = 0$, $t_0 = 0$, $T = 1$. The 'highest' case is when $A_t = 90$ and either bounds or both prices reach the payoff $(S_t - A_t)^+ = 10$ for $t = T = 1$. The 'middle' and 'lowest' cases arise when $A_t = 100$ and $A_t = 110$ respectively: bounds and prices reach $(S_t - A_t)^+ = 0$ then.

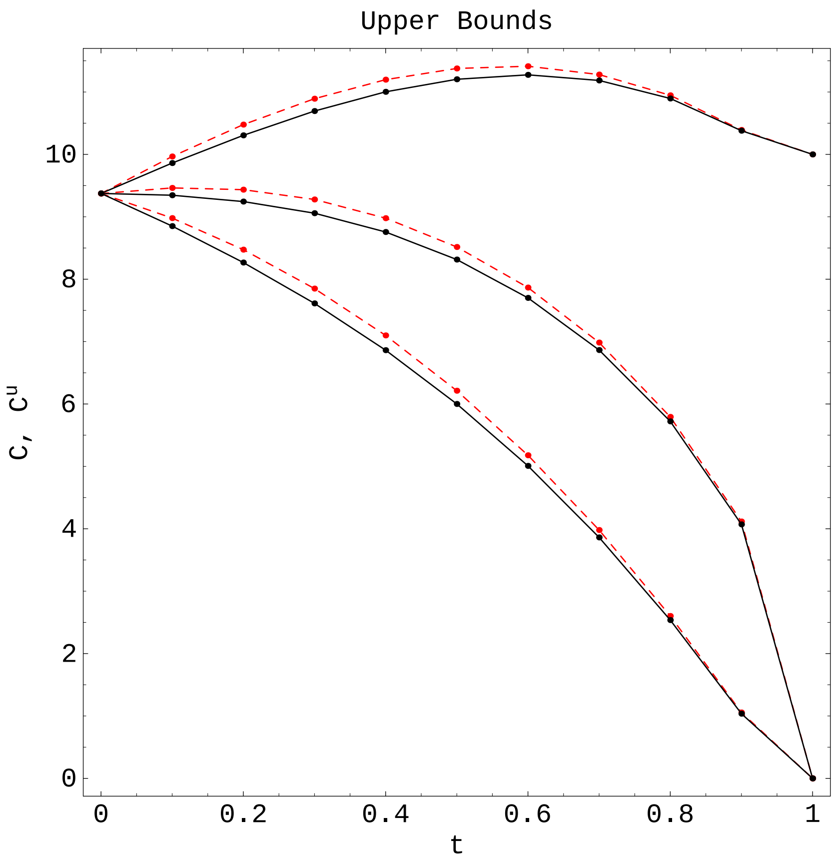

Figure 2: *Upper Bounds* C^u (dashed lines with dots) vs optimized control variate Monte-Carlo estimates of the arithmetic Asian option price C (solid lines with dots). Parameters are $S_t = 100$, $\sigma = 0.3$, $r = 0.1$, $q = 0$, $t_0 = 0$, $T = 1$. The 'highest' case is when $A_t = 90$ and either bounds or both prices reach the payoff $(S_t - A_t)^+ = 10$ for $t = T = 1$. The 'middle' and 'lowest' cases arise when $A_t = 100$ and $A_t = 110$ respectively: bounds and prices reach $(S_t - A_t)^+ = 0$ then.

bound depends only on fixed-strike Asians and vanilla options. Given an efficient method for pricing a fixed-strike Asian option, and our approximation for the optimal weights of the fixed-strike and vanilla, the bound can be calculated immediately in one extra line of code. The accuracy and speed of computation of the bound depends on the choice of algorithm for the fixed-strike option. Using Shaw's [19] implementation, calculations took a few seconds to give prices to within a couple of percent of the Monte Carlo simulation. This approximation may serve as a simple check of the price of a floating-strike Asian option, before a more complex and accurate solution is implemented, and is particularly accurate near the beginning and end of the averaging period.

Ongoing research of the authors includes lower bounds via symmetry, and possible extensions to American style payoffs, see Hansen and Jorgensen [10].

8 Appendix

8.1 Proof of Theorem 1[(iii)]

This proof is an extension of the argument used in Henderson and Wojakowski [11, Theorem 1]. Assume that S has the dynamics given by (1) and that the option is 'starting', so $t = t_0$. We begin by rewriting the price of the generalised Asian option as:

$$
\begin{aligned}
&V_{t_0}(a, b, c; r, \delta; S_{t_0}, \star; t_0, T) = e^{-r(T-t_0)}\mathbb{E}[(aS_T + bA_T + c)^+ | \mathcal{F}_{t_0}] \\
&= S_{t_0} e^{-\delta(T-t_0)}\mathbb{E}_{t_0}\left[\frac{S_T e^{-(r-\delta)(T-t_0)}}{S_{t_0}} \frac{(aS_T + bA_T + c)^+}{S_T}\right]
\end{aligned}
$$

Define the measure $\hat{\mathbb{Q}}$ via

$$
\frac{d\hat{\mathbb{Q}}}{d\mathbb{Q}} = \frac{S_T e^{-(r-\delta)(T-t_0)}}{S_{t_0}} = \exp\left\{\sigma(W_T - W_{t_0}) - \frac{\sigma^2}{2}(T - t_0)\right\}.
$$

Under $\hat{\mathbb{Q}}$, $\hat{W}_u = W_u - \sigma u$ is a Brownian motion and the price becomes

$$
V_{t_0} = S_{t_0} e^{-\delta(T-t_0)}\hat{\mathbb{E}}_{t_0}\left[\left(a + b\frac{A_T}{S_T} + \frac{c}{S_T}\right)^+\right]. \tag{12}
$$

Again under $\hat{\mathbb{Q}}$ we have $dS_u = S_u\{(r - \delta + \sigma^2)du + \sigma d\hat{W}\}$ and for $u \geq t_0$,

$$
\frac{S_u}{S_T} = \exp\left\{\left(r - \delta + \frac{1}{2}\sigma^2\right)(u - T) + \sigma(\hat{W}_u - \hat{W}_T)\right\}.
$$

Now let $(\hat{B}_u)_{t_0 \le u \le T}$ be defined via $\hat{B}_u = \hat{B}_{t_0} + \hat{W}_{T+t_0-u} - \hat{W}_T$ for some constant $\hat{B}_{t_0}$. Then $\hat{B}$ is a time-reversal of a Brownian motion and therefore $\hat{B}$ is again a Brownian motion under $\hat{\mathbb{Q}}$. Also

$$\frac{c}{S_T} = \frac{c}{S_{t_0}} \frac{S_{t_0}}{S_T} = \frac{c}{S_{t_0}} e^{(\delta-r)(T-t_0)} \exp\left\{ \sigma(\hat{B}_T - \hat{B}_{t_0}) - \frac{1}{2}\sigma^2 (T-t_0) \right\} = \frac{c}{S_{t_0}} \frac{\hat{S}_T}{\hat{S}_{t_0}},$$

where $\hat{S}$ solves the stochastic differential equation

$$\frac{d\hat{S}_u}{\hat{S}_u} = (\delta - r)du + \sigma d\hat{B}_u \qquad u \ge t_0 \tag{13}$$

with $\hat{S}_{t_0} \equiv S_{t_0}$. In particular we think of $\hat{S}$ as a stock paying constant rate of dividends r in a market with interest rate δ. Further

$$\begin{aligned}
(T-t_0)\frac{A_T}{S_T} &= \int_{t_0}^T \frac{S_u}{S_T} du = \int_{t_0}^T du \ \exp\left\{ \left(r - \delta + \frac{1}{2}\sigma^2 \right)(u-T) + \sigma(\hat{W}_u - \hat{W}_T) \right\} \\
&= \int_{t_0}^T du \ e^{(\delta-r)(T-u)} \exp\left\{ \sigma(\hat{B}_{T+t_0-u} - \hat{B}_{t_0}) - \frac{1}{2}\sigma^2(T-u) \right\} \\
&= \int_{t_0}^T dv \ e^{(\delta-r)(v-t_0)} \exp\left\{ \sigma(\hat{B}_v - \hat{B}_{t_0}) - \frac{1}{2}\sigma^2(v-t_0) \right\} \\
&= \int_{t_0}^T dv \ \frac{\hat{S}_v}{\hat{S}_{t_0}}.
\end{aligned}$$

We have that

$$S_{t_0} \frac{(aS_T + bA_T + c)^+}{S_T} = \left(\frac{c}{\hat{S}_{t_0}} \hat{S}_T + b \frac{1}{T-t_0} \int_{t_0}^T dv \hat{S}_v + a\hat{S}_{t_0} \right)^+$$

and under $\hat{\mathbb{Q}}$, this last term is the payoff of a generalised Asian option under the dynamics (13). Finally, if we discount this expression at the interest rate δ we get from (12)

$$V_{t_0}(a, b, c; r, \delta; S_{t_0}, \star; t_0, T) = V_{t_0}\left(\frac{c}{S_{t_0}}, b, aS_{t_0}; \delta, r; S_{t_0}, \star; t_0, T \right)$$

as required. □

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A_t	t	C_g	C	δ	C^u	α	$\hat{C}^u$	$\hat{\alpha}$
–	0	14.8329	13.6756	0.00436	13.6729	–	13.6729	1.
90	0.1	15.3068	14.0854	0.00449	14.2607	0.91211	14.2677	0.90839
	0.2	15.6529	14.4306	0.00441	14.7292	0.81985	14.74	0.81396
	0.3	15.8499	14.7114	0.00393	15.0582	0.724	15.0702	0.71722
	0.4	15.8765	14.8765	0.00334	15.224	0.62525	15.2353	0.61856
	0.5	15.7078	14.8806	0.00264	15.197	0.5242	15.2065	0.5183
	0.6	15.3105	14.6794	0.00189	14.9375	0.42136	14.9447	0.41668
	0.7	14.6338	14.205	0.00122	14.3875	0.31717	14.3922	0.31388
	0.8	13.5896	13.3462	0.00062	13.4512	0.21201	13.4538	0.21009
	0.9	12.	11.9074	0.0002	11.9435	0.1062	11.9445	0.10543
	1	10	10	–	10	–	10	0
–	0	14.8329	13.6756	0.00436	13.6729	–	13.6729	1.
100	0.1	14.7731	13.6521	0.00423	13.8452	0.90388	13.8525	0.9
	0.2	14.5808	13.54	0.00396	13.8625	0.80577	13.8727	0.8
	0.3	14.2363	13.3326	0.00337	13.7055	0.70625	13.7159	0.7
	0.4	13.7166	12.976	0.00276	13.3502	0.60579	13.3591	0.6
	0.5	12.9918	12.4235	0.0021	12.7643	0.50476	12.771	0.5
	0.6	12.019	11.6225	0.00143	11.9019	0.40347	11.9063	0.4
	0.7	10.7285	10.4886	0.00087	10.69	0.30217	10.6924	0.3
	0.8	8.9891	8.8749	0.0004	8.9949	0.20105	8.9959	0.2
	0.9	6.4745	6.4442	0.0001	6.49	0.10028	6.4902	0.1
	1	0	0	–	0	–	0	0
–	0	14.8329	13.6756	0.00436	13.6729	–	13.6729	1.
110	0.1	14.2977	13.2319	0.00402	13.4418	0.89589	13.4501	0.89161
	0.2	13.6413	12.6993	0.00361	13.0442	0.79246	13.0565	0.78604
	0.3	12.8466	12.0728	0.00295	12.4643	0.68984	12.4776	0.68278
	0.4	11.8926	11.2887	0.00233	11.6814	0.58814	11.6937	0.58144
	0.5	10.7517	10.3158	0.00171	10.6685	0.48742	10.6787	0.4817
	0.6	9.3853	9.1014	0.00111	9.3872	0.38774	9.3947	0.38332
	0.7	7.7337	7.577	0.00063	7.7793	0.28913	7.7842	0.28612
	0.8	5.6952	5.6295	0.00027	5.7464	0.19163	5.7491	0.18991
	0.9	3.0649	3.051	0.00006	3.0916	0.09525	3.0926	0.09457
	1	0	0	–	0	–	0	0

Table 1: *Upper Bounds* for $S_t = 100$, $\sigma = 0.5$, $r = 0.1$, $q = 0$, $t_0 = 0$, $T = 1$. In the table, A_t is the arithmetic average realized up to time t, C_g is the price at time t of an otherwise identical geometric average option, C is an optimized control variate Monte-Carlo estimate of the arithmetic Asian price, with $N = 100000$ simulated paths and $m = 3000$ sampling points, δ is the standard deviation of the Monte-Carlo estimate, C^u is the upper bound computed by numerically minimizing over α, $\hat{C}^u$ is the approximate upper bound for approximate $\hat{\alpha}$. Note that the average A_t is unknown if $t = t_0 = 0$ and that at this time point, as well as at $t = T = 1$, both bounds $C^u, \hat{C}^u$ are 'exact'.

A_t	t	C_g	C	δ	C^u	α	$\hat{C}^u$	$\hat{\alpha}$
–	0	9.8676	9.3741	0.00159	9.3725	–	9.3725	1.
90	0.1	10.4256	9.8614	0.00174	9.9666	0.91042	9.97	0.90839
	0.2	10.8995	10.3054	0.00177	10.4776	0.81726	10.483	0.81396
	0.3	11.2724	10.6948	0.00164	10.8925	0.72112	10.8988	0.71722
	0.4	11.5293	11.0028	0.00144	11.1978	0.62253	11.2039	0.61856
	0.5	11.655	11.2041	0.00116	11.3775	0.52193	11.383	0.5183
	0.6	11.6326	11.2749	0.00085	11.4128	0.41968	11.4172	0.41668
	0.7	11.4408	11.1851	0.00056	11.2793	0.31611	11.2825	0.31388
	0.8	11.0517	10.8962	0.0003	10.9466	0.2115	10.9485	0.21009
	0.9	10.4468	10.3797	0.0001	10.3915	0.1071	10.3968	0.10543
	1	10	10	–	10	–	10	0
–	0	9.8676	9.3741	0.00159	9.3725	–	9.3725	1.
100	0.1	9.8245	9.3457	0.00156	9.4627	0.90205	9.4659	0.9
	0.2	9.686	9.2441	0.00146	9.4348	0.80304	9.4394	0.8
	0.3	9.4384	9.0576	0.00125	9.2778	0.70329	9.2825	0.7
	0.4	9.0662	8.7564	0.00103	8.9778	0.60304	8.9818	0.6
	0.5	8.5501	8.3147	0.00078	8.5161	0.5025	8.5191	0.5
	0.6	7.8629	7.7003	0.00054	7.8659	0.40182	7.8679	0.4
	0.7	6.9617	6.8647	0.00032	6.9845	0.30114	6.9856	0.3
	0.8	5.7668	5.7215	0.00015	5.793	0.20055	5.7935	0.2
	0.9	4.0824	4.0707	0.00004	4.1165	0.10021	4.1167	0.1
	1	0	0	–	0	–	0	0
–	0	9.8676	9.3741	0.00159	9.3725	–	9.3725	1.
110	0.1	9.2954	8.8508	0.00143	8.979	0.89393	8.983	0.89161
	0.2	8.6481	8.267	0.00124	8.4735	0.78961	8.4798	0.78604
	0.3	7.9171	7.6121	0.00101	7.8495	0.68681	7.8567	0.68278
	0.4	7.0922	6.8623	0.00078	7.0992	0.58537	7.1063	0.58144
	0.5	6.1611	6.0006	0.00056	6.2128	0.48517	6.2191	0.4817
	0.6	5.1084	5.0082	0.00036	5.1782	0.38612	5.1833	0.38332
	0.7	3.9149	3.8619	0.0002	3.9802	0.28813	3.9838	0.28612
	0.8	2.5583	2.537	0.00009	2.6021	0.19115	2.6043	0.18991
	0.9	1.0412	1.0369	0.00002	1.0557	0.09408	1.0564	0.09457
	1	0	0	–	0	–	0	0

Table 2: *Upper Bounds* for $S_t = 100$, $\sigma = 0.3$, $r = 0.1$, $q = 0$, $t_0 = 0$, $T = 1$. In the table, A_t is the arithmetic average realized up to time t, C_g is the price at time t of an otherwise identical geometric average option, C is an optimized control variate Monte-Carlo estimate of the arithmetic Asian price, with $N = 100000$ simulated paths and $m = 3000$ sampling points, δ is the standard deviation of the Monte-Carlo estimate, C^u is the upper bound computed by numerically minimizing over α, $\hat{C}^u$ is the approximate upper bound for approximate $\hat{\alpha}$. Note that the average A_t is unknown if $t = t_0 = 0$ and that at this time point, as well as at $t = T = 1$, both bounds $C^u, \hat{C}^u$ are 'exact'.