# On the pricing and hedging of volatility derivatives

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**Abstract**

We consider the pricing of a range of volatility derivatives, including volatility and variance swaps. Under risk-neutral valuation we provide closed form formulae for volatility-average and variance swaps for a variety of diffusion and jump-diffusion models for volatility. We describe a general partial differential equation framework for derivatives that have an extra dependence on an average of the volatility. We give approximate solutions of this equation for volatility products written on assets for which the volatility process fluctuates on a time-scale that is fast compared with the lifetime of the contracts, analysing both the "outer" region and, by matched asymptotic expansions, the "inner" boundary layer near expiry.

## 1 Introduction

In this paper we discuss derivative products that provide exposure to the realised volatilities or variances of asset returns (or covariances between asset returns), while avoiding direct exposure to the underlying assets themselves. These products are attractive to investors who either wish to hedge volatility risk or who wish to take a view on future realised volatilities. Indeed, much of the investor interest in volatility products seems to have been provided by the LTCM collapse in 1998, which was accompanied by a dramatic increase in volatilities. As a result, a number of recent papers [3, 7, 8, 11, 14] address the evaluation of volatility products; see also Chapter 13 of [16].

Like several of these authors, we take a stochastic volatility model as our starting point; we also provide formulae for the case that the volatility follows a jump-diffusion process of the type described in [17]. The fact that stochastic volatility models are able to fit skews and smiles, while simultaneously providing sensible Greeks, have made these models a popular choice in the pricing of exotic options.

Under this framework, we present a number of formulae for the "fair" delivery price for volatility and variance swaps, and show how other related contracts can be priced. In addition to providing formulae for a range of volatility and variance swaps, we consider an asymptotic analysis under which we derive approximate solutions for volatility and variance swaptions (options on the realised average volatility or variance). The main motivation is the empirical evidence that volatility is mean-reverting over a time-scale which is fast compared with the typical lifetime of options and other contracts. That is, when considering the time-scale of months, stock and index volatility is observed to fluctuate rapidly, see for example the discussion in [9] or [18] and references therein. In this way, we show how to calculate accurate approximations to the price both at $\mathcal{O}(1)$ times before expiry and in a temporal boundary layer near expiry.

In what follows, the asset $S$ is assumed to follow the usual log-normal process

$$\frac{dS_t}{S_t} = \mu_t\left(t, \cdots\right) dt + \sigma_t\left(t, \cdots\right) dW_t, \tag{1}$$

where $W_t$ is Brownian motion. For the rest of the paper we fix notation as follows: the conditional expectation at time $t$ is denoted by $\mathbb{E}_t = \mathbb{E}[\cdot|\mathcal{F}_t]$ where $\mathcal{F}_t$ is the filtration up to time $t$ and $\mathbb{E}_0$ is thus the initial value of the expectation. All expectations are considered with respect to the risk-neutral probability measure. Within a stochastic volatility framework, the market is typically incomplete, which accounts for an infinite number of equivalent martingale measures. This is to say, the market price of volatility risk is not unique, and it is an open question how does one chooses in a optimal way the appropriate measure to price derivatives. The view we take in the present investigation, which is the standard machinery for many practical purposes, is that the risk-neutral probability measure is chosen by the market and this has a number of immediate implications on calibration issues, mainly that parameter estimation is not possible from stock data. Except from a brief discussion in §4.6 we do not further address these issues here.

The rest of the paper is organized as follows: we begin section §2 by briefly discussing the contracts, while the stochastic volatility framework is introduced in §3, and the general pricing equation is given in §3.3. In §4 we present in depth the asymptotic analysis which concludes with second order approximations for the volatility derivatives of interest. In §4.5 we concentrate on pure volatility products, in which case the analysis greatly simplifies. Then we give examples in §5. We conclude in §6.

# 2 Variance and volatility swaps

The variance swap is a forward contract in which the investor who is long pays a fixed amount $K^{var}$/\$1 nominal value at expiry and receives the floating amount $v_R$/\$1 nominal value, where $K^{var}$ is the strike and $v_R = (\sigma^2)_R$, where $\sigma$ is the volatility, is the realized variance. The entering price must be zero, that is, it costs nothing to enter the contract; we use this condition to find the fair value $K^{var}$. The measure of realized variance to be used is defined at the beginning of the contract; a typical formula for it is

$$\frac{1}{T}\sum_{i=1}^{M}\Big(\frac{S_i - S_{i-1}}{S_{i-1}}\Big)^2,$$

which in continuous time we approximate by

$$v_R = \frac{1}{T}\int_0^T \sigma_t^2 dt.$$

The corresponding payoff is then

$$v_R - K^{var}. \tag{2}$$

It is also possible to construct contracts on the realised volatility. One measure of this is

$$\Big(\frac{1}{T}\sum_{i=1}^{M}\Big(\frac{S_i - S_{i-1}}{S_{i-1}}\Big)^2\Big)^{\frac{1}{2}}$$

derived from the standard deviation of the asset price random walk, and the corresponding continuous-time payoff for a volatility swap is

$$(v_R)^{1/2} - K^{s/d} = \Big(\frac{1}{T}\int_0^T \sigma_t^2 dt\Big)^{\frac{1}{2}} - K^{s/d}; \tag{3}$$

we term this contract a standard deviation swap. However, this is not the only possible measure of realised volatility. As discussed in [1], a more robust measure is

$$\sqrt{\frac{\pi}{2MT}}\sum_{i=1}^{M}\Big|\frac{S_i - S_{i-1}}{S_{i-1}}\Big|, \tag{4}$$

and the associated continuous-time contract has payoff

$$\sigma_R - K^{\textit{vol-ave}} = \frac{1}{T}\int_0^T \sigma_t \, dt - K^{\textit{vol-ave}}, \tag{5}$$

which involves the average of realised volatility, rather than the square root of the average realised variance as in (3). We term this contract a volatility-average swap. In addition, we shall consider products based on an average of a suitable implied volatility, for example the implied volatility $\sigma_t^i$ of the at-the-money call options with the same expiry as the volatility derivative; this implied volatility swap has continuous-time payoff

$$\sigma_R^i - K^{\textit{i-vol}} = \frac{1}{T}\int_0^T \sigma_t^i \, dt - K^{\textit{i-vol}}. \tag{6}$$

We could also use a single option throughout the life of the contract, for example the option that is initially at-the-money; we could further construct implied variance swaps, and so on.

Generalizing further, we consider variance and volatility swaptions [16], a typical payoff being

$$\max(\sigma_R - K, 0), \qquad \max(K - \sigma_R, 0)$$

for volatility call and put swaptions, or

$$\max(v_R - K, 0), \qquad \max(K - v_R, 0)$$

for variance swaptions. We can also contemplate contracts whose payoff depends on both a realised volatility or variance and the asset; for example, the payoff

$$\max(Se^{(\sigma_0 - \sigma_R)\sqrt{T}} - K, 0)$$

is a call option which pays more if the asset rises steadily without much volatility than if it rises in a volatile way; here $\sigma_0$ is a reference volatility and $T$ is the contract's lifetime.

# 3 Risk-neutral pricing techniques

Working within the standard risk-neutral pricing framework, we now outline three approaches to the valuation of volatility derivatives. In the case of volatility or variance swaps, this shows that we need to choose

$$K^{var} = \mathbb{E}[v_R] = \mathbb{E}_0\Big[\frac{1}{T}\int_0^T \sigma_t^2 dt\Big],$$

and for a standard-deviation swap we need $K^{s/d} = \mathbb{E}[v_R^{1/2}]$ (which may be less easy to calculate explicitly; an approximation in terms of higher moments of $v_t$ is given in [3, 14]). For the volatility-average swap, we have

$$K^{vol-ave} = \mathbb{E}[\sigma_R] = \mathbb{E}_0\Big[\frac{1}{T}\int_0^T \sigma_t dt\Big]$$

and similarly, for the more complicated implied volatility average swap. Volatility swaptions and so forth are priced in the usual manner; for example, the price of a volatility swaption is

$$\mathbb{E}_0\Big[\max\Big(\frac{1}{T}\int_0^T \sigma_t dt - K, 0\Big)\Big].$$

There are, however, various approaches to the calculation of these prices, which we now consider.

## 3.1 Pricing independently of the volatility model

As observed by [7], it is possible to derive risk-neutral prices for average-variance products without making any assumption on the evolution of the volatility, although the asset is still considered to follow the risk-neutral geometric Brownian motion

$$\frac{dS_t}{S_t} = r\,dt + \sigma_t\,dW_t.$$

Since

$$d(\log S_t) = (r - \frac{1}{2}\sigma_t^2)dt + \sigma_t dW_t,$$

we have that

$$\log S_T - \log S_0 = \int_0^T (r - \frac{1}{2}\sigma_t^2)dt + \int_0^T \sigma_t dW_t.$$

Taking expectations to get risk-neutral prices, we have

$$\frac{1}{2}\mathbb{E}_0\left[\int_0^T \sigma_t^2 dt\right] = rT + \log S_0 - \mathbb{E}_0\left[\log S_T\right],$$

and the final term on the right-hand side is the value of a log-contract, which can be decomposed into a strip of call and put options in a standard way. Hence variance swaps can be easily priced in terms of vanilla options. However, this method does not allow us to compute the Greeks, nor does it give straightforward explicit formulae.

## 3.2 Pricing by expectations in a stochastic volatility framework

It is possible to derive values for the quantities

$$\mathbb{E}[v_R] = \mathbb{E}_0\left[\frac{1}{T}\int_0^T \sigma_t^2 dt\right] = \frac{1}{T}\int_0^T \mathbb{E}_0[\sigma_t^2]dt$$

and

$$\mathbb{E}[\sigma_R] = \mathbb{E}_0\left[\frac{1}{T}\int_0^T \sigma_t dt\right] = \frac{1}{T}\int_0^T \mathbb{E}_0[\sigma_t]dt$$

when either the volatility $\sigma_t$ or the variance $v_t$ follows a quite general random walk. In this way, we can immediately give risk-neutral prices for variance and volatility-average swaps, although standard-deviation swaps are less straightforward in this framework.

In Appendix 1 we show how to calculate $\mathbb{E}[v_R]$ and $\mathbb{E}[\sigma_R]$ when $\sigma_t$ follows the (risk-adjusted) process

$$d\sigma_t = (a_1 + a_2\sigma_t)dt + (a_3 + a_4\sigma_t)dW_t + (a_5 + a_6\sigma_t)dN_t, \tag{7}$$

where $N_t$ is a standard compound Poisson process with constant intensity $\lambda$, and $a_1, \cdots, a_6$ are constants.[1] We also calculate $\mathbb{E}_0[v_R]$ (but not $\mathbb{E}_0[\sigma_R]$) when $v_t$, instead of $\sigma_t$, follows a similar process; lastly we calculate $\mathbb{E}_0[v_R]$ and $\mathbb{E}_0[\sigma_R]$ for the process

$$d\sigma_t = (b_1 + b_2\sigma_t)dt + b_3\sigma_t^{1/2}dW_t \tag{8}$$

[1] We expect $a_1 > 0$, $a_2 < 0$, to model mean-reversion, and we note that for certain choices of the parameters this model allows negative values of $\sigma_t$. Models of volatility with jumps have been considered by [17].

and, as above, $\mathbb{E}_0[v_R]$ if $v_t$ follows a similar process. Specific results for the mean-reverting log-normal process

$$d\sigma_t = \alpha(\bar{\sigma} - \sigma_t)dt + \beta\sigma_t dW_t, \tag{9}$$

for constants $\alpha$, $\bar{\sigma}$ and $\beta$, are reported in [12]. These, and the more general formulae of the Appendix, serve to check the asymptotic approach developed in the following sections.

This approach can also be used to calculate prices and hedge ratios at intermediate times. For example, if we have a volatility-average swap, with payoff

$$\frac{1}{T}\int_0^T \sigma_s^2 ds - K^{vol-ave},$$

the value at earlier times $t$ $(0 \le t < T)$ is

$$\begin{aligned} e^{-r(T-t)}\left(\mathbb{E}_t\left[\frac{1}{T}\int_0^T \sigma_s ds\right] - K^{vol-ave}\right) \\ = \frac{e^{-r(T-t)}}{T}\left(\int_0^t \sigma_s ds + \mathbb{E}_t\left[\int_t^T \sigma_s ds\right] - TK^{vol-ave}\right) \end{aligned}$$

since the contribution $\int_0^t \sigma_s ds$ to the final average of the volatility is known at time $t$. Using the formulae of Appendix 1, the conditional expectation

$$\mathbb{E}_t\left[\int_t^T \sigma_s ds\right] = F(\sigma_t, t),$$

say, is readily evaluated, and then the Vega of the contract is

$$\frac{1}{T}e^{-r(T-t)}\frac{\partial F}{\partial \sigma} \tag{10}$$

which is also readily evaluated.

### 3.2.1 Examples

Here we briefly give some examples for the price of volatility products, the derivation of which is based on the results of the Appendix. The mean-reverting log-normal model

$$d\sigma_t = \alpha(\bar{\sigma} - \sigma_t)dt + \beta\sigma_t dW_t,$$

is a special case of (7), and following the calculation of §A1, under this model, we have

$$K^{vol-ave} = \frac{1 - e^{-\alpha T}}{\alpha T}(\sigma_0 - \bar{\sigma}) + \bar{\sigma}. \tag{11}$$

Similarly, the value at any time $t \le T$ is

$$V_t = \frac{e^{-r(T-t)}}{T}\left\{\int_0^t \sigma_s ds - t\bar{\sigma} + \frac{1}{\alpha}(e^{-\alpha(T-t)} - 1)(\sigma_0 - \sigma_t)\right\}. \tag{12}$$

In addition, for the fair strike of the variance swap we have

$$
\begin{aligned}
K^{var} &= \frac{2\alpha\overline{\sigma}^2}{(2\alpha-\beta^2)T}\Big(T - \frac{1-e^{-(2\alpha-\beta^2)T}}{2\alpha-\beta^2}\Big) \\
&+ \frac{2\alpha\overline{\sigma}(\sigma_0-\overline{\sigma})}{(\alpha-\beta^2)T}\Big(\frac{1-e^{-\alpha T}}{\alpha} - \frac{1-e^{-(2\alpha-\beta^2)T}}{2\alpha-\beta^2}\Big) \\
&+ \frac{\sigma_0^2}{(2\alpha-\beta^2)T}\Big(1-e^{-(2\alpha-\beta^2)T}\Big).
\end{aligned} \tag{13}
$$

The formula for the price of the contract at time $t$ is

$$
\begin{aligned}
V_t = e^{-r(T-t)}\Big\{\frac{1}{T}\int_0^t \sigma_s^2 ds &+ \frac{2\alpha\overline{\sigma}^2}{(2\alpha-\beta^2)T}\Big((T-t) - \frac{1-e^{-(2\alpha-\beta^2)(T-t)}}{2\alpha-\beta^2}\Big) \\
&+ \frac{2\alpha\overline{\sigma}(\sigma_t-\overline{\sigma})}{(\alpha-\beta^2)T}\Big(\frac{1-e^{-\alpha(T-t)}}{\alpha} - \frac{1-e^{-(2\alpha-\beta^2)(T-t)}}{2\alpha-\beta^2}\Big) \\
&+ \frac{\sigma_t^2}{2\alpha-\beta^2}\Big(1-e^{-(2\alpha-\beta^2)(T-t)}\Big) - K^{var}\Big\},
\end{aligned} \tag{14}
$$

with $K^{var}$ given in (13).

Note that if we were to consider the variance $v_t$ as the underlying process satisfying the mean reverting log-normal model, then the relevant expressions for the variance swap would be different. We can illustrate the contrast between volatility and variance driven models by considering the two cases

$$
d\sigma = -k_1(\sigma_t - \sigma_\infty)dt + k_2\sqrt{\sigma}dW_t,
$$

and

$$
dv_t = -k_3\left(v_t - v_\infty\right)dt + k_4\sqrt{v_t}dW_t.
$$

Then, following the calculations in §A2, we have for the volatility-average swap

$$
K^{vol-ave} = \frac{1}{T}\left(\frac{-\sigma_0}{k_1}\left(e^{-k_1T}-1\right) + \sigma_\infty\left(-k_1T - e^{-k_1T}+1\right)\right),
$$

in the first case, and we cannot price this contract explicitly in the second framework. However, the price of the fair variance strike is

$$
K^{var} = \frac{1}{T}\left(-\frac{\delta_1 T}{2b_2} - \frac{\delta_2}{b_2^2}\left(e^{b_2T}-1\right) + \frac{1}{2b_2}\left(\sigma_0^2 + \frac{\delta_2}{b_2} + \frac{\delta_1}{2b_2}\right)\left(e^{2b_2T}-1\right)\right),
$$

for the first model and

$$
K^{var} = \frac{1}{T}\left(-\frac{\sigma_0^2}{k_3}\left(e^{-k_3T}-1\right) + v_t\left(k_3T - e^{-k_3T}+1\right)\right)
$$

for the second; the various constants are as defined in the Appendix. It is apparent that the second model, for $v_t$ rather than $\sigma_t$, leads to considerably simpler formulae.

### 3.3 Pricing via partial differential equations

Explicit formulae are in general only available for pure volatility products, and under the assumption that the coefficients in the process for volatility are independent of $S_t$. For more general cases, we must use either Monte-Carlo methods or numerical/asymptotic solutions of the pricing differential equation. In this section we consider the latter.

From now on we assume that it is $\sigma_t$ that drives the volatility; if instead the underlying process is written in terms of $v_t = \sigma_t^2$ the appropriate modifications are easily made. We assume that $S_t$ and $\sigma_t$ follow the process

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma_t\, dW_t,$$

$$d\sigma_t = M_t dt + \Sigma_t d\widetilde{W}_t \tag{15}$$

where $M_t$ and $\Sigma_t$ may depend on $S_t$ and $\sigma_t$, and where the correlation coefficient of $W_t$ and $\widetilde{W}_t$ is $\rho_t$. As with Asian options, we need to introduce a variable to measure the average volatility to date. The payoff of the derivatives under consideration involves an average of the form

$$I_T = \int_0^T F(\sigma_s) ds;$$

for example,

$$I_T^{var} = \int_0^T \sigma_s^2 ds$$

for a variance swap, for which the payoff (2) is

$$\frac{1}{T} I_T^{var} - K^{var},$$

and the same average is sufficient for the standard-deviation swap payoff (3), namely

$$\left(\frac{1}{T} I_T^{var}\right)^{1/2} - K^{s/d}.$$

Likewise the volatility-average swap and the implied-volatility swap have payoffs (5) (6), namely

$$\frac{1}{T} I_T^{vol-ave} - K^{vol-ave}, \qquad \frac{1}{T} I_T^{i-vol} - K^{i-vol}$$

respectively, where $I_T^{vol-ave} = \int_0^T \sigma_s ds$, $I_t^{i-vol} = \int_0^T \sigma_s^i ds$, and so forth. For times before expiry, we use the running average (denoted generally by $I_t$ regardless of $F$)

$$I_t = \int_0^t F(\sigma_s) ds,$$

and then it is a standard combination of stochastic volatility and Asian options analysis [9, 20] to show that the value $V(S, \sigma, I, t)$ of a derivative satisfies the partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho S \sigma \Sigma \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2}\Sigma^2 \frac{\partial^2 V}{\partial \sigma^2} + F(\sigma)\frac{\partial V}{\partial I} + rS\frac{\partial V}{\partial S} + (M - \Lambda\Sigma)\frac{\partial V}{\partial \sigma} - rV = 0, \qquad (16)$$

where $\Lambda$ is the market price of volatility risk, and we write the general payoff condition in the form

$$V(S, \sigma, I, T) = P(S, I). \qquad (17)$$

It is straightforward to show that the formulae derived in §3 and the Appendix satisfy this equation when $\Lambda = 0$. For example for the mean-reverting log-normal model (9) we have

$$M = \alpha(\bar{\sigma} - \sigma_t), \qquad \Sigma = \beta\sigma_t.$$

and expressions (12), (14) satisfy equation (16).

# 4 Asymptotic analysis for fast mean-reversion

We now present an asymptotic analysis for the case of fast mean-reversion for $\sigma_t$, in the spirit of the Fouque et al. (FPS) analysis [9] for equity and fixed-income derivatives. The novelty here is first in the application to volatility products, and secondly in that we provide a fairly complete description of the solution both at $\mathcal{O}(1)$ times before expiry which FPS do, and in the short boundary layer immediately before expiry which they do not. We initially make the simplifying assumptions that $M_t$ and $\Sigma_t$, the coefficients in the process for $\sigma_t$, are independent of $S_t$, and we let $\rho_t = 0$, $\Lambda_t = 0$. The assumption of zero correlation is not as dramatic as it would be for equity derivatives, and indeed if we consider payoffs depending only on volatility averages it is irrelevant. We indicate briefly at the end of this section how the analysis should be extended for nonzero correlation.

We introduce a small parameter $\epsilon$, where $0 < \epsilon \ll 1$, to measure the ratio of the mean-reversion time-scale to the lifetime of an option, and we assume that $M_t$ and $\Sigma_t$ have the forms

$$M_t = \frac{m_t}{\epsilon}, \qquad \Sigma_t = \frac{\varsigma_t}{\epsilon^{1/2}},$$

the relative sizes of these coefficients being chosen to ensure that $\sigma_t$ has a nontrivial invariant distribution

$$p_\infty(\sigma) = \lim_{t\to\infty} p(\sigma, t|\sigma_0, 0)$$

where $p(\sigma, t|\sigma_0, 0)$ is the transition density function for $\sigma_t$ starting from $\sigma_0$ at time zero.

With these assumptions the pricing equation becomes

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2\epsilon}\varsigma^2 \frac{\partial^2 V}{\partial \sigma^2} + rS\frac{\partial V}{\partial S} + \frac{m}{\epsilon}\frac{\partial V}{\partial \sigma} + F(\sigma)\frac{\partial V}{\partial I} - rV = 0,$$

which we write in the form

$$\Big(\frac{1}{\epsilon}\mathcal{L}_0 + \mathcal{L}_1\Big)V = 0$$

where

$$\mathcal{L}_0 \;=\; \frac{1}{2}\varsigma^2\frac{\partial^2}{\partial\sigma^2} + m\frac{\partial}{\partial\sigma},$$

$$\mathcal{L}_1 \;=\; \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2}{\partial S^2} + rS\frac{\partial}{\partial S} + F(\sigma)\frac{\partial}{\partial I} - r.$$

We note immediately that $p_\infty(\sigma)$ satisfies

$$\mathcal{L}_0^* p_\infty = \frac{\partial}{\partial\sigma^2}\left(\frac{1}{2}\varsigma^2 p_\infty\right) - \frac{\partial}{\partial\sigma}\left(m p_\infty\right) = 0$$

where $\mathcal{L}_0^*$ is the adjoint of $\mathcal{L}_0$, and we assume that $\varsigma^2$, $m$ are such that $p_\infty$ exists; it is then proportional to $e^{-2\int^\sigma m(s)/\varsigma^2(s)ds}/\varsigma^2(\sigma)$. We introduce the notation $\langle\cdot,\cdot\rangle$ for the usual inner product over $0<\sigma<\infty$, and note the identity

$$\langle\mathcal{L}_0 u, v\rangle = -\langle u, \mathcal{L}_0^* v\rangle \tag{18}$$

for suitable functions $u$ and $v$, which we will use repeatedly.

We now expand

$$V(S,\sigma,I,t) \sim V_0(S,\sigma,I,t) + \epsilon V_1(S,\sigma,I,t) + \epsilon^2 V_2(S,\sigma,I,t) + \cdots \qquad ,$$

so that

$$\frac{1}{\epsilon}\mathcal{L}_0 V_0 + (\mathcal{L}_1 V_0 + \mathcal{L}_0 V_1) + \epsilon\left(\mathcal{L}_1 V_1 + \mathcal{L}_0 V_2\right) + \cdots = 0.$$

At lowest order, $\mathcal{O}(1/\epsilon)$, we have

$$\mathcal{L}_0 V_0 = 0$$

and so

$$V_0 = V_0(S,t,I)$$

since the operator $\mathcal{L}_0$ consists of derivatives with respect to $\sigma$ only; however $V_0$ is as yet undetermined.

At $\mathcal{O}(1)$ we find

$$\mathcal{L}_0 V_1 + \mathcal{L}_1 V_0 = 0, \tag{19}$$

a Poisson equation for $V_1$ considering $V_0$ as known. Multiplying by $p_\infty$, integrating and using (18), we find that the solvability (Fredholm Alternative) condition for this equation can be expressed as

$$\langle\mathcal{L}_1 V_0, p_\infty\rangle = 0.$$

That is, $\mathcal{L}_1 V_0$ is orthogonal to $p_\infty$, which, being a solution of the stationary forward Kolmogorov equation for $\sigma$, is an eigenfunction of the adjoint of $\mathcal{L}_0$. Thus,

$$\frac{\partial V_0}{\partial t} + \frac{1}{2}S^2\frac{\partial^2 V_0}{\partial S^2}\int_0^\infty p_\infty(\sigma)\sigma^2 d\sigma + rS\frac{\partial V_0}{\partial S} + \frac{\partial V_0}{\partial I}\int_0^\infty p_\infty(\sigma)F(\sigma)d\sigma - rV_0 = 0,$$

where we have used the result that $V_0$ is independent of $\sigma$. Denoting the integrals by $\overline{\sigma^2}$ and $\overline{F} = \overline{F(\sigma)}$ respectively to represent the fact that they are averages of $\sigma^2$ and $F(\sigma)$, we have

$$\overline{\mathcal{L}}_1 V_0 = \frac{\partial V_0}{\partial t} + \frac{1}{2}\overline{\sigma^2}S^2\frac{\partial^2 V_0}{\partial S^2} + rS\frac{\partial V_0}{\partial S} + \overline{F}\frac{\partial V_0}{\partial I} - rV_0 = 0, \tag{20}$$

(here $\overline{\mathcal{L}}_1 = \langle \mathcal{L}_1, p_\infty \rangle$). Making the transformation

$$\overline{I} = I + (T-t)\overline{F}$$

and writing $V_0(S,t,I) = \overline{V}_0(S,t;\overline{I})$, reduces this further, to

$$\frac{\partial \overline{V}_0}{\partial t} + \frac{1}{2}\overline{\sigma^2}S^2\frac{\partial^2 \overline{V}_0}{\partial S^2} + rS\frac{\partial \overline{V}_0}{\partial S} - r\overline{V}_0 = 0, \tag{21}$$

which is the Black–Scholes equation with volatility $\left(\overline{\sigma^2}\right)^{1/2}$. The dependence on $I$ is retained parametrically via the payoff, which takes the form

$$\overline{V_0}(S,T;\overline{I}) = V_0(S,T,I) = P(S,I),$$

and so we have $V_0(S,t,I) = \overline{V}_0(S,t,I+(T-t)\overline{F})$. Of course, this formula is considerably simpler for pure volatility products, for which $\partial P/\partial S = 0$ so that $\partial V/\partial S = 0$.

The next step is to calculate $V_1$. First, observe that $\mathcal{L}_1 V_0$ can be written as

$$\mathcal{L}_1 V_0 = \frac{1}{2}(\sigma^2 - \overline{\sigma^2})S^2\frac{\partial^2 V_0}{\partial S^2} + (F(\sigma) - \overline{F})\frac{\partial V_0}{\partial I}.$$

Hence, (19) can be written as

$$\mathcal{L}_0 V_1 = \frac{1}{2}(\overline{\sigma^2} - \sigma^2)S^2\frac{\partial^2 V_0}{\partial S^2} + (\overline{F} - F(\sigma))\frac{\partial V_0}{\partial I}.$$

We seek a solution of the form

$$V_1(S,t,\sigma,I) = f_2(\sigma)S^2\frac{\partial^2 V_0}{\partial S^2} + f_1(\sigma)\frac{\partial V_0}{\partial I} + H(S,t,I), \tag{22}$$

where $H$ is independent of $\sigma$. The functions $f_1$ and $f_2$ are then the appropriate solutions of the equations

$$\begin{aligned}
\frac{1}{2}\varsigma^2(\sigma)\frac{d^2 f_2}{d\sigma^2} + m(\sigma)\frac{df_2}{d\sigma} &= \frac{1}{2}(\overline{\sigma^2} - \sigma^2), \\
\frac{1}{2}\varsigma^2(\sigma)\frac{d^2 f_1}{d\sigma^2} + m(\sigma)\frac{df_1}{d\sigma} &= \overline{F} - F(\sigma).
\end{aligned} \tag{23}$$

and can readily be found in integral form; one of the complementary solutions is a constant and can be absorbed into $H$, and the other is unbounded at infinity (because $p_\infty$ exists). However, the function $H(S,t,I)$ can only be determined by proceeding to next order and applying the solvability condition to the equation

$$\mathcal{L}_0 V_2 + \mathcal{L}_1 V_1 = 0. \tag{24}$$

We further see that the solution (22), which depends on $\sigma$, cannot satisfy the payoff condition $V_1(S,\sigma,I,T) = 0$. This discrepancy is resolved by a boundary layer analysis in which $T - t = \mathcal{O}(\epsilon)$ (if the payoff has discontinuities, as for a volatility option, further local analysis near these points is also necessary). We point out that the lowest-order analysis is quite general, and not specific to the random walk (15). Only at higher order do we need to know more about these details, and even then only certain moments need be calculated.

Now, the solvability condition for (24) takes the form:

$$\langle \mathcal{L}_1 V_1, p_\infty \rangle = 0$$

or

$$\langle \mathcal{L}_1 \left( f_2(\sigma) S^2 \frac{\partial^2 V_0}{\partial S^2} \right), p_\infty \rangle + \langle \mathcal{L}_1 \left( f_1(\sigma) \frac{\partial V_0}{\partial I} \right), p_\infty \rangle + \langle \mathcal{L}_1 H, p_\infty \rangle = 0,$$

that is,

$$\langle \mathcal{L}_1 \left( f_2(\sigma) S^2 \frac{\partial^2 V_0}{\partial S^2} \right), p_\infty \rangle + \langle \mathcal{L}_1 \left( f_1(\sigma) \frac{\partial V_0}{\partial I} \right), p_\infty \rangle + \overline{\mathcal{L}}_1 H = 0, \tag{25}$$

where $\overline{\mathcal{L}}_1$ is as in (20). Before solving (25), we note that since $\overline{\mathcal{L}}_1 V_0 = 0$, we also have

$$\overline{\mathcal{L}}_1 \left( S^n \frac{\partial^n V_0}{\partial S^n} \right) = 0, \qquad n \geq 1, \qquad \overline{\mathcal{L}}_1 \left( \frac{\partial V_0}{\partial I} \right) = 0. \tag{26}$$

We can therefore write the first term in (25) as

$$\begin{aligned} \langle \mathcal{L}_1 \left( f_2(\sigma) S^2 \frac{\partial^2 V_0}{\partial S^2} \right), p_\infty \rangle &= \frac{1}{2} S^2 \frac{\partial^2}{\partial S^2} \left( S^2 \frac{\partial^2 V_0}{\partial S^2} \right) \left( \overline{f_2(\sigma)\sigma^2} - \overline{\sigma^2}\; \overline{f_2(\sigma)} \right) \\ &+ \frac{\partial}{\partial I} \left( S^2 \frac{\partial^2 V_0}{\partial S^2} \right) \left( \overline{f_2(\sigma) F(\sigma)} - \overline{F(\sigma)}\; \overline{f_2(\sigma)} \right). \end{aligned} \tag{27}$$

The second term, after some algebra, takes the form

$$\begin{aligned} \langle \mathcal{L}_1 \left( f_1(\sigma) \frac{\partial V_0}{\partial I} \right), p_\infty \rangle &= \frac{1}{2} S^2 \frac{\partial^3 V_0}{\partial I \partial S^2} \left( \overline{f_1(\sigma)\sigma^2} - \overline{f_1(\sigma)}\; \overline{\sigma^2} \right) \\ &+ \frac{\partial^2 V_0}{\partial I^2} \left( \overline{f_1(\sigma) F(\sigma)} - \overline{f_1(\sigma)}\; \overline{F(\sigma)} \right). \end{aligned}$$

As a result we can rewrite equation (25) as

$$\begin{aligned} \overline{\mathcal{L}}_1 H &= A_1 S^2 \frac{\partial^3 V_0}{\partial I \partial S^2} + A_2 \frac{\partial^2 V_0}{\partial I^2} + A_3 S^2 \frac{\partial^2}{\partial S^2} \left( S^2 \frac{\partial^2 V_0}{\partial S^2} \right) \\ &= W(S,t,I), \end{aligned} \tag{28}$$

say, where we have defined

$$A_1 = \left(\overline{F(\sigma)}\ \overline{f_2(\sigma)} - \overline{f_2(\sigma)F(\sigma)}\right) + \frac{1}{2}\left(\overline{f_1(\sigma)}\ \overline{\sigma^2} - \overline{f_1(\sigma)\sigma^2}\right),$$

$$A_2 = \overline{f_1(\sigma)}\ \overline{F(\sigma)} - \overline{f_1(\sigma)F(\sigma)},$$

$$A_3 = \frac{1}{2}\left(\overline{\sigma^2}\ \overline{f_2(\sigma)} - \overline{f_2(\sigma)\sigma^2}\right). \tag{29}$$

However, from (26), $\overline{\mathcal{L}}_1 W = 0$, which implies that $\overline{\mathcal{L}}_1\left((T-t)W\right) = -W$. This allows us to find a particular solution of the form $-(T-t)W$, and we can then express the general solution of (28) as

$$H(S,t,I) = -(T-t)\Bigg[A_1 S^2 \frac{\partial^3 V_0}{\partial I \partial S^2} + A_2 \frac{\partial^2 V_0}{\partial I^2} + A_3 S^2 \frac{\partial^2}{\partial S^2}\left(S^2 \frac{\partial^2 V_0}{\partial S^2}\right)\Bigg] + H_1(S,t,I), \tag{30}$$

where $H_1$ solves the equation

$$\overline{\mathcal{L}}_1 H_1 = 0. \tag{31}$$

The solution of this last equation can be obtained directly via the transformation $\overline{I} = I + \overline{F}(T-t)$, which transforms (31) into the Black-Scholes equation with volatility $\left(\overline{\sigma^2}\right)^{1/2}$.

## 4.1 Boundary Layer Analysis

We have already satisfied the payoff condition with $V_0$, and so we expect that

$$V_1(S,T,\sigma,I) = 0.$$

But this is not possible, since $V_1$ is a function of $\sigma$ whereas the payoff is not. The remedy is to introduce a boundary layer in $t$ near $t = T$, with length of $\mathcal{O}(\epsilon)$. We define the scaled inner variable $\tau$ via

$$t = T + \epsilon\tau, \qquad \tau < 0.$$

Our goal is now to find the expansion in the boundary layer (the *inner solution*), which we subsequently match with the solution outside the boundary layer (the *outer solution*) using Van Dyke's matching principle [19]. We introduce the following operators:

$$\widetilde{\mathcal{L}}_0 = \frac{\partial}{\partial \tau} + \mathcal{L}_0, \qquad \widetilde{\mathcal{L}}_1 = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + rS\frac{\partial}{\partial S} + F(\sigma)\frac{\partial}{\partial I} - r,$$

and we have

$$\mathcal{L} = \frac{1}{\epsilon}\widetilde{\mathcal{L}}_0 + \widetilde{\mathcal{L}}_1,$$

but we note that $\widetilde{\mathcal{L}}_0$, unlike $\mathcal{L}_0$ above, contains the time derivative $\partial/\partial\tau$. We write $\widetilde{V}(S,\tau,\sigma,I)$ for the value of the derivative in the boundary layer, and expand

$$\widetilde{V} \sim \widetilde{V}_0 + \epsilon\widetilde{V}_1 + \cdots$$

As a result equation (16) becomes

$$\frac{1}{\epsilon}\widetilde{\mathcal{L}}_0\widetilde{V}_0 + \left(\widetilde{\mathcal{L}}_1\widetilde{V}_0 + \widetilde{\mathcal{L}}_0\widetilde{V}_1\right) + \cdots = 0. \tag{32}$$

At lowest order,

$$\widetilde{\mathcal{L}}_0\widetilde{V}_0 = 0$$

with the condition

$$\widetilde{V}_0(S,0,\sigma,I) = P(S,I).$$

The solution is easily seen to be

$$\widetilde{V}_0(S,\tau,\sigma,I) = P(S,I). \tag{33}$$

Note that this matches automatically with our outer solution $V_0(S,t,I)$ as $t \to T$, $\tau \to -\infty$.

At the next order we have

$$\widetilde{\mathcal{L}}_0\widetilde{V}_1 = -\widetilde{\mathcal{L}}_1\widetilde{V}_0 = -\widetilde{\mathcal{L}}_1 P,$$

which can be written as

$$\frac{\partial\widetilde{V}_1}{\partial\tau} + \mathcal{L}_0\widetilde{V}_1 = \frac{1}{2}\left(\overline{\sigma^2} - \sigma^2\right)S^2\frac{\partial^2 P}{\partial S^2} + \left(\overline{F(\sigma)} - F(\sigma)\right)\frac{\partial P}{\partial I} - \overline{\mathcal{L}}_1 P \tag{34}$$

with the final condition

$$\widetilde{V}_1(S,0,\sigma,I) = 0$$

(note that $\partial P/\partial\tau = 0$). We now need the solution of (34) as $\tau \to -\infty$ in order to match with the outer solution. A particular solution of (34) is

$$\widetilde{V}_1^{\infty} = f_2(\sigma)S^2\frac{\partial^2 P}{\partial S^2} + f_1(\sigma)\frac{\partial P}{\partial I} - \tau\overline{\mathcal{L}}_1 P + \widetilde{H}(S,I) \tag{35}$$

where $\widetilde{H}(S,I)$ is arbitrary, and we conjecture that this is the correct form for the asymptotic behaviour of $\widetilde{V}_1(S,\tau,\sigma,I)$ as $\tau \to -\infty$. Now

$$\langle\widetilde{\mathcal{L}}_0\widetilde{V}_1, p_\infty\rangle = \langle\frac{\partial\widetilde{V}_1}{\partial\tau} + \mathcal{L}_0\widetilde{V}_1, p_\infty\rangle = \langle\frac{\partial\widetilde{V}_1}{\partial\tau}, p_\infty\rangle$$

since $\langle\mathcal{L}_0\widetilde{V}_1, p_\infty\rangle = 0$. We further note that from (34),

$$\frac{\partial}{\partial\tau}\langle\widetilde{V}_1, p_\infty\rangle = \langle\frac{\partial\widetilde{V}_1}{\partial\tau}, p_\infty\rangle = -\langle\overline{\mathcal{L}}_1 P, p_\infty\rangle = -\overline{\mathcal{L}}_1 P$$

and therefore

$$\langle\widetilde{V}_1, p_\infty\rangle = -\tau\overline{\mathcal{L}}_1 P, \tag{36}$$

where we have used the condition $\widetilde{V}_1(S,0,\sigma,I)=0$. Now from (35) we have

$$\langle \widetilde{V}_1^\infty, p_\infty\rangle = -\tau\overline{\mathcal{L}}_1 P + \overline{f_2(\sigma)} S^2 \frac{\partial^2 P}{\partial S^2} + \overline{f_1(\sigma)}\frac{\partial P}{\partial I} + \widetilde{H}(S,I). \tag{37}$$

We now compare (36) with (37) and we immediately have

$$\widetilde{H}(S,I) = -S^2 \frac{\partial^2 P}{\partial S^2}\overline{f_2(\sigma)} - \frac{\partial P}{\partial I}\overline{f_1(\sigma)}.$$

Thus, as $\tau \to -\infty$,

$$V_1 \sim \widetilde{V}_1^\infty = S^2\left(f_2(\sigma) - \overline{f_2(\sigma)}\right)\frac{\partial^2 P}{\partial S^2} + \left(f_1(\sigma) - \overline{f_1(\sigma)}\right)\frac{\partial P}{\partial I} - \tau\overline{\mathcal{L}}_1 P, \tag{38}$$

since subtracting of the particular solution leaves a 'complementary function' $\widetilde{V}_1 - \widetilde{V}_1^\infty$ whose inner product with $p_\infty$ vanishes, which satisfies the homogeneous version of the parabolic equation (34), and which therefore vanishes as $\tau \to -\infty$.

## 4.2 Matching

We now consider the matching. Expanding the outer solution to $\mathcal{O}(\epsilon)$ in the inner variable $\tau$, we have

$$\begin{aligned}
V_0(S,t,I) &= \widetilde{V}_0\left(S, T+\epsilon\tau, I + \overline{F(\sigma)}(T-t)\right) \\
&= \widetilde{V}_0\left(S, T+\epsilon\tau, I - \epsilon\tau\overline{F}\right) \\
&\sim \overline{V}_0(S,T,I) + \epsilon\tau\frac{\partial \overline{V}_0}{\partial t}\left(S,T,\overline{I}\right) - \epsilon\tau\overline{F}\frac{\partial \overline{V}_0}{\partial \overline{I}}(S,t,I) \\
&= P(S,T,I) + \epsilon\tau\frac{\partial \overline{V}_0}{\partial t}(S,T,I) - \epsilon\tau\overline{F}\frac{\partial \overline{V}_0}{\partial \overline{I}}(S,t,I)
\end{aligned}$$

(note that for $t=T$ we have $I=\overline{I}$). Using (20) this reduces to

$$V_0(S,t,I) \sim P(S,I) - \epsilon\tau\overline{\mathcal{L}}_1 P,$$

and so, combining with (22), the two-term inner expansion of $V_0 + \epsilon V_1$ is

$$P(S,I) + \epsilon\left(-\tau\overline{\mathcal{L}}P + f_1(\sigma)\frac{\partial P}{\partial I} + f_2(\sigma)S^2\frac{\partial^2 P}{\partial S^2} + H(S,T,I)\right). \tag{39}$$

Similarly, from (33) and (38), the two-term expansion of the inner solution in terms of the outer variable $t$ is

$$P(S,I) + \epsilon\left(\left(f_2(\sigma) - \overline{f_2(\sigma)}\right)S^2\frac{\partial^2 P}{\partial S^2} + \left(f_1(\sigma) - \overline{f_1(\sigma)}\right)\frac{\partial P}{\partial I} - \tau\overline{\mathcal{L}_1}P\right). \tag{40}$$

Matching (40) and (39), we see that the final condition for $H(S,t,I)$ is

$$H(S,T,I) = -\overline{f_2(\sigma)}S^2\frac{\partial^2 P}{\partial S^2} - \overline{f_1(\sigma)}\frac{\partial P}{\partial I},$$

and therefore (30) becomes

$$H_1(S,T,I) = -\overline{f_2(\sigma)}S^2\frac{\partial^2 P}{\partial S^2} - \overline{f_1(\sigma)}\frac{\partial P}{\partial I},$$

and the boundary layer analysis thus leads to the missing final condition for $H_1$. As stated above, we can reduce $\overline{\mathcal{L}}H_1 = 0$ to the Black-Scholes equation using the substitution $\overline{I} = I + (T-t)\overline{F}$, and hence we find $\overline{H}_1(S,t;\overline{I}) = H_1(S,t,I)$ by solving

$$\mathcal{L}^{BS}\overline{H}_1 = 0,$$

with

$$\overline{H}_1(S,T;\overline{I}) = H_1(S,T,\overline{I}) = -\overline{f_2(\sigma)}S^2\frac{\partial^2 P}{\partial S^2} - \overline{f_1(\sigma)}\frac{\partial P}{\partial I}.$$

Recalling that $\overline{V}_0(S,t;\overline{I})$ satisfies $\mathcal{L}^{BS}\overline{V_0} = 0$ with $\overline{V_0}(S,T;\overline{I}) = P(S,\overline{I})$, and (26), we see that for $t \leq T$,

$$\overline{H_1}(S,t;\overline{I}) = -\overline{f_2(\sigma)}S^2\frac{\partial^2\overline{V_0}}{\partial S^2} - \overline{f_1(\sigma)}\frac{\partial\overline{V_0}}{\partial I}.$$

## 4.3 Summary

In summary, the outer expansion takes the form

$$V(S,t,\sigma,I) \sim V_0(S,t,I) + \epsilon V_1(S,t,\sigma,I)$$

where

$$V_0(S,t,I) = \overline{V_0}(S,t;I + (T-t)\overline{F})$$

and $\overline{V_0}(S,t;\overline{I})$ satisfies the Black-Scholes problem

$$\mathcal{L}^{BS}\overline{V_0} = 0, \qquad \overline{V_0}(S,T;\overline{I}) = P(S,\overline{I})$$

with volatility $(\overline{\sigma^2})^{1/2}$. The correction $V_1(S,t,\sigma,I)$ takes the form

$$\begin{aligned} V_1(S,t,\sigma,I) = {} & S^2\frac{\partial^2 V_0}{\partial S^2}\left(f_2(\sigma) - \overline{f_2(\sigma)}\right) + \frac{\partial V_0}{\partial I}\left(f_1(\sigma) - \overline{f_1(\sigma)}\right) \\ & - (T-t)\left(A_1 S^2\frac{\partial^3 V_0}{\partial I\partial S^2} + A_2\frac{\partial^2 V_0}{\partial I^2} + A_3 S^2\frac{\partial^2}{\partial S^2}\left(S^2\frac{\partial^2 V_0}{\partial S^2}\right)\right), \end{aligned} \quad (41)$$

where the functions $f_1(\sigma)$, $f_2(\sigma)$ are defined in (23) and the constants $A_1$, $A_2$, $A_3$ in (29). In the boundary layer, the solution takes the form

$$\widetilde{V}(S,\tau,\sigma,I) \sim P(S,I) + \epsilon\widetilde{V}_1,$$

where $\widetilde{V}_1$ satisfies the parabolic problem (34) with final data $\widetilde{V}_1(S,0,\sigma,I) = 0$; this solution cannot be found explicitly but its large-time behaviour is sufficient to allow matching with the outer solution to this order.

## 4.4 Extensions

### 4.4.1 Payoff singularities

It is implicit in our analysis that the payoff is smooth enough for the expansion to be valid. For example, in (30) we have terms such as $\partial^2 V_0/\partial I^2$ which is a delta function at expiry if the contract is a volatility call swaption with payoff $\max(I/T - K, 0)$. Roughly speaking, such a discontinuity will propagate along the line $I + (T - t)\overline{F} = KT$, where $K$ is the strike, and a separate analysis is necessary in a small region around this line; we do not pursue this further here, nor do we consider the implications of discontinuities in the $S$-dependence of the payoff, which (because of the second $S$-derivative in the pricing equation) may be expected to induce small "square root of time to expiry" region in which the expansion is not valid.

### 4.4.2 Correlation and the market price of risk

We also briefly consider the effects of correlation. If $\rho_t$ or $\Lambda_t$ is of $\mathcal{O}(\epsilon^{1/2})$ then the effect of it is to modify the operator $\mathcal{L}_1$ (we do not consider this special case here), but if they are of $\mathcal{O}(1)$ then our pricing equation takes the form

$$\Big(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\epsilon^{1/2}}\mathcal{L}_{\frac{1}{2}} + \mathcal{L}_1\Big)V = 0$$

where $\mathcal{L}_0$ and $\mathcal{L}_1$ are as before, and

$$\mathcal{L}_{\frac{1}{2}} = \rho\varsigma\sigma S\frac{\partial^2}{\partial S\partial\sigma} - \Lambda\varsigma\frac{\partial}{\partial\sigma}.$$

The expansion for $V$ is now in the form

$$V \sim V_0 + \epsilon^{1/2}V_{\frac{1}{2}} + \epsilon V_1 + \epsilon^{3/2}V_{\frac{3}{2}} + \cdots,$$

and the first four equations are, in order,

$$\mathcal{L}_0 V_0 = 0,$$

$$\mathcal{L}_0 V_{\frac{1}{2}} + \mathcal{L}_{\frac{1}{2}} V_0 = 0,$$

$$\mathcal{L}_0 V_1 + \mathcal{L}_{\frac{1}{2}} V_{\frac{1}{2}} + \mathcal{L}_1 V_0 = 0, \tag{42}$$

$$\mathcal{L}_0 V_{\frac{3}{2}} + \mathcal{L}_{\frac{1}{2}} V_1 + \mathcal{L}_1 V_{\frac{1}{2}} = 0. \tag{43}$$

We still have that $V_0$ is a function of $(S, I, t)$ alone, hence $\mathcal{L}_{\frac{1}{2}} V_0 = 0$ and so $V_{\frac{1}{2}}$ is also a function of $(S, I, t)$. Thus $\mathcal{L}_{\frac{1}{2}} V_{\frac{1}{2}} = 0$ and the solvability condition for (42) gives

$$\langle \mathcal{L}_1 V_0, p_\infty \rangle = 0$$

just as before, so also as before $V_0 = \overline{V_0}(S, t; I + (T - t)\overline{F})$. We then have

$$V_1 = f_2(\sigma)S^2\frac{\partial^2 V_0}{\partial S^2} + f_1(\sigma)\frac{\partial V_0}{\partial I} + H(S, t, I),$$

also as before, and we note that $\mathcal{L}_{\frac{1}{2}}V_1 \neq 0$ but $\mathcal{L}_{\frac{1}{2}}H = 0$. Proceeding to the solvability condition for (43), we have $\langle \mathcal{L}_{\frac{1}{2}}V_1 + \mathcal{L}_1 V_{\frac{1}{2}}, p_\infty\rangle = 0$, which becomes

$$\overline{\mathcal{L}}V_{\frac{1}{2}} = -\overline{\rho\varsigma\sigma f_2'}S\frac{\partial}{\partial S}\left(S^2\frac{\partial^2 V_0}{\partial S^2}\right) - \overline{\rho\varsigma\sigma f_1'}S\frac{\partial^2 V_0}{\partial S\partial I} + \overline{\Lambda\varsigma f_2'}S^2\frac{\partial^2 V_0}{\partial S^2} + \overline{\Lambda\varsigma f_1'}\frac{\partial V_0}{\partial I}, \quad (44)$$

where we denote by $f_1'$ and $f_2'$ the corresponding derivatives with respect to $\sigma$. The solution for $V_{\frac{1}{2}}$ that vanishes at expiry is readily found to be

$$\begin{aligned} V_{\frac{1}{2}}(S,I,t) = (T-t)\Bigg(&\overline{\rho\varsigma\sigma f_2'}\left(S^3\frac{\partial^3 V_0}{\partial S^3} + 2S^2\frac{\partial^2 V_0}{\partial S^2}\right) + \overline{\rho\varsigma\sigma f_1'}S\frac{\partial^2 V_0}{\partial S\partial I} \\ &- \overline{\Lambda\varsigma f_2'}S^2\frac{\partial^2 V_0}{\partial S^2} - \overline{\Lambda\varsigma f_1'}\frac{\partial V_0}{\partial I}\Bigg) \end{aligned} \quad (45)$$

where $V_0$ is already known. The calculation of $V_1$, from a solvability condition at $\mathcal{O}(\epsilon^2)$, is straightforward but even more cumbersome and we do not give details here; in the next section we give them for a pure volatility product. In the boundary layer, we have

$$\left(\frac{1}{\epsilon}\widetilde{\mathcal{L}}_0 + \frac{1}{\epsilon^{1/2}}\mathcal{L}_{\frac{1}{2}} + \widetilde{\mathcal{L}}_1\right)\widetilde{V} = 0, \qquad \widetilde{V} \sim \widetilde{V}_0 + \epsilon^{1/2}\widetilde{V}_{\frac{1}{2}} + \epsilon\widetilde{V}_1 + \cdots$$

where $\widetilde{\mathcal{L}}_0$ and $\widetilde{\mathcal{L}}_1$ are as before. Again $\widetilde{V}_0 = P(S,I)$, matching automatically with $V_0$, and the problem for $\widetilde{V}_{\frac{1}{2}}$ is

$$\widetilde{\mathcal{L}}_0\widetilde{V}_{\frac{1}{2}} = -\mathcal{L}_{\frac{1}{2}}\widetilde{V}_0 = 0;$$

the solution is simply $\widetilde{V_{\frac{1}{2}}} = 0$, and this is consistent with matching with the two-term outer expansion $V_0 + \epsilon^{1/2}V_{\frac{1}{2}}$, since in inner variables the $T-t$ in $V_{\frac{1}{2}}$ means that this term only contributes $\mathcal{O}(\epsilon^{3/2})$ to the inner expansion of the outer solution. Again, $\widetilde{V}_1$ can be calculated and matched although we do not give details here.

In summary, with nonzero $\mathcal{O}(1)$ correlation, the outer solution is given by

$$V(S,t,\sigma,I) \sim V_0(S,t,I) + \epsilon^{1/2}V_{\frac{1}{2}}(S,t,I)$$

where $V_0$ is the solution of a Black-Scholes problem and $V_{\frac{1}{2}}$ is given in (45).

## 4.5 Pure volatility products

The analysis above is greatly simplified for pure volatility products for which all $S$-derivatives vanish. Again using $\sigma_t$ as the underlying volatility variable, the pricing equation is simply

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\varsigma^2}{\epsilon}\frac{\partial^2 V}{\partial\sigma^2} + \left(\frac{m}{\epsilon} - \frac{\Lambda\varsigma}{\epsilon^{1/2}}\right)\frac{\partial V}{\partial\sigma} + F(\sigma)\frac{\partial V}{\partial I} - rV = 0$$

with payoff

$$V(\sigma,I,T) = P(I).$$

For certain models and products it is possible to write down explicit solutions not only for swaps—as in §3.2—but also for swaptions, since corresponding results for Asian options can simply be transferred. In particular, for a Hull-White type model in which $\Sigma = \Sigma_0 \sigma$ and $M - \Lambda\Sigma = M_0\sigma$, so that $\sigma_t$ follows Geometric Brownian motion, there is a series solution for a volatility-average swaption and, if one were to construct a swaption based on the continuously sampled geometric mean of $\sigma_t$, that too would have an explicit solution because the running geometric average of a Geometric Brownian motion is log-normally distributed.

### 4.5.1 Asymptotics for pure volatility products

As mentioned above, the approximate analysis is rather simpler for a pure volatility product, since then all $S$-derivatives vanish. We therefore give more details of the analysis of §4.4, in Appendix 2; in summary, the outer expansion takes the form

$$V \sim V_0 + \epsilon^{1/2} V_{1/2} + \epsilon V_1 + \cdots$$

where

$$V_0(I,t) = e^{-r(T-t)} P(I + (T-t)\overline{F}), \tag{46}$$

$$\begin{aligned} V_{1/2}(I,t) &= -(T-t)\, \overline{\Lambda\varsigma f_1'} \frac{\partial V_0}{\partial I} \\ &= -(T-t)\, e^{-r(T-t)} \overline{\Lambda\varsigma f_1'} P'(I + (T-t)\overline{F}), \end{aligned} \tag{47}$$

$$\begin{aligned} V_1(I,t) &= \left( f_1(\sigma) - \overline{f_1(\sigma)} - (T-t)\, \overline{\Lambda\varsigma f_3'} \right) \frac{\partial V_0}{\partial I} \\ &\quad + \left( -A_2\,(T-t) + \left(\overline{\Lambda\varsigma f_1'}\right)^2 \frac{(T-t)^2}{2} \right) \frac{\partial^2 V_0}{\partial I^2} \\ &= e^{-r(T-t)} \left( f_1(\sigma) - \overline{f_1(\sigma)} - (T-t)\, \overline{\Lambda\varsigma f_3'} \right) P'(I + (T-t)\overline{F}) \\ &\quad + e^{-r(T-t)} \left( -A_2\,(T-t) + \left(\overline{\Lambda\varsigma f_1'}\right)^2 \frac{(T-t)^2}{2} \right) P''(I + (T-t)\overline{F}). \end{aligned} \tag{48}$$

The function $f_3(\sigma)$ is defined in Appendix 2. The boundary-layer solution is as before.

Notice that the leading order term in the outer solution has a natural financial interpretation. If, say, the contract is a call swaption, the outer solution is zero for

$$I + (T-t)\,\overline{F} < KT;$$

since $I$ is the contribution to the payoff already "in the bank" and $(T-t)\overline{F}$ is the approximate expected remaining contribution (as in a fast mean-reverting model the fluctuations in $\sigma$ average out at this order), the option is unlikely to payout if the sum of these is an $\mathcal{O}(1)$ amount less than $KT$. The discontinuity at $I + (T-t)\overline{F} = KT$ must be resolved by an inner expansion which we do not deal with here.

### 4.6 Calibration

We note briefly that, like the original FPS [9] model, a great deal of calibration of the current framework can be accomplished without reference to a specific model. The first two terms in (45) can be calibrated to pure equitie option volatility smiles as in the FPS scheme. It is more likely that calibration is neseccary for pure volatility products, and gives, say, a series of volatility swaptions of different prices, (46) and (47) can be used to calculate $\overline{F}$ and $\overline{\Lambda\sigma f_1'}$. The calibration to $\mathcal{O}(\epsilon)$ is more complicated and we do not discuss it here, although it is in principle possible.

## 5 Examples

In this section we illustrate the theory with several different products. The first three are the pure volatility swaps described in §2 where explicit formulae are available (see §3.2.1). The asymptotic results can be shown to be correct. For the implied-volatility swap, there is some $S$-dependence via $\sigma^i$, although the payoff is still independent of $I$. Finally we look briefly at volatility-average swaptions.

### 5.1 The variance swap

For this contract, we have $F(\sigma) = \sigma^2$ and so $\overline{F} = \overline{\sigma^2}$, the average variance to be used in the Black-Scholes equation (21). The payoff is $I^{var}/T - K^{var} = \overline{I}^{var}/T - K^{var}$ and the first three terms in the expression for the variance swap value are

$$V(t,\sigma,I) \sim e^{-r(T-t)}\Big(\frac{I_t^{var} + \overline{\sigma^2}(T-t)}{T} - K^{var}\Big) - \epsilon^{1/2}e^{-r(T-t)}\frac{(T-t)\,\overline{\Lambda\varsigma f_1'}}{T}$$
$$+\,\epsilon e^{-r(T-t)}\frac{\Big(f_1(\sigma) - \overline{f_1(\sigma)} - (T-t)\overline{\Lambda\varsigma f_3'}\Big)}{T}.$$

For the random walk (9) for which $\overline{\sigma^2} = 2\alpha\bar{\sigma}^2/(2\alpha - \beta^2) = 2a\bar{\sigma}^2/(2a - b^2)$, it is easily confirmed that we recover the $\mathcal{O}(1)$ term of the exact result (14).

### 5.2 The standard-deviation swap

For the standard-deviation swap payoff (3), we still have $F(\sigma) = \sigma^2$ but now the payoff is $(I^{var}/T)^{1/2} - K^{s/d}$. Hence, the standard-deviation swap price to order $\mathcal{O}(\epsilon)$ is

$$V(t,\sigma,I) \sim e^{-r(T-t)}\Bigg(\Big(\frac{I_t^{var} + \overline{\sigma^2}(T-t)}{T}\Big)^{1/2} - K^{s/d}\Bigg)$$
$$-\,\epsilon^{1/2}\frac{e^{-r(T-t)}}{2T}(T-t)\,\overline{\Lambda\varsigma f_1'}\left(\frac{I_t^{var} + \overline{\sigma^2}(T-t)}{T}\right)^{-1/2}$$
$$+\,\epsilon\frac{e^{-r(T-t)}}{4T^2}\Bigg[2T\left(f_1(\sigma) - \overline{f_1(\sigma)} - (T-t)\,\overline{\Lambda\varsigma f_3'}\right)\left(\frac{I_t^{var} + \overline{\sigma^2}(T-t)}{T}\right)^{-1/2}$$
$$+\,(T-t)\left(A_2 - \left(\overline{\Lambda\varsigma f_1'}\right)^2\frac{(T-t)}{2}\right)\left(\frac{I_t^{var} + \overline{\sigma^2}(T-t)}{T}\right)^{-3/2}\Bigg].$$

### 5.3 The volatility-average swap

For the payoff (5) we have $F(\sigma)=\sigma$ and we find that

$$V(t,\sigma,I) \sim e^{-r(T-t)}\Big(\frac{I_t^{vol-ave}+\overline{\sigma}(T-t)}{T}-K^{vol\text{-}ave}\Big)$$
$$-\,\epsilon^{1/2}\frac{e^{-r(T-t)}}{T}\,(T-t)\,\overline{\Lambda\varsigma f_1'}$$
$$+\,\epsilon e^{-r(T-t)}\frac{\Big(f_1(\sigma)-\overline{f_1(\sigma)}+(T-)\,\overline{\Lambda\varsigma f_3'}\Big)}{T}$$

Again here, we recover the $\mathcal{O}(1)$ term of the exact solution given by (11) and (12) for large mean reversion coefficient.

### 5.4 The implied volatility swap

In this case $F(\sigma)$ is the implied volatility of an at the money option, say a call, with price $C(S,t,\sigma)$. We can of course apply the same procedure to pure equity options (this is the FPS analysis) to give

$$C(S,t,\sigma) \sim C_{BS}\left(S,t,\left(\overline{\sigma^2}\right)^{1/2}\right)$$
$$+\epsilon^{1/2}\,(T-t)\left[\overline{\rho\varsigma\sigma f_2'}\left(S^3\frac{\partial^3 C_{BS}}{\partial S^3}+2S^2\frac{\partial^2 C_{BS}}{\partial S^2}\right)-\overline{\Lambda\varsigma f_2'}S^2\frac{\partial^2 C_{BS}}{\partial S^2}\right]+\mathcal{O}(\epsilon). \qquad (49)$$

The implied volatility of this option can also be expanded in the form

$$\sigma^i \sim \sigma_0^i+\epsilon^{1/2}\sigma_1^i+\cdots$$

and since by definition

$$C(S,t,\sigma)=C_{BS}(S,t,\sigma^i),$$

we have

$$C(S,t,\sigma) \sim C_{BS}(S,t,\sigma_0^i)+\epsilon^{1/2}\sigma_1^i\frac{\partial C_{BS}}{\partial\sigma}|_{\sigma_0^i}+\mathcal{O}(\epsilon). \qquad (50)$$

Comparing (49) to (50) we clearly have

$$\sigma_0^i=\left(\overline{\sigma^2}\right)^{1/2}$$

and, setting the strike of $C$ equal to $S$ and substituting from the Black-Scholes formulae,

$$\sigma_1^i=\frac{\overline{\rho\varsigma\sigma f_2'}\left(\frac{1}{2}\overline{\sigma^2}-r\right)}{\left(\overline{\sigma^2}\right)^{3/2}}-\frac{\overline{\Lambda\varsigma f_2'}}{\left(\overline{\sigma^2}\right)^{1/2}}.$$

Hence we see an additional complication in that the averaging function $F(\sigma)$ itself has an expansion

$$F(\sigma) \sim \left(\overline{\sigma^2}\right)^{1/2} + \epsilon^{1/2}\sigma_1^i + \mathcal{O}(\epsilon).$$

Thus the price operator takes the form

$$\frac{1}{\epsilon}\left(\mathcal{L}_0 + \epsilon^{1/2}\mathcal{L}_{\frac{1}{2}} + \epsilon\mathcal{L}_1 + \epsilon^{3/2}\mathcal{L}_{\frac{3}{2}} + \cdots\right)$$

where

$$\mathcal{L}_{\frac{3}{2}} = \sigma_1^i \frac{\partial}{\partial I}.$$

This means in turn that the right-hand side of (44) has an extra term

$$-\langle \mathcal{L}_{\frac{3}{2}} V_0, p_\infty\rangle = -\sigma_1^i \frac{\partial V_0}{\partial I}$$

and so there is an extra term $(T-t)\sigma_1^i \partial V_0/\partial I$ on the right hand side of (45), that is an extra $\mathcal{O}(\epsilon^{3/2})$ correction to $V$; it has the obvious financial interpretation as the vega with respect to the average $I$. The $\mathcal{O}(\epsilon)$ correction, however, is more complicated because of the $S$-dependence in the implied volatility and we do not give details here.

### 5.5 Volatility-average swaptions

For the volatility-average swaption we have $F(\sigma) = \sigma$ so $\overline{F} = \overline{\sigma}$, and

$$\begin{aligned}
V(t,\sigma,I) \sim\; & e^{-rT-t)} \max\left(\frac{I + \overline{\sigma}(T-t)}{T} - K, 0\right) \\
& - \epsilon^{1/2} e^{-r(T-t)} (T-t)\, \overline{\Lambda\varsigma f_1'}\, \mathcal{H}\left(I + \overline{\sigma}(T-t) - TK\right) \\
& + \epsilon e^{-r(T-t)} \Bigg[ \left(f_1(\sigma) - \overline{f_1(\sigma)} - (T-t)\, \overline{\Lambda\varsigma f_3'}\right) \mathcal{H}(I + \overline{\sigma}(T-t) - TK) \\
& + \left(-A_2\,(T-t) + \left(\overline{\Lambda\varsigma f_1'}\right)^2 \frac{(T-t)^2}{2}\right) \frac{\delta(I + \overline{\sigma}(T-t) - TK)}{T} \Bigg],
\end{aligned}$$

where $\mathcal{H}$ is the Heaviside function with jump $1/T$:

$$\mathcal{H}(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{T} & \text{if } x > 0 \end{cases}$$

and $\delta(x)$ is the delta function. As discussed above, the $\mathcal{O}(\epsilon)$ contribution is singular on the line $I + (T-t)\overline{F}$ and the expansion is not valid in this region.

## 6 Conclusion

We have described a range of approaches to the pricing and hedging problem for a variety of products depending on realised volatility. Some of these, especially those based on realised variance, are already traded; but as we point out in §2, from a statistical point of view the realised first variation (4) is a more robust estimator and hence we have described the application of the theory to volatility-averaged options as well as to implied volatility averages, which are the type measured by the VIX index [6]. We have presented an asymptotic analysis which leads to a description of the derivative price even in the time leading up to expiry, and this should be an accurate approximation whenever volatility is fast mean-reverting, as well as straightforward to calibrate from implied volatility smiles.

# A Appendix 1: Derivation of certain expectations

## A.1 The random walk (7)

Let $y_t$ satisfy the stochastic differential equation

$$dy_t = (a_1 + a_2 y_t)dt + (a_3 + a_4 y_t)dW_t + (a_5 + a_6 y_t)dN_t. \tag{51}$$

Define

$$E_{1\tau} = \mathbb{E}[y_\tau | y_0], \qquad \overline{E}_{1\tau} = \int_0^\tau E_{1t} dt,$$

and

$$E_{2\tau} = \mathbb{E}[y_\tau^2 | y_0], \qquad \overline{E}_{2\tau} = \int_0^\tau E_{2t} dt.$$

Recalling that $\mathbb{E}[dN_t] = \lambda dt$, we have

$$\begin{aligned} dE_{1t} &= (a_1 + a_2 E_{1t} + \lambda (a_5 + a_6 E_{1t}))\, dt \\ &= (\alpha_0 + \alpha_1 E_{1t}) dt, \end{aligned}$$

where

$$\alpha_0 = a_1 + \lambda \alpha_5, \qquad \alpha_1 = a_2 + \lambda a_6,$$

from which, since $E_{10} = y_0$,

$$E_{1\tau} = y_0 e^{\alpha_1 \tau} - \frac{\alpha_0}{\alpha_1}(1 - e^{\alpha_1 \tau}), \tag{52}$$

and

$$\overline{E}_{1\tau} = \frac{y_0}{\alpha_1}(e^{\alpha_1 \tau} - 1) - \frac{\alpha_0}{\alpha_1^2}(\alpha_1 \tau - e^{\alpha_1 \tau} + 1). \tag{53}$$

For a mean-reverting process $\alpha_1 < 0$, so the unconditional expectation of $y_t$ is

$$y_\infty = \lim_{\tau \to \infty} E_{1\tau} = -\frac{\alpha_0}{\alpha_1},$$

and the long-term average of $y_t$ is the same:

$$\lim_{\tau \to \infty} \tau^{-1} \overline{E}_{1\tau} = -\frac{\alpha_0}{\alpha_1};$$

note the relatively slow (algebraic) decay of the contribution to $\overline{E}_{1\tau}$ from the initial value $y_0$.

Now define $z_t = y_t^2$. We easily find that

$$\begin{aligned} dz_t = & \left(2y_t(a_1 + a_2 y_t) + (a_3 + a_4 y_t)^2\right) dt \\ & + 2y_t(a_3 + a_4 y_t) dW_t + (a_5 + a_6 y_t) \left(a_5 + (a_6 + 2) y_t\right) dN_t. \end{aligned} \tag{54}$$

Thus $E_{2\tau} = \mathbb{E}[z_\tau | z_0 = y_0^2]$ satisfies

$$dE_{2t} = (\beta_0 + \beta_1 E_{1t} + \beta_2 E_{2t})\, dt$$

where

$$\begin{aligned} \beta_0 &= a_3^2 + \lambda a_5^2, \\ \beta_1 &= 2(a_1 + \lambda a_5) + 2(\lambda a_5 a_6 + a_3 a_4) = 2(\alpha_0 + \gamma_1), \\ \beta_2 &= 2(a_2 + \lambda a_6) + \lambda a_6^2 + a_4^2 = 2(\alpha_1 + \gamma_2), \end{aligned} \tag{55}$$

with $\alpha_0$, $\alpha_1$ as before, and

$$\gamma_1 = \lambda a_5 a_6 + a_3 a_4, \qquad \gamma_2 = \frac{\lambda a_6^2 + a_4^2}{2}.$$

Using (53), we find that

$$E_{2\tau} = \left(\frac{\alpha_0\beta_1 - \alpha_1\beta_0}{\alpha_1\beta_2}\right)\left(1 - e^{\beta_2\tau}\right) - \frac{\beta_1(\alpha_1 y_0 + \alpha_0)}{\alpha_1(\alpha_1 - \beta_2)}\left(e^{\beta_2\tau} - e^{\alpha_1\tau}\right) + y_0^2 e^{\beta_2\tau},$$

and we note that $E_{2\tau}$ grows exponentially in $\tau$ unless $\beta_2 < 0$, a condition analogous to $\alpha_1 < 0$ for $E_{1\tau}$. In this case, the unconditional expectation of $E_{2\tau}$, namely

$$z_\infty = \lim_{\tau\to\infty} E_{2\tau},$$

is $(\alpha_0\beta_1 - \alpha_1\beta_0)/\alpha_1\beta_2$.

Lastly, we calculate the averaged-expectation of $z_t$,

$$\begin{aligned} \overline{E}_{2\tau} = & \left(\frac{\alpha_0\beta_1 - \alpha_1\beta_0}{\alpha_1\beta_2}\right)\tau \\ & + \left(\frac{\alpha_0\beta_1 - \alpha_1\beta_0}{\alpha_1\beta_2} + \frac{\beta_1(\alpha_1 y_0 + \alpha_0)}{\alpha_1(\alpha_1 - \beta_2)} - y_0^2\right)\left(\frac{1 - e^{\beta_2\tau}}{\beta_2}\right) \\ & - \frac{\beta_1(\alpha_1 y_0 + \alpha_0)}{\alpha_1^2(\alpha_1 - \beta_2)}\left(1 - e^{\alpha_1\tau}\right). \end{aligned} \tag{56}$$

### A.1.1 Special cases

We note some special cases:

**(i)** The case $\alpha_1 = 0$, $\beta \neq 0$ corresponds to the condition $\lambda = -\alpha_2/\alpha_6$, and then we have

$$E_{1\tau} = y_0 + \alpha_0\tau, \qquad \overline{E}_{1\tau} = \tau y_0 + \frac{1}{2}\alpha_0\tau^2,$$

$$E_{2\tau} = \frac{\beta_0 + \beta_1 y_0 + \beta_2 y_0^2}{\beta_2}\left(e^{\beta_2\tau} + 1\right) + \frac{\beta_2 y_0^2 - \beta_1\alpha_0\tau}{\beta_2},$$

and

$$\overline{E}_{2\tau} = \frac{\beta_0 + \beta_1 y_0 + \beta_2 y_0^2}{\beta_2^2}(e^{\beta_2\tau} - 1) - \frac{\tau}{\beta_2}(\beta_0 + \beta_1 y_0) - \frac{\beta_1}{2\beta_2}\alpha_0\tau^2.$$

**(ii)** The case $\alpha_1 = \beta_2 \neq 0$ corresponds to $\alpha_1 + \lambda a_6^2 + a_4^2 = 0$, and then we have $E_{1\tau}$, $\overline{E}_{1\tau}$ as in the general case, whereas

$$E_{2\tau} = \left(\frac{\alpha_0\beta_1 - \alpha_1\beta_0}{\alpha_1^2}\right)(1 - e^{\alpha_1\tau}) + \frac{\beta_1(\alpha_1 y_0 + \alpha_0)\tau}{\alpha_1}e^{\alpha_1\tau} + y_0^2 e^{\alpha_1\tau},$$

and

$$\overline{E}_{2\tau} = \left(\frac{\alpha_0\beta_1 - \alpha_1\beta_0}{\alpha_1^2}\right)\tau + \frac{\beta_1\alpha_0 - \alpha_1\beta_0 - \alpha_1^2 y_0^2}{\alpha_1^3}\left(1 - e^{\alpha_1\tau}\right).$$

**(iii)** The case $\alpha_1 = \beta_2 = 0$ corresponds to $a_4 = a_6 = 0$, and we have

$$E_{2\tau} = y_0^2 + (\beta_0 + 2\alpha_0 y_0)\,\tau + \alpha_0^2\tau^2,$$

$$\overline{E}_{2\tau} = y_0^2\tau + \frac{(\beta_0 + 2\alpha_0 y_0)\tau^2}{2} + \frac{\alpha_0^2\tau^3}{3}.$$

**(iv)** Finally, consider the case $\alpha_1 \neq 0$, $\beta_2 = 0$. Then we have that $2a_2 + 2\lambda a_6 + \lambda a_6^2 + a_4^2 = 0$ and we have $E_{1\tau}$ and $\overline{E}_{1\tau}$ as in the general case, and

$$E_{2\tau} = \frac{\tau\left(\alpha_1\beta_0 - \alpha_0\beta_1\right)}{\alpha_1} - \frac{\beta_1\left(\alpha_1 y_0 + \alpha_0\right)}{\alpha_1^2}\left(1 - e^{\alpha_1\tau}\right) + y_0^2$$

and

$$\overline{E}_{2\tau} = \frac{\left(\alpha_1\beta_0 - \alpha_0\beta_1\right)\tau^2}{2\alpha_1} - \frac{\beta_1\left(\alpha_1 y_0 + \alpha_0\right)}{\alpha_1^3}\left(\alpha_1\tau - e^{\alpha_1\tau} + 1\right) + y_0^2\tau.$$

### A.1.2 Derivatives pricing

Using the general expressions (52), (53), (56) and (56), or any of the particular cases described above, either volatility-average or variance swaps can be priced by taking $y_t = \sigma_t$ (in which case $z_t = \sigma_t^2 = v_t$) or by taking $y_t = v_t$ directly, depending on whether the volatility model is for $\sigma_t$ or $v_t$. For example, the strike for a volatility-average swap is

$$\begin{aligned} K^{vol-ave} &= \frac{1}{T}\mathbb{E}_0\Big[\int_0^T \sigma_t dt\Big] \\ &= \frac{1}{T}\overline{E}_{1T}|_{y_0=\sigma_0}, \end{aligned}$$

where it is understood that the process for $y_t$ is the required model for volatility. In a similar way, the expectation needed to calculate the vega (10), namely,

$$\mathbb{E}_t\Big[\int_t^T \sigma_s ds\Big],$$

is equal to

$$\overline{E}_{1(T-t)}\big|_{y_0=\sigma_t}$$

that is, $y_0$ is replaced by $\sigma_t$ and $\tau$ by $T-t$.

## A.2 The process (8)

Suppose now that

$$dy_t = (b_1 + b_2 y_t)dt + b_3 y_t^{1/2} dW_t \tag{57}$$

and define $E_{1\tau}$, $\overline{E}_{1\tau}$, $E_{2\tau}$, $\overline{E}_{2\tau}$ as before. Also define $z_t = y_t^2$ so that

$$\begin{aligned} dz_t &= (2y_t(b_1 + b_2 y_t) + b_3^2 y_t)dt + 2b_3 y_t^{3/2} dW_t \\ &= ((2b_1 + b_3^2)y_t + 2b_2 z_t)dt + 2b_3 y_t^{3/2} dW_t. \end{aligned}$$

Then, proceeding as above, we find linear ordinary differential equations first for $E_{1\tau}$, then $\overline{E}_{1\tau}$, $E_{2\tau}$ and $\overline{E}_{2\tau}$, yielding

$$dE_{1t} = (b_1 + b_2 E_{1t})dt,$$

so that

$$\begin{aligned} E_{1\tau} &= y_0 e^{b_2\tau} - \frac{b_1}{b_2}(1 - e^{b_2\tau}), \\ \overline{E}_{1\tau} &= \frac{y_0}{b_2}(e^{b_2\tau} - 1) - \frac{b_1}{b_2^2}(b_2\tau - e^{b_2\tau} + 1), \end{aligned} \tag{58}$$

and then

$$\begin{aligned} \frac{dE_{2t}}{dt} &= (2b_1 + b_3^2)E_{1t} + 2b_2 E_{2t} \\ &= (2b_1 + b_3^2)\Big(y_0 e^{b_2 t} - \frac{b_1}{b_2}(1 - e^{b_2 t})\Big) + 2b_2 E_{2t} \\ &= \delta_1 + \delta_2 e^{b_2 t} + 2b_2 E_{2t}, \end{aligned}$$

where

$$\delta_1 = -\frac{b_1(2b_1 + b_3^2)}{b_2}, \qquad \delta_2 = y_0(2b_1 + b_3^2) - \delta_1.$$

Integrating, we have

$$E_{2\tau} = -\frac{\delta_1}{2b_2} - \frac{\delta_2}{b_2}e^{b_2\tau} + \Big(y_0^2 + \frac{\delta_2}{b_2} + \frac{\delta_1}{2b_2}\Big)e^{2b_2\tau}$$

and so

$$\overline{E}_{2\tau} = -\frac{\delta_1\tau}{2b_2} - \frac{\delta_2}{b_2^2}(e^{b_2\tau} - 1) + \frac{1}{2b_2}\Big(y_0^2 + \frac{\delta_2}{b_2} + \frac{\delta_1}{2b_2}\Big)(e^{2b_2\tau} - 1).$$

## A.3 Popular Stochastic Volatility Models

For completeness we give the results of these calculations for some popular stochastic volatility models:
**(i)** The Hull-White model [13].
This is a geometric Brownian motion for the variance $\sigma_t^2$:

$$d\sigma_t^2 = \kappa\sigma_t^2 dt + \theta\sigma_t^2 dW_t.$$

We use (51) with $y_t = \sigma_t^2$, so that

$$\alpha_0 = 0, \qquad \alpha_1 = \kappa.$$

We obtain

$$E_{1\tau} = \sigma_0^2 e^{\kappa\tau}, \qquad \overline{E}_{1\tau} = \frac{\sigma_0^2}{\kappa}\left(e^{\kappa\tau} - 1\right). \tag{59}$$

**(ii)** Analogous to (**i**), we consider a geometric random walk for the volatility

$$d\sigma_t = \kappa\sigma_t dt + \theta\sigma_t dW_t.$$

Then in (51) we have $\alpha_0 = 0$, $\alpha_1 = \kappa$, $\beta_0 = 0$, $\beta_1 = 0$, $\beta_2 = 2\kappa + \theta^2$, and we have

$$E_{1\tau} = \sigma_0 e^{\kappa\tau}, \qquad \overline{E}_{1\tau} = \frac{\sigma_0}{\kappa}\left(e^{\kappa\tau} - 1\right),$$

and

$$E_{2\tau} = \sigma_0^2 e^{(2\kappa+\theta^2)\tau}, \qquad \overline{E}_{2\tau} = -\frac{\sigma_0^2}{2\kappa+\theta^2}\left(1 - e^{(2\kappa+\theta^2)\tau}\right).$$

**(iii)** The mean reverting-version of the Ornstein-Uhlenbeck model for the volatility is

$$d\sigma_t = \kappa\left(\nu - \sigma_t\right)dt + \theta dW_t.$$

We take $y_t = \sigma_t$, $\alpha_0 = \kappa\nu$, $\alpha_1 = -\kappa$, $\beta_0 = \theta^2$, $\beta_1 = 2\kappa\nu$ and $\beta_2 = -2\kappa$ in (51). Then it is straightforward to show that

$$E_{1\tau} = (\sigma_0 - \nu)e^{-\kappa\tau} + \nu, \qquad \overline{E}_{1\tau} = \frac{\sigma_0 - \nu}{\kappa}\left(1 - e^{-\kappa\tau}\right) + \nu\tau;$$

and that

$$E_{2\tau} = \frac{2\kappa\nu^2 + \theta^2}{2\kappa}\left(1 - e^{-2\kappa\tau}\right) + 2\nu\left(\nu - \sigma_0\right)\left(e^{-2\kappa\tau} - e^{-\kappa\tau}\right) + \sigma_0^2 e^{-2\kappa\tau},$$

$$\begin{aligned}\overline{E}_{2\tau} = &\left(\frac{2\kappa\nu^2 + \theta^2}{2\kappa}\right)\tau + \left(\sigma_0^2 - 2\nu\left(\sigma_0 - \nu\right) - \frac{2\kappa\nu^2 + \theta^2}{2\kappa}\right)\left(\frac{1 - e^{-2\kappa\tau}}{2\kappa}\right) \\ &- \frac{2\nu\left(\nu - \sigma_0\right)}{\kappa}\left(1 - e^{-\kappa\tau}\right).\end{aligned}$$

**(iv)** The Heston model [10].
This is

$$d\sigma_t^2 = \kappa\left(\nu - \sigma_t^2\right)dt + \theta\sigma_t dW_t.$$

In this case we use (57) with $y_t = \sigma_t^2$,

$$b_1 = \kappa\nu, \qquad b_2 = -\kappa, \qquad \text{and} \qquad b_3 = \theta.$$

Then from (58) we have:

$$E_{1\tau} = \sigma_0^2 e^{-\kappa\tau} + \nu\left(1 - e^{-\kappa\tau}\right), \qquad \overline{E}_{1\tau} = \frac{\sigma_0^2 - \nu}{\kappa}\left(1 - e^{-\kappa\tau}\right) + \nu\tau.$$

**(v)** Finally we consider the mean reverting log-normal model [12]

$$d\sigma_t = \alpha(\overline{\sigma} - \sigma_t)dt + \beta_t\sigma_t dW_t.$$

We use (51) with $y_t = \sigma_t$, $\alpha_0 = \alpha\overline{\sigma}$, $\alpha_1 = -\alpha$, $\beta_0 = 0$, $\beta_1 = 2\alpha\overline{\sigma}$, $\beta_2 = -2\alpha + \beta^2$. As expected, we obtain the following expressions:

$$\begin{aligned} E_{1\tau} &= \sigma_0 e^{-\alpha\tau} + \overline{\sigma}\left(1 - e^{-\alpha\tau}\right), \\ \overline{E}_{1\tau} &= \frac{\sigma_0 - \overline{\sigma}}{\alpha}\left(1 - e^{-\alpha\tau}\right) + \overline{\sigma}\tau, \end{aligned}$$

$$\begin{aligned} E_{2\tau} = & \frac{2\alpha\overline{\sigma}^2}{2\alpha - \beta^2}\left(1 - e^{-(2\alpha-\beta^2)\tau}\right) \\ & + \frac{2\alpha\overline{\sigma}\left(\sigma_0 - \overline{\sigma}\right)}{\alpha - \beta^2}\left(e^{-\alpha\tau} - e^{-(2\alpha-\beta^2)\tau}\right) + \sigma_0^2 e^{-(2\alpha-\beta^2)\tau}, \end{aligned}$$

$$\begin{aligned} \overline{E}_{2\tau} = & \frac{2\alpha\overline{\sigma}^2}{2\alpha - \beta^2}\tau + \left(\frac{2\alpha\overline{\sigma}^2}{2\alpha - \beta^2} + \frac{2\overline{\sigma}(\overline{\sigma} - \sigma_0)\alpha}{\alpha - \beta^2} - \sigma_0^2\right)\left(\frac{e^{-(2\alpha-\beta^2)\tau} - 1}{2\alpha - \beta^2}\right) \\ & - \frac{2\overline{\sigma}(\overline{\sigma} - \sigma_0)}{\alpha - \beta^2}\left(1 - e^{-\alpha\tau}\right). \end{aligned}$$

# B Appendix 2: Analysis for pure volatility products

Following §4.4, we have $\mathcal{L}_0 V_0 = 0$, so $V_0 = V_0(S,t)$; similarly $\mathcal{L}_0 V_{\frac{1}{2}} = -\mathcal{L}_{\frac{1}{2}} V_0 = 0$, so $V_{\frac{1}{2}} = V_{\frac{1}{2}}(I,t)$. From the solvability condition for (42), we have

$$\overline{\mathcal{L}}_1 V_0 = \frac{\partial V_0}{\partial t} + \overline{F}\frac{\partial V_0}{\partial I} - rV_0 = 0$$

whose solution with $V_0(I,T) = P(I)$ is

$$V_0(I,t) = e^{-r(T-t)} P\left(I + (T-t)\overline{F}\right).$$

Then, solving (42),

$$V_1 = f_1(\sigma)\frac{\partial V_0}{\partial I} + H(I,t).$$

where $H$ is as yet undetermined.

Now the solvability condition for (43) gives

$$\overline{\mathcal{L}} V_{\frac{1}{2}} = \overline{\Lambda \varsigma f_1'} \frac{\partial V_0}{\partial I}$$

so that the solution satisfying $V_{\frac{1}{2}}(I,T) = 0$ is

$$V_{\frac{1}{2}}(I,t) = -(T-t)\,\overline{\Lambda \varsigma f_1'}\frac{\partial V_0}{\partial I}.$$

Thus far we have paralleled the analysis of §4.4. Now we continue by finding the solution of (43): it is

$$\begin{aligned}
\mathcal{L}_0 V_{\frac{3}{2}} &= -\mathcal{L}_{\frac{1}{2}} V_1 - \mathcal{L}_1 V_{\frac{1}{2}} \\
&= \left(\Lambda \varsigma f_1' - \overline{\Lambda \varsigma f_1'}\right)\frac{\partial V_0}{\partial I} - (T-t)\,\overline{\Lambda \varsigma f_1'}\left(\overline{F} - F(\sigma)\right)\frac{\partial^2 V_0}{\partial I^2}
\end{aligned}$$

so that

$$V_{\frac{3}{2}} = f_3(\sigma)\frac{\partial V_0}{\partial I} - (T-t)\,\overline{\Lambda \varsigma f_1'} f_1(\sigma)\frac{\partial^2 V_0}{\partial I^2} + H_{\frac{3}{2}}(I,t)$$

where $f_3(\sigma)$ satisfies

$$\frac{1}{2}\varsigma^2 \frac{d^2 f_3}{d\sigma^2} + m\frac{df_3}{d\sigma} = \Lambda \varsigma f_1'(\sigma) - \overline{\Lambda \varsigma f_1'}$$

and $H_{\frac{3}{2}}$ is arbitrary. Now at $\mathcal{O}(\epsilon)$,

$$\mathcal{L}_0 V_2 + \mathcal{L}_{\frac{1}{2}} V_{\frac{3}{2}} + \mathcal{L}_1 V_1 = 0$$

and our final application of the solvability condition, in the form $\langle \mathcal{L}_{\frac{1}{2}} V_{\frac{3}{2}} + \mathcal{L}_1 V_1, p_\infty\rangle = 0$, gives

$$\langle \mathcal{L}_1\left(f_1(\sigma)\frac{\partial V_0}{\partial I} + H\right), p_\infty\rangle = -\langle \mathcal{L}_{\frac{1}{2}} V_{\frac{3}{2}}, p_\infty\rangle,$$

that is,

$$\overline{\mathcal{L}}_1 H = \left(A_2 - (T-t)\left(\overline{\Lambda \varsigma f_1'}\right)^2\right)\frac{\partial^2 V_0}{\partial I^2} + \overline{\Lambda \varsigma f_3'}\frac{\partial V_0}{\partial I},$$

where $A_2$ is defined in (29). The solution is

$$\begin{aligned}
H(I,t) = &\left(-A_2(T-t) + \left(\overline{\Lambda \varsigma f_1'}\right)^2 \frac{(T-t)^2}{2}\right)\frac{\partial^2 V_0}{\partial I^2} \\
&- \overline{\Lambda \varsigma f_3'}\,(T-t)\frac{\partial V_0}{\partial I} + H_1(I,t),
\end{aligned}$$

where $\overline{\mathcal{L}}_1 H_1 = 0$, so that $H_1 = H_1\left(I + (T-t)\overline{F}\right)$; this last unknown function is determined by mathcing with the boundary layer.

In the boundary layer, we have

$$\left(\frac{1}{\epsilon}\widetilde{\mathcal{L}}_0 + \frac{1}{\epsilon^{1/2}}\mathcal{L}_{\frac{1}{2}} + \widetilde{\mathcal{L}}_1\right)\left(\widetilde{V}_0 + \epsilon^{1/2}\widetilde{V}_{\frac{1}{2}} + \epsilon\widetilde{V}_1 + \cdots\right) = 0,$$

as before, and it is easy to see that $\widetilde{V}_0(I,t) = P(I)$, $\widetilde{V}_{\frac{1}{2}}(I,t) = 0$, and that, as before,

$$\frac{\partial \widetilde{V}_1}{\partial \tau} + \mathcal{L}_0\widetilde{V}_1 = \left(\overline{F(\sigma)} - F(\sigma)\right)\frac{dP}{dI} - \overline{\mathcal{L}}_1 P$$

so that, as above,

$$\widetilde{V}_1(I,\tau) \sim \left(f_1(\sigma) - \overline{f_1(\sigma)}\right)\frac{dP}{dI} - \tau\overline{\mathcal{L}}_1 P$$

as $\tau \to -\infty$. Hence the matching condition is unaffected at this order by the market price of risk term, and the required final condition for $H_1(I,t)$ is

$$H_1(I,T) = -\overline{f_1(\sigma)}\frac{dP}{dI}.$$

Hence $H_1(I,t) = -e^{-r(T-t)}\overline{f_1(\sigma)}P'(I + (T-t)\overline{F})$, where $' = d/dI$.

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