An Option Pricing Formula for the GARCH Diffusion Model*

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Abstract

We derive analytically the first four conditional moments of the integrated variance implied by the GARCH diffusion process. From these moments we obtain an analytical closed-form approximation formula to price European options under the GARCH diffusion model. Using Monte Carlo simulations, we show that this approximation formula is accurate for a large set of reasonable parameters. Finally, we use the closed-form option pricing solution to shed light on the qualitative properties of implied volatility surfaces induced by GARCH diffusion models.

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1 Introduction

In this paper we study European option prices in stochastic volatility models where the underlying asset follows a geometric Brownian motion with instantaneous variance driven by a GARCH diffusion process. Precisely, we derive analytically a closed-form approximation for European option prices under the GARCH diffusion model.

Stochastic volatility models were first introduced by Hull and White (1987), Scott (1987) and Wiggins (1987) to overcome the drawbacks of the Black and Scholes (1973) and Merton (1973) model. Volatilities, stochastically changing over time, account for random behaviours of implied and historical variances and generate some of the log-return features observed in empirical studies[1]. Unfortunately, in the stochastic volatility setting it is difficult to derive closed or analytically tractable option pricing formulas even for European options. The Hull and White (1987) and the Heston (1993) models have an analytical approximation and a quasi-analytical formula to price European options, respectively. For other stochastic volatility models numerical methods are available but these procedures are highly computationally intensive[2]. In this paper, we derive an analytical closed-form approximation for European option prices based on the conditional moments of the integrated variance when the variance is driven by an uncorrelated GARCH diffusion process[3]. Our approximation is very accurate and easy to implement, it can be used to study the implied volatility and the volatility risk premium associated to GARCH diffusion models.

The GARCH diffusion process has several desirable properties. It is positive, mean reverting, with a stationary inverse Gamma distribution and it satisfies the restriction that both historical and implied variances be positive. It also fits the observation that variances seem to be stationary and mean reverting; cf. Scott (1987), Taylor (1994), Jorion (1995) and Guo (1996, 1998). Moreover, the GARCH diffusion model allows for rich pattern behaviours of volatilities and asset prices. For instance, as observed in empirical studies, it produces large autocorrelation in the squared log-returns, arbitrary large kurtosis and finite unconditional moments of log-return distributions up to a given order; cf. Genon-Catalot, Jeantheau and Laredo (2000). Furthermore, Nelson (1990) and Drost and Werker (1996) showed that under the GARCH diffusion model discrete time returns of asset prices follow a GARCH(1,1) process. Hence, the nasty problem of making inference on continuous-time parameters is reduced to the inference on the parameters

[1]See , for instance, Mandelbrot (1963) and Fama (1965).

[2]When large trading books have to be quickly and frequently evaluated many procedures are practically not feasible.

[3]This model was first introduced by Wong (1964) and popularized by Nelson (1990).

of a GARCH(1,1) model[4]; cf., for instance, Engle and Lee (1996), Lewis (2000) and Melenberg and Werker (2001). This is an important advantage over other stochastic volatility models for which the parameter estimation is much more involved. Finally, the GARCH diffusion model is the 'mean reverting' extension of the Hull and White (1987) model where the variance process follows an uncorrelated log-normal process without drift. The GARCH diffusion model makes a marked improvement over the Hull and White model because the mean reverting drift gives stationary variance and log-return processes (cf. Genon-Catalot, Jeantheau and Laredo (2000)) and it can include the volatility risk premium in the variance process. By contrast, for the Hull and White model the analytical option pricing approximation is available only when the drift is equal to zero[5]. Furthermore, the mean reversion of the variance allows to approximate long maturity option prices, while in the Hull and White model the option pricing approximation holds only for short maturity options; cf. Hull and White (1987) and Gesser and Poncet (1997).

Our approximation for option prices under the GARCH diffusion model is based on the Hull and White (1987) formula, which holds when the asset price and the instantaneous variance are uncorrelated. This assumption implies symmetric volatility 'smiles', i.e. symmetric shapes of implied volatilities plotted versus strike prices; cf. Hull and White (1987) and Renault and Touzi (1996). Typically, foreign currency option markets are characterized by symmetric volatility smiles; cf., for instance, Chesney Scott (1989), Melino and Turbull (1990), Taylor and Xu (1994) and Bollerslev and Zhou (2002). Therefore, the present model can be appropriate to price currency options. Furthermore, also in some index option markets the non zero correlation between price and variance can be neglected without increasing option pricing errors; cf. Chernov and Ghysels (2000) and Melenberg and Werker (2001) for studies on Standard & Poor's 500 and Dutch EOE index options, respectively.

The specific contributions of this paper are the following. We derive analytically the first four conditional moments of the integrated variance implied by the GARCH diffusion process. This result has several important implications. Firstly and foremost, these conditional moments allow to obtain an analytical closed-form approximation for European option prices under the GARCH diffusion model. This approximation can be easily implemented in any software package (such as Excel spread sheets). Then, just plugging in the model parameters, it provides option prices without any computational efforts. As we will show by Monte Carlo simulations, this approximation is very accurate across different strikes and maturities for a large set of reasonable parameters. Secondly, we propose an analytical approximation for implied volatilities

[4]Chou (1988) verified the empirical consistency of Nelson's theory.

[5]The volatility risk premium seems to be a significant component of the risk premia in many currency markets; cf. Guo (1998), Melenberg and Werker (2001) and references therein.

based on the conditional moments of the integrated variance, which allows us to easily study volatility surfaces induced by GARCH diffusion models. Thirdly, the conditional moments of the integrated variance can be used to estimate the continuous time parameters of the GARCH diffusion model using high frequency data. Precisely, by matching the sample moments of the realized volatility with the conditional moments of the integrated variance one has a standard and easy-to-compute GMM-type estimator for the underlying model parameters; cf. Bollerslev and Zhou (2002). Finally, the conditional moments of the integrated variance implied by the GARCH diffusion process generalize the conditional moments derived by Hull and White (1987) for log-normal variance processes.

The rest of the paper is organized as follows. Section 2 introduces the GARCH diffusion model. Section 3 presents the analytical approximation formula to price European vanilla options under the GARCH diffusion model. In Section 4, using Monte Carlo simulations, the accuracy of the approximation is investigated across different strike prices and time to maturities for different parameter choices. Section 5 studies implied volatility surfaces induced by the model and Section 6 concludes.

2 The Model

Let $S = (S_t)_{t\geq 0}$ be the underlying currency price and $V = (V_t)_{t\geq 0}$ its latent instantaneous variance. We assume that $(S_t, V_t)_{t\geq 0}$ satisfies the two-dimensional GARCH diffusion model

$$dS_t = \mu\, S_t\, dt + \sqrt{V_t}\, S_t\, dB_t, \tag{1}$$

$$dV_t = (c_1 - c_2\, V_t)dt + c_3\, V_t\, dW_t, \tag{2}$$

where c_1, c_2 and c_3 are positive constants, μ is the positive constant drift of dS_t/S_t, B and W are mutually independent one-dimensional Brownian motions on some filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ and $\mathbb{P}$ is the objective measure. We set the initial time $t = 0$ and $(S_0, V_0) \in \mathbb{R}^+ \times \mathbb{R}^+$.

The V process is mean reverting, c_1/c_2 determines the run mean value and c_2 is the reversion rate (see also equation (4)). For 'small' c_2 the mean reversion is 'weak' and V_t tends to stay above (or below) the run mean value for long periods, i.e. to volatility cluster. The parameter c_3 determines the random behaviour of the volatility: for $c_3 = 0$ the volatility process is deterministic, for $c_3 > 0$ the kurtosis of log-return distributions is larger than 3. When $c_1 = c_2 = 0$, the GARCH diffusion process reduces to the log-normal process without drift in the Hull and White (1987) model.

Given $V_0 > 0$, V_t is positive $\mathbb{P}$-almost surely $\forall t \geq 0$, and the strong solution is

$$V_t = V_0 \, e^{-(c_2 + \frac{1}{2}c_3^2)t + c_3 W_t} + \; c_1 \int_0^t e^{(c_2 + \frac{1}{2}c_3^2)(s-t) + c_3(W_t - W_s)} ds, \tag{3}$$

see Karatzas and Shreve (1991), p. 360. The stationary distribution of V is the Inverse Gamma distribution (cf. Nelson (1990)) with parameters $1 + 2c_2/c_3^2$ and $c_3^2/2c_1$, i.e. $1/V_t \rightsquigarrow \Gamma(1 + 2c_2/c_3^2, c_3^2/2c_1)$. Hence V_t has finite moments up to order r if and only if $r < 1 + 2c_2/c_3^2$. This implies that log-return distributions have finite unconditional moments up to order $2r$. Empirical studies showed that log-return distributions have finite moments up to some given order. Moreover, when $c_3^2 \to 2c_2^+$, the kurtosis of log-return distributions tends to infinity and the correlation between squared log-returns approaches to 1/3. By contrast, when the variance follows a square root process (cf. Heston (1993)) the corresponding stationary Gamma distribution implies log-return distributions with finite unconditional moments of any order, excess kurtosis at most equal to 3 and autocorrelation of squared log-returns at most 1/5; cf. Genon-Catalot, Jeantheau and Laredo (2000).

When $2c_2 > c_3^2$, the V process is strictly stationary, ergodic with conditional mean and variance

$$\mathbb{E}[V_t \mid V_0] = \frac{c_1}{c_2} + (V_0 - \frac{c_1}{c_2}) \, e^{-c_2 t}, \tag{4}$$

$$\begin{aligned} \mathbb{V}ar[V_t \mid V_0] \quad = \quad & \frac{(c_1/c_2)^2}{2c_2/c_3^2 - 1} + \; e^{-c_2 t} \, \frac{2(c_1/c_2)(V_0 - (c_1/c_2))}{c_2/c_3^2 - 1} - \; e^{-2c_2 t} \, (V_0 - (c_1/c_2))^2 \\ & + e^{(c_3^2 - 2c_2)t} \left(V_0^2 - \frac{2V_0(c_1/c_2)}{1 - c_3^2/c_2} + \frac{(c_1/c_2)^2}{(1 - c_3^2/2c_2)(1 - c_3^2/c_2)} \right). \end{aligned} \tag{5}$$

The unconditional expectation of (4) and (5) give the unconditional mean and variance of V

$$\mathbb{E}[V_1] = \frac{c_1}{c_2}, \quad \mathbb{V}ar[V_1] = \frac{(c_1/c_2)^2}{2c_2/c_3^2 - 1}. \tag{6}$$

In Section 4 we will infer some reasonable parameters for the variance process using equations (6). Higher order unconditional moments of V can be derived by the stationary Inverse Gamma distribution.

In the following section we will derive an analytical approximation formula for European options when the underlying currency price satisfies equations (1)–(2).

3 The Option Pricing Formula

Given the model (1)–(2), a foreign currency option price $f(S,V,t)$ satisfies the following partial differential equation

$$\frac{1}{2}\,V S^2\,\frac{\partial f^2}{\partial S^2}+\frac{1}{2}\,c_3^2\,V^2\,\frac{\partial f^2}{\partial V^2}+(r_d-r_f)\,S\,\frac{\partial f}{\partial S}+((c_1-c_2)\,V-\lambda(S,V,t))\frac{\partial f}{\partial V}-(r_d-r_f)\,f+\frac{\partial f}{\partial t}=0,$$

where r_d and r_f are the domestic and the foreign interest rates, respectively, and the unspecified term $\lambda(S,V,t)$ represents the market price of risk associated to the variance V. As V is not a traded asset, arbitrage arguments are not enough to determine the option price $f(S,V,t)$. To specify $\lambda(S,V,t)$ one introduces an equilibrium market model to set investor risk preferences. As in Chesney and Scott (1989) and Heston (1993) we can specify the volatility risk premium $\lambda(V,S,t)=\lambda V$ or as an affine function of V, $\lambda(V)=c+\lambda V$. In both cases, the risk-adjusted process is still a GARCH diffusion process

$$dS_t=(r_d-r_f)\,S_t\,dt+\sqrt{V_t}\,S_t\,dB_t^*, \tag{7}$$

$$dV_t=(c_1^*-c_2^*\,V_t)dt+c_3\,V_t\,dW_t^*, \tag{8}$$

where $c_1^*=c_1-c$, $c_2^*=c_2+\lambda$, B^* and W^* are mutually independent Brownian motions under the risk-adjusted measure $\mathbb{P}^*$.

For the risk-adjusted dynamics in equations (7)–(8) the notable option pricing result in Hull and White (1987) holds: the fair price value C_{sv} for a European call with time to maturity T and strike price K is given by

$$C_{sv}=\int_0^\infty C_{bs}(\overline{V}_T)f(\overline{V}_T\mid V_0)\,d\overline{V}_T, \tag{9}$$

where C_{bs} is the Black and Scholes (1973) option price, $\overline{V}_T$ is the integrated variance over the time to maturity T, i.e.

$$\overline{V}_T:=\frac{1}{T}\int_0^T V_t\,dt \tag{10}$$

and $f(\overline{V}_T\mid V_0)$ is the conditional density function of $\overline{V}_T$ given V_0. The integrated variance density $f(\overline{V}_T\mid V_0)$ is not known and the option price C_{sv} is not available in closed-form. The expectation in equation (9) can be computed by Monte Carlo simulation but such a procedure is very time-consuming. Hull and White (1987) provided an analytical approximation for C_{sv} in (9). Precisely, they computed the Taylor expansion of C_{bs} in equation (9) around the conditional mean of $\overline{V}_T$ obtaining a series option pricing formula that involves only the conditional moments of $\overline{V}_T$ and the sensitivities of the Black and Scholes price to the variance. Denoting by $M_1:=$

$\mathbb{E}[\overline{V}_T \mid V_0]$ the conditional mean of $\overline{V}_T$ and $M_{ic} := \mathbb{E}[(\overline{V}_T - M_1)^i \mid V_0]$ $i \geq 2$ the i-th centered conditional moment of $\overline{V}_T$, the option pricing series is

$$C_{sv} = C_{bs}(M_1) + \frac{1}{2} M_{2c} \left. \frac{\partial^2 C_{bs}}{\partial \overline{V}_T^2} \right|_{\overline{V}_T = M_1} + \frac{1}{6} M_{3c} \left. \frac{\partial^3 C_{bs}}{\partial \overline{V}_T^3} \right|_{\overline{V}_T = M_1} + \frac{1}{24} M_{4c} \left. \frac{\partial^4 C_{bs}}{\partial \overline{V}_T^4} \right|_{\overline{V}_T = M_1} + ..., \quad (11)$$

where the derivatives are

$$\frac{\partial C_{bs}}{\partial \overline{V}_T} = \frac{e^{-r_f T} S_0 \sqrt{T} e^{-d_1^2/2}}{\sqrt{8\pi \overline{V}_T}}, \quad (12)$$

$$\frac{\partial^2 C_{bs}}{\partial \overline{V}_T^2} = \frac{\partial C_{bs}}{\partial \overline{V}_T} \left[\frac{1}{2} \frac{m^2}{(\overline{V}_T T)^2} - \frac{1}{2 \overline{V}_T T} - \frac{1}{8} \right] T,$$

$$\frac{\partial^3 C_{bs}}{\partial \overline{V}_T^3} = \frac{\partial C_{bs}}{\partial \overline{V}_T} \left[\frac{m^4}{4(\overline{V}_T T)^4} - \frac{m^2 (12 + \overline{V}_T T)}{8(\overline{V}_T T)^3} + \frac{48 + 8\overline{V}_T T + (\overline{V}_T T)^2}{64(\overline{V}_T T)^2} \right] T^2,$$

$$\frac{\partial^4 C_{bs}}{\partial \overline{V}_T^4} = \frac{\partial C_{bs}}{\partial \overline{V}_T} \left[\frac{1}{8} \frac{m^6}{(\overline{V}_T T)^6} - \frac{3}{32} \frac{m^4 (20 + \overline{V}_T T)}{(\overline{V}_T T)^5} + \frac{3}{128} \frac{m^2 (240 + 24\overline{V}_T T + (\overline{V}_T T)^2)}{(\overline{V}_T T)^4} - \frac{(960 + 144\overline{V}_T T + 12(\overline{V}_T T)^2 + (\overline{V}_T T)^3)}{512(\overline{V}_T T)^3} \right] T^3,$$

and $m := \log(S_0/K) + (r_d - r_f)T$. So far, the conditional moments of the integrated variance have been calculated analytically only for few specifications of the variance process

1. for the mean reverting Ornstein-Uhlenbeck process[6] Cox and Miller (1972, Sec. 5.8) derived the first two conditional moments of $\overline{V}_T$;

2. for the geometric Brownian motion with drift Hull and White (1987) derived the first two conditional moments of $\overline{V}_T$ and the first three conditional moments of $\overline{V}_T$ for the variance process without drift;

3. for the squared root process Bollerslev and Zhou (2002) derived the first two conditional moments. Lewis (2000a) derived the first four conditional moments of the integrated variance for the general class of affine processes (including the squared root process).

Given the analytical conditional moments of $\overline{V}_T$ it is very easy to price European options by the series approximation (11). Garcia, Lewis and Renault (2001) use this formula to price European options under the Heston model notwithstanding the Heston option pricing formula; cf. also Ball and Roma (1994). Indeed, implementing integral solutions for option prices, such as the Heston formula, can be very delicate due to divergence of the integrand in some regions of the parameter space.

[6] We recall that the mean reverting Ornstein-Uhlenbeck process is normally distributed and then can not ensure positive variance.

We derive the first four conditional moments of $\overline{V}_T$ when the variance V is driven by the GARCH diffusion process (2). The first conditional moment is already known in the literature. The second, the third and the fourth are believed to be new. Higher order moments are essential to capture the 'smile' effect of implied volatilities; cf., for instance, Bodurtha and Courtadon (1987) for PHLX foreign currency options and Lewis (2000). We denote these conditional moments by M_1^{gd}, M_{2c}^{gd}, M_{3c}^{gd} and M_{4c}^{gd}. Here we state M_1^{gd}, M_{2c}^{gd} and the calculations are given in Appendix A. The third and the fourth conditional moments are more involved and are available from the authors on request.

Proposition 3.1 Let $V = (V_t)_{t\geq 0}$ to satisfy the stochastic differential equation (2). Given $(V_0, c_1) \in \mathbb{R}^+ \times \mathbb{R}^+$ and $c_2 > c_3^2$, the first and the second conditional moment of the integrated variance $\overline{V}_T$ are

$$M_1^{gd} := \mathbb{E}[\overline{V}_T|V_0] = \frac{c_1}{c_2} + (V_0 - \frac{c_1}{c_2})\,\frac{1-e^{-c_2 T}}{c_2 T}, \tag{13}$$

$$\begin{aligned} M_{2c}^{gd} := \mathbb{E}[(\overline{V}_T - M_1^{gd})^2|V_0] = & -\frac{e^{-2\,T\,c_2}\,(c_2\,V_0 - c_1)^2}{T^2\,c_2^4} \\ & +\frac{2\,e^{(c_3^2-2\,c_2)\,T}\,(2\,c_1^2 + 2\,c_1\,(c_3^2 - 2\,c_2)\,V_0 + (2\,c_2^2 - 3\,c_2\,c_3^2 + c_3^4)\,V_0^2)}{T^2\,(c_2 - c_3^2)^2\,(2\,c_2 - c_3^2)^2} \\ & -\frac{c_3^2\,(c_1^2\,(4\,c_2\,(3 - T\,c_2) + (2\,T\,c_2 - 5)\,c_3^2) + 2\,c_1\,c_2\,(-2\,c_2 + c_3^2)\,V_0 + c_2^2\,(-2\,c_2 + c_3^2)\,V_0^2)}{T^2\,c_2^4\,(-2\,c_2 + c_3^2)^2} \\ & +\frac{2\,e^{-T\,c_2}\,c_3^2\,(2\,c_1^2\,(T\,c_2^2 - (1 + T\,c_2)\,c_3^2) + 2\,c_1\,c_2^2\,(1 - T\,c_2 + T\,c_3^2)\,V_0 + c_2^2\,(c_3^2 - c_2)\,V_0^2)}{T^2\,c_2^4\,(c_2 - c_3^2)^2}. \end{aligned} \tag{14}$$

These moments are obtained using properties of Brownian motion such as independence and stationarity of non-overlapping increments and the linearity of dV_t in V_t. As already observed, for $c_1 = 0$ the GARCH diffusion process reduces to the log-normal process with drift and then M_1^{gd}, M_{2c}^{gd} reduce to the conditional mean and variance of $\overline{V}_T$ in Hull and White (1987), p. 287.

Given the first four conditional moments of $\overline{V}_T$, under the GARCH diffusion model the call price is

$$\widetilde{C}^{gd} = C_{bs}(M_1^{gd}) + \frac{1}{2}\,M_{2c}^{gd}\,\frac{\partial^2 C_{bs}}{\partial \overline{V}_T^2}\bigg|_{\overline{V}_T = M_1^{gd}} + \frac{1}{6}\,M_{3c}^{gd}\,\frac{\partial^3 C_{bs}}{\partial \overline{V}_T^3}\bigg|_{\overline{V}_T = M_1^{gd}} + \frac{1}{24}\,M_{4c}^{gd}\,\frac{\partial^4 C_{bs}}{\partial \overline{V}_T^4}\bigg|_{\overline{V}_T = M_1^{gd}}. \tag{15}$$

Although M_1^{gd}, M_{2c}^{gd}, M_{3c}^{gd} and M_{4c}^{gd} are rather nasty, the closed-form approximation formula (15) can be easily implemented in any software package (such as Excel spread sheets) providing option prices by just plugging in model parameters without any computational efforts.

As we will show in the next section, our approximation formula (15) is very accurate for a large set of reasonable parameters. Intuitively, when the time to maturity T is 'short', $\overline{V}_T$ is not too far from $M_1^{gd} := \mathbb{E}[\overline{V}_T|V_0]$, then we expect approximation (15) to converge quickly. When the time to maturity T increases, M_1^{gd} tends to the run mean value of V, $\mathbb{E}[M_1^{gd}] = c_1/c_2$, and M_{2c}, M_{3c} and M_{4c} go to zero. Therefore, we expect the approximation formula (15) to work well also for long maturities. By contrast, in the Hull and White (1987) model, where the variance V_t follows a log-normal process without drift, M_{2c} and M_{3c} tend to infinity when T increases and the series (11) fails to give the right price; cf. Hull and White (1987) and Gesser and Poncet (1997). The effect of moving to a mean reverting process from a log-normal process is to avoid that the variance explodes or goes to zero when T increases.

Lewis (2000) derived a closed-form approximation for European option prices for a large class of stochastic volatility models including the GARCH diffusion model (7)–(8). Lewis's approximation formula for European option prices is based on second order Taylor expansion of some complex integrals around $c_3 = 0$; see Lewis (2000), p. 77–84. When $c_3 = 0$, $\overline{V}_T$ is deterministic and equals to M_1^{gd}. Indeed, it can be shown that Lewis's approximation is a particular case of our approximation (15) and is obtained by (15) neglecting terms $o(c_3^2)$. Therefore, for the GARCH diffusion model, our approximation is more accurate than the Lewis's one.

In the following section, by Monte Carlo simulations we study the accuracy of the approximation formula (15).

4 Monte Carlo Simulations

In order to verify the accuracy of the approximation (15) we compute by Monte Carlo simulations European option prices. The advantage of using Monte Carlo estimates is that the standard error of estimates is known. Precisely, we compute put option prices[7] implied by (9) using the conditional Monte Carlo method; cf. Boyle, Bradie and Glasserman (1997).

Specifically, we divide the time interval $[0, T]$ into s equal subintervals and we draw s independent standard normal variables $(\upsilon_i)_{i=1,\dots,s}$. We simulate the random variable V_t in (8) over the discrete time iT/s, for $i = 1, \dots, s$, using the Milstein scheme (cf. Kloeden and Platen (1992))

$$V_i = c_1^* \Delta t + V_{i-1}(1 - c_2^* \Delta t + c_3 \sqrt{\Delta t}\, \upsilon_i) + \frac{1}{2} c_3^2 V_{i-1}^2 ((\sqrt{\Delta t}\, \upsilon_i)^2 - \Delta t),$$

where $\Delta t := T/s$. Then, we compute the Black and Scholes put option price $P_{bs}^{(n)}$ with squared

[7]Monte Carlo standard errors are generally smaller for put option prices than for call option prices as in the first case payoffs are bounded. Using the put-call parity call option prices are readily computed.

volatility $s^{-1}\sum_{i=1}^{s} V_i$. Finally, iterating this procedure N times we obtain the Monte Carlo estimate for the put option price

$$P_{mc} := N^{-1}\sum_{n=1}^{N} P_{bs}^{(n)},$$

with the corresponding Monte Carlo standard error

$$e_{mc} := \frac{\sqrt{N^{-1}\sum_{n=1}^{N}(P_{bs}^{(n)} - P_{mc})^2}}{\sqrt{N}}.$$

When N goes to infinity, P_{mc} converges in probability to the put option price implied by (9). Notice that we do not need to simulate the price process S.

To simulate the variance process (8) we use parameter values inferred from empirical estimates of model (7)–(8). Typically, for currency and index daily log-returns the unconditional mean of V, c_1^*/c_2^*, ranges from 0.01 to 0.1 per year. The 'half life'[8] varies from few days to about a half year; cf. Chesney and Scott (1989), Taylor and Xu (1994), Xu and Taylor (1994), Guo (1996, 1998) and Fouque, Papanicolaou and Sircar (2000). This implies that c_2^* ranges from 1 to 40. Moreover, empirical estimates of discrete GARCH(1,1) model on currency and index daily log-returns imply values of c_3 ranging from about 1 to 4; cf., for instance, Hull and White (1987a, 1988) and Guo (1996,1998). For stock log-returns, estimates of c_3 are generally smaller.

For the Monte Carlo simulations, we consider time to maturities for European put options ranging from 30 to 504 days. We wrote a Matlab code to run $N = 10^6$ simulations. The computation time goes from about 14 hours for $T = 30$ days to 15 hours for $T = 504$ days on a PC Pentium IV 1GHz, running Windows XP.

In Table 1 we simulate the risk-adjusted variance process (8) using parameter values that we infer (cf. Nelson (1990)) from the GARCH(1,1) estimates in Guo (1996) for the dollar/yen exchange rates[9], i.e. $c_1^* = 0.16$, $c_2^* = 18$ and $c_3 = 1.8$. The variance process is quickly mean reverting (the half life is about 10 days) and rather volatile, the two-standard deviation range for V is from 0.003 to 0.016; see equations (6). Table 1 shows the Monte Carlo put price P_{mc}, the put price $\widetilde{P}^{gd}$ given by the series approximation (15), the pricing error $e_p\%$ defined as $e_p\% := 100\times(P_{mc}-\widetilde{P}^{gd})/P_{mc}$ and the Monte Carlo standard error e_{mc}. All errors are practically negligible across all strikes and maturities and the average error is -0.025%. Although the variance process is rather volatile, the high mean reversion rate c_2^* implies that the integrated variance process $\overline{V}_T$ tends to stay around $\mathbb{E}[\overline{V}_T|V_0]$ and then the approximation (15) works well.

[8] The 'half life' is the time necessary after a shock to halve the deviation of V_t from its run mean value, given that there are no more shocks. For this model the half life is equal to $252 \times \ln(2)/c_2^*$ days.

[9] As in Guo (1996) we assume the volatility risk premium $\lambda(S, V, t) = 0$.

In Table 2 we simulate the variance process (8) using the risk-neutral parameters estimated by Melenberg and Werker (2001) for the Dutch EOE index. The volatility risk premium was estimated using European call options on the Dutch index. The correlation between price and volatility was negligible. The risk-neutral coefficients are $c_1^* = 0.53$, $c_2^* = 29.23$ and $c_3 = 3.65$. The run mean value of the variance is 0.018 and the two-standard deviation range for V is 0–0.03. Table 2 is organized as Table 1. Also in this case pricing errors $e_p\%$ are almost always lower than 1% (except for one case). The average error is -0.011%.

In Table 3 and Table 4 we use parameter values that give reasonable variance process as discussed in Hull and White (1988). In Table 3 we set $c_1^* = 0.18$, $c_2^* = 2$ and $c_3 = 0.8$. The parameter value c_2^* is quite small and implies a 'slow' mean reverting variance process (8), the half life of about 88 days. The unconditional mean and standard deviation of V are 0.09 and 0.03, respectively, and the two-standard deviation range for V is 0.01–0.16. As the volatility of V_t is not too large, the process $\overline{V}_T$ tends to stay around $\mathbb{E}[\overline{V}_T|V_0]$ and hence the series approximation (15) is very accurate. The average pricing error is -0.044%. In Table 4 we set c_1^* and c_2^* as in Table 3 and $c_3 = 1.2$. This implies that the standard deviation of V is 0.06 and the two-standard deviation range for V is 0–0.22. Table 4 shows that pricing errors $e_p\%$ are still very small (the average pricing error is -0.2%) but slightly larger than in Table 3 as the variance process is more volatile than in the previous case.

Finally, in Table 5 we set $c_1^* = 0.09$, $c_2^* = 4$ and $c_3 = 1.2$ as in Lewis (2000). The unconditional mean of V is 0.02, the 'half life' is about 43 days and the two-standard deviation range for V is 0.001–0.04. Also in this case errors $e_p\%$ are generally quite small and the average pricing error is -0.018%.

We simulate the variance process (8) also for other reasonable parameter choices (not reported here) and we found similar results. The approximation formula (15) induces pricing errors less than 1% for at the money options and less than 2% for out of the money options. Bid-ask spreads on currency option prices are larger than 2% of the prices for out of the money options and about 1% for more liquid, at the money currency options. Then, the approximation formula (15) gives accurate prices within the tolerance imposed by market frictions.

5 The Implied Volatility Surface

In this section we study the implied volatility induced by the GARCH diffusion model (7)–(8), i.e. the volatility σ^2_{imp} which gives the Black and Scholes option price equals to the GARCH diffusion option price, $C_{bs}(\sigma^2_{imp}) = \widetilde{C}^{gd}$. Typically, to solve the implicit equation $C_{bs}(\sigma^2_{imp}) = \widetilde{C}^{gd}$ the

Newton-Raphson method is used[10]. Hence, we propose to compute σ^2_{imp} as

$$\sigma^2_{imp} = M_1^{gd} + \frac{\widetilde{C}^{gd} - C_{bs}(M_1^{gd})}{\partial C_{bs}/\partial \overline{V}_T\big|_{\overline{V}_T = M_1^{gd}}}, \tag{16}$$

that is a one-step Newton-Raphson algorithm starting at M_1^{gd}. As $\sigma^2_{imp} \to M_1^{gd}$ when $T \to \infty$, M_1^{gd} is a sensible starting point for the algorithm and one iteration gives very accurate results[11]. Given $\widetilde{C}^{gd}$ implementing (16) is straightforward and the model (7)–(8) can be easily calibrated to the market implied volatilities.

Renault and Touzi (1996) show that, for *any* stochastic volatility process, the assumption of no correlation between price and variance induces symmetric 'volatility smiles', i.e. symmetric shape with respect to the forward price of the implied volatility plotted as a function of the strike price; cf. also Hull and White (1987). The functional dependence of implied volatilities on time to maturities, i.e. the 'term structure patterns', depends on the specific variance process. In the following we qualitatively study the volatility smile and the term structure pattern induced by the GARCH diffusion model. As in Table 5 we set $c_1^* = 0.09$, $c_2^* = 4$ and $c_3 = 1.2$ and we compute the GARCH diffusion option prices (15). Then, by formula (16) we obtain the implied volatilities for different strikes and maturities. Figure 1 shows volatility smiles for time to maturities equal to 30, 60, 90 and 120 days. Figure 2 shows the volatility surface for time to maturity between 0 and 120 days and strike prices between 90 and 110. As expected, volatility smiles are quite symmetric with respect to the forward price. Moreover, the convexity of the volatility surface increases when the time to maturity decreases. These features of implied volatility surface were observed for all parameter choices (positive parameters). When the time to maturity increases the volatility surface flattens because the random variable $\overline{V}_T$ converges to the the run mean value c_1^*/c_2^* by the Ergodic theorem and $\sigma^2_{imp} \to c_1^*/c_2^*$ for all strike prices. These results are in qualitative agreement with the empirical evidence on volatility surfaces observed in currency option markets, where volatility smiles are quite symmetric with respect to the forward price, very pronounced at short maturities and almost flat for long maturities; cf., for instance, Chesney and Scott (1989), Melino and Turbull (1990), Taylor and Xu (1994), and Bollerslev and Zhou (2002).

[10] See for instance the Matlab function `blsimpvdiv` and the Mathematica function `BlackScholesCallImpVol`.

[11] We compared implied volatilities given by (16) with implied volatilities returned by the Matlab function `blsimpvdiv` and the errors were less than 0.01% for all the parameters used in Section 4.

6 Conclusions

We derive analytically the first four conditional moments of the integrated variance under the GARCH diffusion model. Using these conditional moments we obtain an analytical closed-form approximation formula $\widetilde{C}^{gd}$ (15) which allows us to price European options under the GARCH diffusion model. This formula can be easily implemented in any software package and provides option prices without any computational efforts. Monte Carlo simulations show that this approximation is accurate across different strikes and maturities for a large set of reasonable parameters. Finally, using the approximation formula (15) we study implied volatility surfaces induced by GARCH diffusion models. We find that volatility smiles and term structure patterns of implied volatilities are in qualitative agreement with volatility surfaces typically observed in the foreign exchange option markets.

A Proof of Proposition 3.1

In the following, we derive the first two conditional moments of the integrated variance $\overline{V}_T$ for the GARCH diffusion process,

$$\overline{V}_T = \frac{V_0}{T}\int_0^T dt\, e^{-(c_2+\frac{1}{2}\,c_3^2)\,t}\, e^{c_3 W_t} + \frac{c_1}{T}\int_0^T dt \int_0^t ds\, e^{(c_2+\frac{1}{2}\,c_3^2)(s-t)}\, e^{c_3\,(W_t-W_s)}. \tag{17}$$

To prove Proposition 3.1 we recall that, if w is a normal random variable $w \sim \mathcal{N}(0,t)$

$$\mathbb{E}[e^{\lambda w}] = e^{\frac{\lambda^2\,t}{2}}. \tag{18}$$

We also need the following lemma

Lemma A.1

$\forall\ x > y > 0$,

$$F(x,y) = e^{-\left(c_2+\frac{1}{2}\,c_3^2\right)(x+y)}\,\mathbb{E}\big[e^{c_3(W_x+W_y)}\big] = e^{-c_2\,x} e^{(c_3^2-c_2)y}. \tag{19}$$

$\forall\ x > y > \alpha > 0$,

$$G(x,y,\alpha) = e^{-\left(c_2+\frac{1}{2}\,c_3^2\right)(x+y-\alpha)}\,\mathbb{E}\big[e^{c_3(W_x+W_y-W_\alpha)}\big] = e^{-c_2\,x} e^{(c_3^2-c_2)y} e^{(c_2-c_3^2)\alpha}. \tag{20}$$

$\forall\ x > \alpha > y > 0$,

$$H(x,y,\alpha) = e^{-\left(c_2+\frac{1}{2}\,c_3^2\right)(x+y-\alpha)}\,\mathbb{E}\big[e^{c_3(W_x+W_y-W_\alpha)}\big] = e^{-c_2\,(x+y-\alpha)}. \tag{21}$$

$\forall\ x > y > \alpha > \beta > 0$,

$$L(x,y,\alpha,\beta) = e^{-\left(c_2+\frac{1}{2}c_3^2\right)(x+y-\alpha-\beta)}\mathbb{E}\big[\,e^{c_3\left(W_x+W_y-W_\alpha-W_\beta\right)}\,\big] = e^{-c_2\,x} e^{(c_3^2-c_2)y} e^{(c_2-c_3^2)\alpha} e^{c_2\beta}. \tag{22}$$

$\forall\ x > \alpha > y > \beta > 0$,

$$M(x,y,\alpha,\beta) = e^{-\left(c_2+\frac{1}{2}c_3^2\right)(x+y-\alpha-\beta)}\mathbb{E}\big[\,e^{c_3\left(W_x+W_y-W_\alpha-W_\beta\right)}\,\big] = e^{-c_2\,(x+y-\alpha-\beta)}. \tag{23}$$

Proof. To prove (19) we write $W_x + W_y = (W_x - W_y) + 2\,W_y$. As $(W_x - W_y)$ and W_y are non-overlapping increments of the Brownian motion W, $(W_x - W_y) \sim \mathcal{N}(0, x-y)$ and $2W_y \sim \mathcal{N}(0,4y)$ one has

$$\mathbb{E}[e^{c_3(W_x+W_y)}] = \mathbb{E}[e^{c_3\,(W_x-W_y)+2W_y}] = \mathbb{E}[e^{c_3\,(W_x-W_y)}]\,\mathbb{E}[e^{c_3\,2W_y}],$$

then formula (19) follows directly from (18).

To prove (20), use $W_x + W_y - W_\alpha = (W_x - W_y) + 2\,(W_y - W_\alpha) + W_\alpha$.

To prove (21), use $W_x + W_y - W_\alpha = (W_x - W_\alpha) + W_y$.

To prove (22), use $W_x + W_y - W_\alpha - W_\beta = (W_x - W_y) + 2\,(W_y - W_\alpha) + W_\alpha - W_\beta$.

To prove (23), use $W_x + W_y - W_\alpha - W_\beta = (W_x - W_\alpha) + (W_y - W_\beta)$. □

A.1 First conditional moment

The first conditional moment of $\overline{V}_T$ is given by

$$\begin{aligned} M_1^{gd} &:= \mathbb{E}[\overline{V}_T \mid V_0] \\ &= \frac{V_0}{T}\int_0^T dt\; e^{-(c_2+\frac{1}{2}c_3^2)\,t}\;\mathbb{E}[e^{c_3\,W_t}] + \frac{c_1}{T}\int_0^T dt\;\int_0^t ds\; e^{(c_2+\frac{1}{2}c_3^2)\,(s-t)}\,\mathbb{E}[e^{c_3\,(W_t-W_s)}]. \end{aligned}$$

As $W_t \sim \mathcal{N}(0,t)$ and $W_t - W_s \sim \mathcal{N}(0, t-s)$, using (18) we get the first conditional moment (13) in Proposition 3.1,

$$\begin{aligned} M_1^{gd} &:= \frac{V_0}{T}\int_0^T dt\, e^{-(c_2+\frac{1}{2}c_3^2)t}\, e^{\frac{1}{2}c_3^2\,t} + \frac{c_1}{T}\int_0^T dt\int_0^t ds\; e^{(c_2+\frac{1}{2}c_3^2)(s-t)}\, e^{\frac{1}{2}c_3^2(t-s)} \\ &= V_0\int_0^T e^{-c_2\,t}\,dt + c_1\int_0^T dt\int_0^t ds\; e^{c_2(s-t)} = \frac{V_0}{c_2}\left(\frac{1-e^{-c_2T}}{T}\right) + \frac{c_1}{T}\int_0^T \frac{1-e^{-c_2\,t}}{c_2}dt \\ &= \frac{c_1}{c_2} + \left(V_0 - \frac{c_1}{c_2}\right)\frac{1-e^{-c_2T}}{c_2T}. \end{aligned}$$

A.2 Second conditional moment

The second conditional moment of $\overline{V}_T$ is given by

$$\begin{aligned} \mathbb{E}[\overline{V}_T^2 \mid V_0] &= \mathbb{E}\left[\frac{1}{T^2}\int_0^T dr_2\int_0^T dr_1\,(V_{r_1}\,V_{r_2})\right] = \frac{1}{T^2}\int_0^T dr_2\int_0^T dr_1\,\mathbb{E}[V_{r_1}\,V_{r_2}] \\ &= \frac{2!}{T^2}\int_0^T dr_2\int_0^{r_2} dr_1\,\mathbb{E}[V_{r_1}\,V_{r_2}] \\ &= \frac{2!}{T^2}\int_0^T dr_2\int_0^{r_2} dr_1\,\big(\mathbb{E}[A]+\mathbb{E}[B]+\mathbb{E}[C]+\mathbb{E}[D]\big), \end{aligned} \tag{24}$$

where

$$\begin{aligned} A &:= V_0^2\, e^{-\left(c_2+\frac{1}{2}c_3^2\right)(r_1+r_2)+c_3\left(W_{r_1}+W_{r_2}\right)}, \\ B &:= c_1\,V_0\,e^{-\left(c_2+\frac{1}{2}c_3^2\right)r_1+c_3\,W_{r_1}}\int_0^{r_2} ds_2\,e^{\left(c_2+\frac{1}{2}c_3^2\right)(s_2-r_2)+c_3\left(W_{r_2}-W_{s_2}\right)}, \\ C &:= c_1\,V_0\,e^{-\left(c_2+\frac{1}{2}c_3^2\right)r_2+c_3W_{r_2}}\int_0^{r_1} ds_1\,e^{\left(c_2+\frac{1}{2}c_3^2\right)(s_1-r_1)+c_3\left(W_{r_1}-W_{s_1}\right)}, \\ D &:= c_1^2\int_0^{r_1} ds_2\int_0^{r_2} ds_1\,e^{\left(c_2+\frac{1}{2}c_3^2\right)(s_1-r_1+s_2-r_2)}e^{c_3\left(W_{r_1}-W_{s_1}+W_{r_2}-W_{s_2}\right)}. \end{aligned}$$

We compute each addend in (24).

- Calculation of

$$\begin{aligned} &\frac{2}{T^2}\int_0^T dr_2\int_0^{r_2} dr_1\,\mathbb{E}[A] = \\ &\quad \frac{2}{T^2}\int_0^T dr_2\int_0^{r_2} dr_1\,V_0^2\,e^{-\left(c_2+\frac{1}{2}c_3^2\right)(r_1+r_2)}\mathbb{E}[e^{c_3\left(W_{r_1}+W_{r_2}\right)}]. \end{aligned}$$

As $r_2 > r_1 > 0$, we use formula (19) with $x = r_2$ and $y = r_1$

$$\frac{2}{T^2}\int_0^T dr_2 \int_0^{r_2} dr_1\, \mathbb{E}[A] = \frac{2\,V_0^2}{T^2}\int_0^T dr_2 \int_0^{r_2} dr_1\, F(r_2, r_1),$$

and iterating integrations

$$\frac{2}{T^2}\int_0^T dr_2 \int_0^{r_2} dr_1\, \mathbb{E}[A] = \frac{2\,V_0^2}{T^2}\left[\frac{e^{-(2c_2-c_3^2)T}}{(c_3^2-2c_2)(c_3^2-c_2)} + \frac{e^{-c_2 T}}{c_2\,(c_3^2-c_2)} - \frac{1}{c_2\,(c_3^2-2c_2)}\right]. \tag{25}$$

- Calculation of

$$\frac{2}{T^2}\int_0^T dr_2 \int_0^{r_2} dr_1\, \mathbb{E}[B] =$$
$$\frac{2\,c_1\,V_0}{T^2}\int_0^T dr_2 \int_0^{r_2} dr_1 \int_0^{r_1} ds_1\, e^{-\left(c_2+\frac{1}{2}c_3^2\right)(r_2+r_1-s_1)}\, \mathbb{E}[e^{c_3\left(W_{r_2}+W_{r_1}-W_{s_1}\right)}].$$

As $r_2 > r_1 > s_1 > 0$, we use formula (20) with $x = r_2$, $y = r_1$ and $\alpha = s_1$ and we get

$$\frac{2}{T^2}\int_0^T dr_2 \int_0^{r_2} dr_1\, \mathbb{E}[B] = \frac{2\,c_1\,V_0}{T^2}\int_0^T dr_2 \int_0^{r_2} dr_1 \int_0^{r_1} ds_1\, G(r_2, r_1, s_1)$$
$$= \frac{c_1\,V_0}{T^2\,c_2^4\,(c_2-c_3^2)^2\,(-2\,c_2+c_3^2)} \times$$
$$\Big[-c_2\,e^{-T\,c_2}\,(-2\,c_2+c_3^2)\,\left(c_2^2\,(-2+T\,c_2)+2\,c_2\,c_3^2-(2+T\,c_2)\,c_3^4\right)$$
$$+c_2\,(c_2-c_3^2)^2\,\left(-2\,c_2\,(-1+T\,c_2)+(-2+T\,c_2)\,c_3^2\right)+2\,c_2^4\,e^{T\,(c_3^2-2\,c_2)}\Big]. \tag{26}$$

- Calculation of

$$\frac{2}{T^2}\int_0^T dr_2 \int_0^{r_2} dr_1\, \mathbb{E}[C].$$

Simply notice that

$$\int_0^T dr_2 \int_0^T dr_1\, \mathbb{E}[B] = \int_0^T dr_2 \int_0^T dr_1\, \mathbb{E}[C]. \tag{27}$$

- Calculation of

$$\frac{2}{T^2}\int_0^T dr_2 \int_0^{r_2} dr_1\, \mathbb{E}[D] =$$
$$\frac{2\,c_1^2}{T^2}\int_0^T dr_2 \int_0^{r_2} dr_1 \int_0^{r_2} ds_2 \int_0^{r_1} ds_1 \Big(e^{-\left(c_2+\frac{1}{2}c_3^2\right)(r_2+r_1-s_2-s_1)}\mathbb{E}[\,e^{c_3\left(W_{r_2}+W_{r_1}-W_{s_1}-W_{s_2}\right)}\,]\Big).$$

We divide the integration domain of s_2 and s_1 as follows

$$\frac{2}{T^2}\int_0^T dr_2 \int_0^{r_2} dr_1\, \mathbb{E}[D] =$$

$$\frac{2\ c_1^2}{T^2}\int_0^T dr_2 \int_0^{r_2} dr_1 \int_0^{r_1} ds_2 \int_0^{s_2} ds_1 \Big(...\Big) + \tag{28}$$

$$+\frac{2\ c_1^2}{T^2}\int_0^T dr_2 \int_0^{r_2} dr_1 \int_0^{r_1} ds_2 \int_{s_2}^{r_1} ds_1 \Big(...\Big) + \tag{29}$$

$$+\frac{2\ c_1^2}{T^2}\int_0^T dr_2 \int_0^{r_2} dr_1 \int_{r_1}^{r_2} ds_2 \int_0^{r_1} ds_1 \Big(...\Big). \tag{30}$$

The previous partition allows us to use

formula (22) with $x = r_2, y = r_1, \alpha = s_2, \beta = s_1$ in (28) as $T > r_2 > r_1 > s_2 > s_1 > 0$;

formula (22) with $x = r_2, y = r_1, \alpha = s_1, \beta = s_2$ in (29) as $T > r_2 > r_1 > s_1 > s_2 > 0$;

formula (23) with $x = r_2, y = r_1, \alpha = s_2, \beta = s_1$ in (30) as $T > r_2 > s_2 > r_1 > s_1 > 0$;

then

$$\begin{aligned}
\frac{2}{T^2}\int_0^T dr_2 \int_0^{r_2} dr_1\, \mathbb{E}[D] &= \\
&= \frac{2\ c_1^2}{T^2}\int_0^T dr_2 \int_0^{r_2} dr_1 \int_0^{r_1} ds_2 \int_0^{s_2} ds_1\, L(r_2, r_1, s_2, s_1) \\
&+\frac{2\ c_1^2}{T^2}\int_0^T dr_2 \int_0^{r_2} dr_1 \int_0^{r_1} ds_2 \int_{s_2}^{r_1} ds_1\, L(r_2, r_1, s_1, s_2) \\
&+\frac{2\ c_1^2}{T^2}\int_0^T dr_2 \int_0^{r_2} dr_1 \int_{r_1}^{r_2} ds_2 \int_0^{r_1} ds_1\, M(r_2, r_1, s_2, s_1),
\end{aligned}$$

and iterating integrations

$$\begin{aligned}
\frac{2}{T^2}\int_0^T \int_0^{r_2} \mathbb{E}[D]\, dr_1\, dr_2 &= \frac{c_1^2}{T^2\, c_2^4\, (c_2 - c_3^2)^2\, (-2\, c_2 + c_3^2)^2}\times \\
&\Big[-2\, e^{-T\, c_2}\, \big(-2\, c_2 + c_3^2\big)^2\, \big(c_2^2 - T\, c_2^3 - 2\, c_2\, c_3^2 + (3 + T\, c_2)\, c_3^4\big) \\
&+ \big(c_2 - c_3^2\big)^2\, \big(\big(4\, c_2^2\, (-1 + T\, c_2)^2 - 4\, c_2\, (4 + T\, c_2\, (-3 + T\, c_2))\, c_3^2 + (6 + T\, c_2\, (-4 + T\, c_2))\, c_3^4\big) + \\
&+ 4\, e^{T\, (c_3^2 - 2\, c_2)}\, c_2^4\Big].
\end{aligned} \tag{31}$$

Summing (25), (26), (27) and (31) we get the second conditional moment of $\overline{V}_T$:

$$
\begin{aligned}
M_2^{gd} := \mathbb{E}[\overline{V}_T^2 \mid V_0] \quad = \quad & \tfrac{1}{T^2\, c_2^4\, (c_2-c_3^2)^2\, (-2\, c_2+c_3^2)^2} \\
& \Big[e^{-2\,T\,c_2} \Big(-2\, e^{T\, c_2} \left(-2\, c_2 + c_3^2 \right)^2 \\
& \left(c_1^2 \left(c_2^2 - T\, c_2^3 - 2\, c_2\, c_3^2 + (3 + T\, c_2)\, c_3^4 \right) \right. \\
& +c_1\, c_2 \left(c_2^2\, (-2 + T\, c_2) + 2\, c_2\, c_3^2 - (2 + T\, c_2)\, c_3^4 \right) V_0 + \\
& \left. c_2^3 \left(c_2 - c_3^2 \right) V_0^2 \right) + e^{2\,T\,c_2} \left(c_2 - c_3^2 \right)^2 \left(c_1^2 \left(4\, c_2^2\, (-1 + T\, c_2)^2 \right. \right. \\
& \left. -4\, c_2\, (4 + T\, c_2\, (-3 + T\, c_2))\, c_3^2 + (6 + T\, c_2\, (-4 + T\, c_2))\, c_3^4 \right) + \\
& 2\, c_1\, c_2 \left(2\, c_2 - c_3^2 \right) \left(2\, c_2\, (-1 + T\, c_2) - (-2 + T\, c_2)\, c_3^2 \right) \\
& \left. V_0 + 2\, c_2^3 \left(2\, c_2 - c_3^2 \right) V_0^2 \right) + 2\, e^{T\, c_3^2}\, c_2^4 \left(2\, c_1^2 - 2\, c_1 \left(2\, c_2 - c_3^2 \right) V_0 + \right. \\
& \left. \left(2\, c_2^2 - 3\, c_2\, c_3^2 + c_3^4 \right) V_0^2 \right) \Big) \Big].
\end{aligned} \tag{32}
$$

The second central conditional moment of $\overline{V}_T$, M_{2c}^{gd}, stated in (14) Proposition 3.1 is given by $M_{2c}^{gd} = M_2^{gd} - (M_1^{gd})^2$.

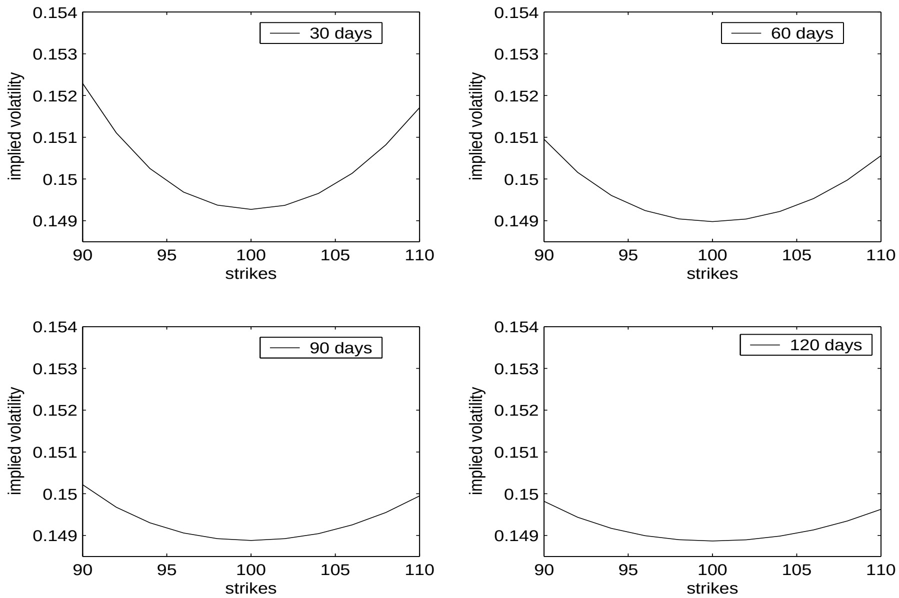

Figure 1: Volatility smiles for maturities of 30, 60, 90 and 120 days and the parameter choice $S_0 = 100$, $r = 0$, $d = 0$; $dV = (0.09 - 4\,V)dt + 1.2\,V\,dW$, $V_0 = 0.0225$, as in Table 5.

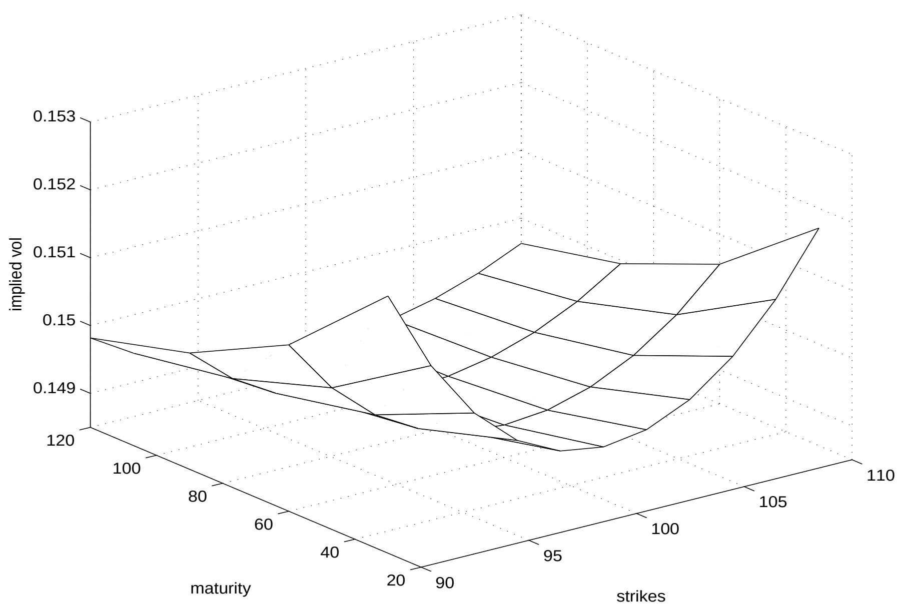

Figure 2: Volatility Surface for maturities $T \in [30, 120]$ days, strikes $K \in [90, 110]$ and the parameter choice $S_0 = 100$, $r = 0$, $d = 0$; $dV = (0.09 - 4\,V)dt + 1.2\,V dW$, $V_0 = 0.0225$, as in Table 5.

maturity	**strike**	P_{mc}	$\widetilde{P}^{gd}$	$e_p\%$	$e_{mc} \times 10^4$
	90	0.0008	0.0008	−0.6840	0.0129
	95	0.0800	0.0800	0.1025	0.3610
30 days	100	1.2921	1.2924	−0.0267	1.1997
	105	5.0999	5.0998	0.0020	0.4231
	110	10.0023	10.0024	−0.0002	0.0294
	90	0.0184	0.0184	−0.2445	0.1330
	95	0.2993	0.2992	0.0199	0.8289
60 days	100	1.8284	1.8290	−0.0281	1.5411
	105	5.3497	5.3496	0.0008	0.9215
	110	10.0363	10.0364	−0.0002	0.2224
	90	0.0664	0.0664	0.0189	0.30666
	95	0.5401	0.5401	−0.0003	1.0964
90 days	100	2.2411	2.2414	−0.0166	1.678
	105	5.6155	5.6155	−0.0003	1.1971
	110	10.1131	10.1131	0.0004	0.45367
	90	0.1393	0.1393	0.0210	0.47265
	95	0.7760	0.7760	−0.0093	1.2603
120 days	100	2.5890	2.5893	−0.0125	1.7489
	105	5.8721	5.8722	−0.0015	1.3634
	110	10.2187	10.2187	0.0004	0.65406
	90	0.3295	0.3294	0.0171	0.7371
	95	1.2163	1.2163	−0.0038	1.4470
180 days	100	3.1727	3.1729	−0.0052	1.8186
	105	6.3464	6.3465	−0.0009	1.5509
	110	10.4734	10.4734	0.0005	0.9500
	90	0.5928	0.5929	−0.0096	0.9571
	95	1.6919	1.6921	−0.0108	1.5684
252 days	100	3.7550	3.7553	−0.0071	1.8615
	105	6.8550	6.8552	−0.0029	1.1821
	110	10.8062	10.8063	−0.0007	1.1863
	90	1.5438	1.5440	−0.0101	1.3327
	95	3.0577	3.0579	−0.0076	1.7336
504 days	100	5.3116	5.3119	−0.0050	1.9141
	105	8.3052	8.3054	−0.0029	1.8354
	110	11.9443	11.9445	−0.0015	1.5584

Table 1: P_{mc} Monte Carlo put prices computed by $N = 10^6$ simulations; $\widetilde{P}^{gd}$ put prices given

maturity	**strike**	P_{mc}	$\widetilde{P}^{gd}$	$e_p\%$	$e_{mc} \times 10^4$
30 days	90	0.0236	0.0240	−1.5083	0.3503
	95	0.3093	0.3071	0.7134	1.5033
	100	1.8344	1.8389	−0.2473	2.6253
	105	5.3602	5.3580	0.0398	1.6594
	110	10.0436	10.0435	0.0016	0.5242
60 days	90	0.1508	0.1495	0.8793	0.9954
	95	0.7899	0.7898	0.0177	2.3314
	100	2.6014	2.6052	−0.1464	3.1471
	105	5.8867	5.8868	−0.0017	2.5150
	110	10.2325	10.2307	0.0176	1.3279
90 days	90	0.3449	0.3438	0.3171	1.4788
	95	1.2348	1.2355	−0.0563	2.7146
	100	3.1915	3.1942	−0.0859	3.3572
	105	6.3659	6.3668	−0.0139	2.9048
	110	10.4914	10.4903	0.0108	1.8716
120 days	90	0.5668	0.5661	0.1128	1.8267
	95	1.6394	1.6403	−0.0526	2.9359
	100	3.6891	3.6911	−0.0555	3.4717
	105	6.7987	6.7997	−0.0147	3.1284
	110	10.7723	10.7717	0.0049	2.2435
180 days	90	1.0310	1.0311	−0.0083	2.2774
	95	2.3550	2.3561	−0.0457	3.1795
	100	4.5229	4.5245	−0.0361	3.5916
	105	7.5601	7.5613	−0.0156	3.3733
	110	11.3372	11.3375	−0.0024	2.7073
252 days	90	1.5777	1.5780	−0.0217	2.6024
	95	3.0980	3.0990	−0.0314	3.3406
	100	5.3549	5.3561	−0.0239	3.6742
	105	8.3478	8.3488	−0.0126	3.5354
	110	11.9831	11.9836	−0.0042	3.0333
504 days	90	3.2715	3.2724	−0.0271	3.0858
	95	5.1528	5.1540	−0.0228	3.5523
	100	7.5754	7.5767	−0.0172	3.7759
	105	10.5187	10.5199	−0.0118	3.7472
	110	13.9310	13.9320	−0.0075	3.5050

Table 2: P_{mc} Monte Carlo put prices computed by $N = 10^6$ simulations; $\widetilde{P}^{gd}$ put prices given

maturity	**strike**	P_{mc}	$\widetilde{P}^{gd}$	$e_p\%$	$e_{mc} \times 10^4$
30 days	90	0.7901	0.7900	0.0099	1.6735
	95	2.0011	2.0010	0.0062	2.5395
	100	4.1169	4.1169	0.0022	2.9438
	105	7.1841	7.1840	0.0018	2.7010
	110	11.0473	11.0472	0.0010	2.0293
60 days	90	1.9061	1.9061	−0.0017	4.0187
	95	3.5125	3.5133	−0.0222	5.0139
	100	5.8083	5.8095	−0.0199	5.4663
	105	8.7862	8.7871	−0.0097	5.3017
	110	12.3644	12.3646	−0.0015	4.6523
90 days	90	2.8985	2.8990	−0.0155	6.0747
	95	4.7109	4.7128	−0.0390	7.0986
	100	7.1015	7.1040	−0.0346	7.5819
	105	10.0522	10.0542	−0.0198	7.4915
	110	13.5044	13.5052	−0.0059	6.9242
120 days	90	3.7858	3.7873	−0.0410	7.8285
	95	5.7337	5.7372	−0.0606	8.8537
	100	8.1892	8.1935	−0.0523	9.3602
	105	11.1311	11.1348	−0.0334	9.3346
	110	14.5125	14.5147	−0.0148	8.8543
180 days	90	5.3378	5.3424	−0.0868	10.6045
	95	7.4645	7.4717	−0.0963	11.6077
	100	10.0110	10.0192	−0.0824	12.1468
	105	12.9552	12.9628	−0.0590	12.2263
	110	16.2621	16.2678	−0.0351	11.9015
252 days	90	6.9387	6.9458	−0.1024	13.0138
	95	9.2060	9.2158	−0.1065	13.9836
	100	11.8297	11.8406	−0.0924	14.5487
	105	14.7891	14.7995	−0.0702	14.7208
	110	18.0562	18.0646	−0.0469	14.5415
504 days	90	11.3404	11.3521	−0.1031	17.3351
	95	13.8865	13.9001	−0.0982	18.2109
	100	16.6835	16.6980	−0.0871	18.8137
	105	19.7148	19.7292	−0.0730	19.1580
	110	22.9623	22.9757	−0.0581	19.2664

Table 3: P_{mc} Monte Carlo put prices computed by $N = 10^6$ simulations; $\widetilde{P}^{gd}$ put prices given

maturity	**strike**	P_{mc}	$\widetilde{P}^{gd}$	$e_p\%$	$e_{mc} \times 10^4$
30 days	90	0.7901	0.7894	0.0882	2.5219
	95	1.9925	1.9924	0.0036	3.8025
	100	4.1038	4.1043	−0.0133	4.4041
	105	7.1747	7.1746	0.0004	4.0438
	110	11.0457	11.0450	0.0067	3.0519
60 days	90	1.8942	1.8934	0.0395	6.0288
	95	3.4865	3.4906	−0.1182	7.4909
	100	5.7761	5.7826	−0.1131	8.1602
	105	8.7583	8.7629	−0.0518	7.9202
	110	12.3481	12.3483	−0.0011	6.9710
90 days	90	2.8713	2.8737	−0.0855	9.0975
	95	4.6673	4.6791	−0.2543	10.5933
	100	7.0509	7.0669	−0.2268	11.3051
	105	10.0057	10.0186	−0.1283	11.1786
	110	13.4706	13.4754	−0.0353	10.3598
120 days	90	3.7432	3.7529	−0.2592	11.7084
	95	5.6738	5.6972	−0.4126	13.1994
	100	8.1220	8.1511	−0.3588	13.9428
	105	11.0676	11.0927	−0.2267	13.9152
	110	14.4615	14.4753	−0.0955	13.2316
180 days	90	5.2682	5.3010	−0.6238	15.8534
	95	7.3776	7.4311	−0.7247	17.3052
	100	9.9165	9.9784	−0.6242	18.0949
	105	12.8633	12.9202	−0.4422	18.2261
	110	16.1813	16.2224	−0.2536	17.7801
252 days	90	6.8458	6.9104	−0.9427	19.4864
	95	9.0969	9.1869	−0.9895	20.8891
	100	11.7133	11.8138	−0.8583	21.7182
	105	14.6740	14.7694	−0.6502	21.9888
	110	17.9501	18.0272	−0.4297	21.7612
504 days	90	11.2078	11.3528	−1.2940	26.2295
	95	13.7415	13.9123	−1.2428	27.5161
	100	16.5319	16.7143	−1.1035	28.4149
	105	19.5621	19.7423	−0.9214	28.9460
	110	22.8135	22.9793	−0.7267	29.1419

Table 4: P_{mc} Monte Carlo put prices computed by $N = 10^6$ simulations; $\widetilde{P}^{gd}$ put prices given

maturity	**strike**	P_{mc}	$\widetilde{P}^{gd}$	$e_p\%$	$e_{mc} \times 10^4$
30 days	90	0.0416	0.0418	−0.4024	0.2796
	95	0.4271	0.4269	0.0559	1.2305
	100	2.0543	2.0546	−0.0118	2.0377
	105	5.4912	5.4910	0.0044	1.3532
	110	10.0743	10.0744	−0.0011	0.4321
60 days	90	0.2403	0.2395	0.3186	1.2698
	95	1.0088	1.0088	0.0013	2.7060
	100	2.8976	2.8995	−0.0661	3.5178
	105	6.1229	6.1230	−0.0019	2.9091
	110	10.3537	10.3527	0.0097	1.6629
90 days	90	0.5059	0.5040	0.3736	2.2925
	95	1.5236	1.5243	−0.0482	3.8280
	100	3.5450	3.5484	−0.0953	4.5905
	105	6.6746	6.6756	−0.0146	4.0842
	110	10.6943	10.6924	0.0180	2.8388
120 days	90	0.7895	0.7873	0.2753	3.1594
	95	1.9843	1.9862	−0.0935	4.6713
	100	4.0917	4.0965	−0.1164	5.3827
	105	7.1657	7.1679	−0.0305	4.9653
	110	11.0435	11.0416	0.0170	3.8040
180 days	90	1.3561	1.3550	0.0806	4.4457
	95	2.7925	2.7964	−0.1395	5.8221
	100	5.0114	5.0181	−0.1319	6.4495
	105	8.0238	8.0281	−0.0539	6.1656
	110	11.7203	11.7200	0.0026	5.2043
252 days	90	2.0035	2.0036	−0.0071	5.4574
	95	3.6288	3.6333	−0.1240	6.6719
	100	5.9334	5.9400	−0.1108	7.2297
	105	8.9091	8.9140	−0.0551	7.0508
	110	12.4759	12.4770	−0.0084	6.2867
504 days	90	3.9613	3.9637	−0.0605	7.0744
	95	5.9352	5.9394	−0.0710	7.9420
	100	8.4032	8.4082	−0.0592	8.3767
	105	11.3436	11.3481	−0.0396	8.3715
	110	14.7118	14.7148	−0.0208	7.9875

Table 5: P_{mc} Monte Carlo put prices computed by $N = 10^6$ simulations; $\widetilde{P}^{gd}$ put prices given

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