Purely discontinuous Lévy Processes and Power Variation: inference for integrated volatility and the scale parameter

Jeannette H.C. Woerner
University of Oxford

Oxford Centre for Industrial and Applied Mathematics, Mathematical Institute, 24-29 St Giles, Oxford OX1 3LB, UK. email: woerner@maths.ox.ac.uk

Abstract

This paper provides consistency and a distributional result for an estimate of the integrated volatility in different Lévy type stochastic volatility models based on high frequency data. As an estimator we consider the p-th power variation, i.e. the sum of the p-th power of the absolute value of the log-price returns, allowing irregularly spaced data. Furthermore, we derive conditions on the mean process under which it is negligible. This allows us more flexibility in modelling, namely to include further jump components or even to leave the framework of semimartingales by adding a certain fractional Brownian motion. As a special case our method includes an estimating procedure for the scale parameter of discretely observed Lévy processes.

key words and phrases: fractional Brownian motion, high frequency data, Lévy process, limit theorem, power variation, quadratic variation, semimartingale, stochastic volatility,

1 Introduction

Recently there have been proposed different Lévy type stochastic volatility models in the financial literature in an attempt to merge the desirable properties, such as being able to fit fat tails, skews and smiles, of the pure jump and the stochastic volatility models. In this paper we will derive estimators of the integrated volatility for both types of stochastic volatility models, one where in the classical setting of a stochastic volatility model the Brownian motion is replaced by a purely discontinuous Lévy process (cf. Eberlein et.al (2003)), and another one where a time changed Lévy process is used (cf. Carr et.al. (2003)). The integrated volatility is an important quantity for financial applications, since it is needed for pricing volatility derivatives, such as volatility and variance swaps and swaptions, cf. Howison et.al. (2003). These financial instruments became increasingly popular to investors, since they avoid direct exposure to the underlying asset, but make it possible to hedge volatility risk.

For our analysis we will use the concept of normed power variation, i.e. taking $\sum_i |t_i - t_{i-1}|^\gamma |X_{t_i} - X_{t_{i-1}}|^p$ as $\max_i |t_i - t_{i-1}| \to 0$, where X_t denotes the log-price process, which has already been studied in the classical stochastic volatility setting by Barndorff-Nielsen and Shephard (2002, 2003) and Woerner (2003b), and also when the Brownian motion is replaced by a stable process by Barndorff-Nielsen and Shephard (2002).

We will provide consistency and a distributional result for the power variation estimator of the integrated volatility in the model

$$X_t = Y_t + \int_0^t \sigma_s dL_s. \tag{1}$$

Here Y_t is a fairly general mean process, possibly with jumps or even a fractional Brownian motion, L_t is a purely discontinuous Lévy process and σ_t some stochastic volatility process, e.g. a geometric Brownian motion, an Ornstein-Uhlenbeck or Ornstein-Uhlenbeck type process.

A key step in the derivation of power variation estimates is to find the correct exponent γ for the norming sequence to obtain non-trivial limits. It turns out that in our general setting the important quantity is the Blumenthal-Getoor index β of L_t, which measures the activity of the jumps. We obtain $\gamma = 1 - p/\beta$, which is in line with the existing results, namely for the classical setting $\gamma = 1 - p/2$, where the Blumenthal-Getoor index of a Brownian motion is 2 and $1 - p/\alpha$ for α-stable processes, where the Blumenthal-Getoor index is equal to the index of stability.

Furthermore, we will examine what kind of mean process Y_t we can add without affecting the estimate of the integrated volatility. It turns out that we can add jump components, if the jump activity is not too high, and even specific fractional Brownian motions, which means that we can leave the framework of semimartingales. This freedom in the mean process can either be interpreted as more flexibility in modelling or robustness against additive noise.

In addition as a special case with constant volatility process our method can be used to estimate the scale parameter of a discretely observed Lévy process. In the context of finance, especially for the CGMY process, this can be interpreted as an estimate for the overall activity, cf. Carr et.al. (2002).

The outline of the paper is the following, first we will introduce our models and notation, then prove consistency and asymptotic normality for the estimate of the integrated volatility in (1). Next we consider the special case of estimation the scale parameter and finally look at the stochastic volatility model based on a time changed Lévy process.

2 Models and Notation

First let us now briefly review the different classes of stochastic processes which we will need in the following. Lévy processes are stochastic processes with independent and stationary increments. They are given by the characteristic function via the Lévy-Khichin formula

$$E[e^{iuX_t}] = \exp\{t(i\alpha u - \frac{\sigma^2 u^2}{2} + \int (e^{iux} - 1 - iuh(x))\nu(dx))\},$$

or for short by the Lévy triplet $(t\alpha, t\sigma^2, t\nu)_h$, where α denotes the drift, σ^2 the Gaussian part, ν the Lévy measure and h is a truncation function, behaving like x around the origin and hence ensuring the existence of the integral. Obviously σ^2 determines the continuous part and the Lévy measure the frequency and size of jumps. If $\int(1 \wedge |x|)\nu(dx) < \infty$ the process has bounded variation, if $\int \nu(dx) < \infty$ the process jumps only finitely many times in any finite time-interval, called finite activity, it is a compound Poisson process. Furthermore, the support of ν determines the size and direction of jumps. A popular example in finance are subordinators, where the the support of the Lévy measure is restricted to the positive half line, hence the process does not have negative jumps and the process is of bounded variation in addition. For more details see Sato (1999).

We will look at integrals of the form $X_t = \int_0^t \sigma_s dL_s$ where L_t is a Lévy process and σ_t some stochastic process independent of L_t. This implies that conditionally under σ_t X_t is an additive process (i.e. a process with independent increments) and unconditionally it is a semimartingale.

For an overview over semimartingales both under financial and theoretical aspects see Shiryaev (1999). In its canonical representation a semimartingale may be written as

$$X_t = X_0 + B(h) + X^c + h * (\mu - \nu) + (x - h(x)) * \mu,$$

or for short with the predictable characteristic triplet $(B(h), \langle X^c \rangle, \nu)_h$, where X^c denotes the continuous local martingale component, $B(h)$ is predictable of bounded variation and h is a truncation function. Furthermore, $\mu((0,t] \times A; \omega) = \sum(I_A(J(X_s)), 0 < s \leq t)$, where $J(X_s) = X_s - X_{s-}$ and $A \in \mathcal{B}(I\!R - \{0\})$ is a random measure, the jump measure, and ν denotes its compensator, satisfying $(x^2 \wedge 1) * \nu \in \mathcal{A}_{loc}$, i.e. the process $(\int_{(0,t] \times I\!R}(x^2 \wedge 1)d\nu)_{t \geq 0}$ is locally integrable.

A measure for the activity of the jump component of a semimartingale is the generalized Blumenthal-Getoor index,

$$\beta = \inf\{\delta > 0 : (|x|^\delta \wedge 1) * \nu \in \mathcal{A}_{loc}\},$$

where $\mathcal{A}_{loc}$ is the class of locally integrable processes. This index β also determines, that for $p > \beta$ the sum of the p-th power of jumps will be finite. Note that if we are in the framework of Lévy processes, being an element of a locally integrable process reduces to finiteness of the integral with respect to the Lévy measure, since the jump measure is deterministic.

As mean processes or noise components we will consider seminartingales, especially with jumps, and outside the framework of semimartingales also fractional Brownian motion. A fractional Brownian motion with Hurst exponent $H \in (0,1)$ is defined by

$$\frac{1}{\Gamma(H+1/2)}(\int_{-\infty}^{0}((t-s)^{H-1/2}-(-s)^{H-1/2})dW_s+\int_0^t(t-s)^{H-1/2}dW_s),$$

where $(W_s, -\infty < s < \infty)$ denotes a Wiener process extended to the real line. Fractional Brownian motion is popular in finance to model clustering and long range dependence, cf. Shiryaev (1999).

Classical stochastic volatility models can be written in the form

$$X_t = Y_t + \int_0^t \sigma_s dB_s,$$

where Y_t is the mean process, B_t a Brownian motion and σ_t a volatility process, e.g. a geometric Brownian motion (cf. Hull and White (1987)), an Ornstein-Uhlenbeck process (cf. Scott (1987), Stein and Stein (1991)), or an Ornstein-Uhlenbeck type process (cf. Barndorff-Nielsen and Shephard (2001)). We now replace the Brownian motion by a purely discontinuous Lévy process and obtain a stochastic volatility model

$$X_t = Y_t + \int_0^t \sigma_s dL_s.$$

This type of model when L_t is a generalized hyperbolic Lévy motion was proposed by Eberlein et.al (2003) in the context of risk management. A slight modification, namely the representation as a time changed Lévy process

$$X_t = Y_t + L_{\int_0^t \sigma_s ds}$$

was studied in Carr et.al. (2003). With our technique based on power variation both models can be treated similarly.

Now we have reviewed our models and associated processes and can introduce the method of power variation estimates. The concept of power variation was introduced in the context of studying the path behaviour of stochastic processes in the 1960ties and 1970ties, cf. Berman (1965), Hudson and Mason (1976) for additive processes or Lepingle (1976) for semimartingales. Assume that we are given a stochastic process X on some finite time interval $[0,t]$. Let n be a positive integer and denote by $S_n = \{0 = t_{n,0}, t_{n,1}, \cdots, t_{n,n} = t\}$ a partition of $[0,t]$, such that $0 < t_{n,1} < t_{n,2} < \cdots < t_{n,n}$ and $\max_{1\le k\le n}\{t_{n,k} - t_{n,k-1}\} \to 0$ as $n \to \infty$. Now the p-th power variation is defined to be

$$\sum_i |X_{t_{n,i}} - X_{t_{n,i-1}}|^p = V_p(X, S_n).$$

Well established are as $n \to \infty$ for convergence in probability the cases for $p = 1$, where finiteness of the limit means that the processes has bounded variation, and $p = 2$, called quadratic variation, which is finite for all semimartingale processes. However, in the classical stochastic volatility setting, i.e. for the moment assuming that our process has the form $\int_0^t \sigma_s dB_s$, only the case $p = 2$ leads to a non-trivial limit. Obviously, for $p > 2$ the limit is zero and for $p < 2$ the limit is infinity.

An extension of the concept of power variation is to consider an appropriate norming sequence, as it was done in Barndorff-Nielsen and Shephard (2003), which allows to find non-trivial limits even in the cases where the non-normed power variation limit would be zero or infinity. Let us introduce the following notation for the normed p-th power variation

$$\sum_i \Delta_{n,i}^{\gamma} |X_{t_{n,i}} - X_{t_{n,i-1}}|^p = V_p(X, S_n, \Delta_n^{\gamma}),$$

where $\gamma \in I\!R$ and $t_{n,i} - t_{n,i-1} = \Delta_{n,i}$ denotes the distance between the i-th and the $i-1$-th time-point. When we have equally spaced observations, $\Delta_{n,i}$ is independent of i and the normed power variation reduces to $\Delta_n^{\gamma} V(X, S_n)$. It turns out that the important quantity for determining the appropriate norming sequence is the Blumenthal-Getoor index. For the classical stochastic volatility setting Barndorff-Nielsen and Shephard (2003) derived that $\gamma = 1 - p/2$, where 2 is the Blumenthal-Getoor index of a Brownian motion. In our framework with $\int_0^t \sigma_s dL_s$ the Blumenthal-Getoor index β of L will determine the norming sequence, namely $\gamma = 1 - p/\beta$ and $p < \beta$.

In the following we need a measure of regularity for the sequence of partitions. We use the term of ϵ-balanced partitions, $\epsilon \in (0, 1)$, which was introduced by Barndorff-Nielsen and Shephard (2002) and is defined by

$$\frac{\max_i \Delta_{n,i}}{(\min_i \Delta_{n,i})^{\epsilon}} \to 0,$$

as $n \to \infty$. This means we compare how fast the minimum distance in the partition converges to zero compared with the sequence of maxima, e.g. if $\max_i \Delta_{n,i} = O(1/n)$ and $\min_i \Delta_{n,i} = O(1/n^2)$, then the partition is ϵ-balanced for $\epsilon \in (0, 1/2)$. Obviously, an equally spaced partition is ϵ-balanced for all $\epsilon \in (0, 1)$. If a partition is ϵ-balanced for some $\epsilon \in (0, 1)$, then it is also δ-balanced for $\delta \in (0, \epsilon]$. Hence the larger ϵ the closer the partition is to an equally-spaced one. We will denote with $\epsilon = 1$ the equally spaced partition. In the following we will show how the regularity of the sampling scheme together with the regularity of the mean process Y will determine the choice of the power variation exponent p.

3 Estimating the integrated volatility

In this section we will provide an estimate for the integrated volatility when the underlying driving process is a purely discontinuous Lévy process with Blumenthal-Getoor index $\beta \in (0, 2)$, whose Lévy measure possesses a Lebesgue density. This means that we obtain results for infinite activity processes, including most Lévy processes popular in finance, such as generalized hyperbolic Levy motions, Normal inverse Gaussian processes, stable processes and CGMY processes for $Y > 0$. Note that the limiting case $\beta = 2$ is the Brownian motion case which has been extensively studied by Barndorff-Nielsen and Shephard (2002, 2003) and Woerner (2003b).

The key step to derive the results for purely discontinuous Lévy processes is to split them into a stable part and some remainder, which can be done by considering a Taylor series expansion of the density of the Lévy measure around the origin. This method indicates why we do not get a result for finite activity processes or infinite activity processes with $\beta = 0$, such as compound Poisson processes and Gamma processes. Namely their first order term of the density of the Lévy measure is not the Lévy measure of a stable process.

Our result implies that from the point of view of estimating the integrated volatility the most important parameter is the Blumenthal-Getoor index. Hence all processes with the same Blumenthal-Getoor index, i.e. the same level of activity lead, up to the scale parameter, to the same result. We can see that the generalized hyperbolic Lévy motion (including the hyperbolic Lévy motion, the Normal inverse Gaussian process and the Student-t Lévy motion), the CGMY process with $Y = 1$ and the Cauchy process form an equivalence class for estimating the integrated volatility.

Theorem 1 *Let*

$$X_t = Y_t + \int_0^t \sigma_s dL_s^\beta, \tag{2}$$

where Y_t is some stochastic process satisfying for some $p < \beta$

$$V_p(Y, S_n, \Delta_n^{1-p/\beta}) \xrightarrow{p} 0. \tag{3}$$

Denote by L_t^β a purely discontinuous Lévy process with Blumenthal-Getoor index $2 > \beta > 0$. Assume that the Lévy triplet is given in the following form:
a) if $\beta < 1$, we take $h = 0$ and the Lévy triplet is $(0, 0, tg(x))_0$,
b) if $\beta = 1$ and $\int |x| 1_{|x|>1}(x) g(x) dx < \infty$, we take $h(x) = x$ and the Lévy triplet is $(0, 0, tg(x))_x$,
c) if $\beta = 1$ and $g(x)$ is symmetric and even, we take h symmetric and odd and the Lévy triplet is $(0, 0, tg(x))_h$,
d) if $\beta > 1$, we can take any h and the Lévy triplet is $(0, 0, tg(x))_h$.
Assume furthermore that the density of the Lévy measure is Riemann integrable and can be expanded in the following Taylor series expansion as $x \to 0$

$$\begin{aligned} g(x) &= \sum_{i=0}^{n} \frac{c_i x^i + k_i |x|^i}{|x|^{1+\beta}} + o(\frac{|x|^n}{|x|^{1+\beta}}) \\ &= \frac{c_0}{|x|^{1+\beta}} + f(x), \end{aligned} \tag{4}$$

with $\beta - n \leq 0$ and $k_0 = 0$.

Assume that the process σ is locally bounded Riemann integrable, nonnegative and bounded away from zero. Furthermore, assume that σ is adapted and stochastically independent of L^β. Then for any $t > 0$, we obtain for $0 < p < \beta$, if L_t is a stable process, or $0 < p < \beta$, $p \neq \beta - 1$

$$V_p(X, S_n, \Delta_n^{1-p/\beta}) \xrightarrow{p} \mu_{p,\beta}(\frac{c_0}{\beta})^{p/\beta} \int_0^t \sigma_s^p ds, \tag{5}$$

as $n \to \infty$, where $\mu_{p,\beta} = E(|U|^p)$ and $U \sim S^\beta$ is a symmetric β-stable random variable with Lévy triplet $(0, 0, \beta|x|^{-1-\beta})$.

Proof. For the proof we proceed in the following way, first we will prove (5) assuming that Y is zero and then show that under the condition (3) on Y the process is actually negligible for (5).

First we have to calculate the characteristic triplet of $\int_0^t \sigma_s dL_s^\beta$ conditional under σ. We can approximate the integral by Riemann-Ito sums $\sum_{j=1}^m \sigma_{m,j-1}(L_{t_{m,j}}^\beta - L_{t_{m,j-1}}^\beta)$ as $m \to \infty$, where $L_{t_{m,j}}^\beta - L_{t_{m,j-1}}^\beta \sim P_{\Delta_{m,j}}$ with Lévy triplet $(0, 0, \Delta_{m,j} g(x))_h$ and $\Delta_{m,j} = t_{m,j} - t_{m,j-1}$. We obtain

$$\begin{aligned}
&E(\exp(iu\sum_{j=1}^m \sigma_{m,j-1}(L_{t_{m,j}}^\beta - L_{t_{m,j-1}}^\beta))|\sigma) \\
= \quad & \exp(\sum_{j=1}^m \Delta_{m,j} \int (e^{iu\sigma_{m,j-1}x} - 1 - iu\sigma_{m,j-1}h(x))g(x)dx) \\
= \quad & \exp(\sum_{j=1}^m \Delta_{m,j} \int (e^{iuy} - 1 - iu\sigma_{m,j-1}h(y/\sigma_{m,j-1}))\frac{g(y/\sigma_{m,j-1})}{\sigma_{m,j-1}}dy) \\
= \quad & \exp(\sum_{j=1}^m \Delta_{m,j} iu \int (h(y) - \sigma_{m,j-1}h(y/\sigma_{m,j-1}))\frac{g(y/\sigma_{m,j-1})}{\sigma_{m,j-1}}dy) \\
& \times \exp(\sum_{j=1}^m \Delta_{m,j} \int (e^{iuy} - 1 - iuh(y))\frac{g(y/\sigma_{m,j-1})}{\sigma_{m,j-1}}dy) \\
= \quad & \exp(\sum_{j=1}^m \Delta_{m,j} iu \int (h(y) - \sigma_{m,j-1}h(y/\sigma_{m,j-1}))\frac{g(y/\sigma_{m,j-1})}{\sigma_{m,j-1}}dy) \\
& \times \exp(\sum_{j=1}^m \Delta_{m,j} \int (e^{iuy} - 1 - iuh(y))\frac{c_0\sigma_{m,j-1}^\beta}{|y|^{1+\beta}}dy) \\
& \times \exp(\sum_{j=1}^m \Delta_{m,j} \int (e^{iuy} - 1 - iuh(y))\frac{f(y/\sigma_{m,j-1})}{\sigma_{m,j-1}}dy)
\end{aligned}$$

Hence letting $m \to \infty$ and using the conditions on σ and g, we can split $\int_{t_{n,i-1}}^{t_{n,i}} \sigma_s dL_s^\beta$ into a part S_i^β which is conditional under σ stable with scale parameter $(c_0/\beta)\int_{t_{n,i-1}}^{t_{n,i}} \sigma_s^\beta ds$ and characteristic triplet $(0, 0, c_0 \int_{t_{n,i-1}}^{t_{n,i}} \sigma_s^\beta ds/|x|^{1+\beta})_h$

and a remainder R_i which is conditional under σ additive with characteristic triplet $(\int_{t_{n,i-1}}^{t_{n,i}} \int (h(y) - \sigma_s h(y/\sigma_s)) \frac{g(y/\sigma_s)}{\sigma_s} dyds, 0, \int_{t_{n,i-1}}^{t_{n,i}} \frac{f(x/\sigma_s)}{\sigma_s} ds)_h$. By construction S_i^β and R_i conditional under σ are sequences of independent random variables. Note that the drift of R_i vanishes under conditions a)-c).

Now we can proceed with the proof of (5) for $p \leq 1$

$$
\begin{aligned}
& P(|V_p(X, S_n, \Delta^{1-p/\beta}) - (\frac{c_0}{\beta})^{p/\beta} \mu_{p,\beta} \int_0^t \sigma_s^p ds| > \gamma) \\
= \; & P(|\sum_{i=1}^n \Delta_{n,i}^{1-p/\beta} |\int_{t_{n,i-1}}^{t_{n,i}} \sigma_s dL_s^\beta|^p - (\frac{c_0}{\beta})^{p/\beta} \mu_{p,\beta} \int_0^t \sigma_s^p ds| > \gamma) \\
= \; & P(|\sum_{i=1}^n \Delta_{n,i} |\frac{(S_i^\beta + R_i)}{\Delta_{n,i}^{1/\beta}}|^p - (\frac{c_0}{\beta})^{p/\beta} \mu_{p,\beta} \int_0^t \sigma_s^p ds| > \gamma) \\
\leq \; & P(|\sum_{i=1}^n \Delta_{n,i} |\frac{(S_i^\beta + R_i)}{\Delta_{n,i}^{1/\beta}}|^p - \sum_{i=1}^n \Delta_{n,i} |\frac{S_i^\beta}{\Delta_{n,i}^{1/\beta}}|^p| > \gamma/2) \\
& + P(|\sum_{i=1}^n \Delta_{n,i} |\frac{S_{t_i}^\beta}{\Delta_{n,i}^{1/\beta}}|^p - (\frac{c_0}{\beta})^{p/\beta} \mu_{p,\beta} \int_0^t \sigma_s^p ds| > \gamma/2) \\
\leq \; & P(|\sum_{i=1}^n \Delta_{n,i} |\frac{R_i}{\Delta_{n,i}^{1/\beta}}|^p| > \gamma/2) \\
& + P(|\sum_{i=1}^n \Delta_{n,i} |\frac{S_i^\beta}{\Delta_{n,i}^{1/\beta}}|^p - (\frac{c_0}{\beta})^{p/\beta} \mu_{p,\beta} \int_0^t \sigma_s^p ds| > \gamma/2) < \epsilon \qquad (6)
\end{aligned}
$$

For the second inequality we used that for $p \leq 1$ applying $|a + b|^p \leq |a|^p + |b|^p$, we have

$$
|\sum_i |a_i + b_i|^p - \sum_i |b_i|^p| \leq \sum_i |a_i|^p.
$$

For the second term in (6) we can apply the LLN conditional under σ together with the scaling relation for stable processes and obtain that in probability

$$
\lim_{n\to\infty} (\sum_{i=1}^n \Delta_{n,i} |\frac{S_i^\beta}{\Delta_{n,i}^{1/\beta}}|^p - \sum_{i=1}^n \Delta_{n,i}^{1-p/\beta} (\frac{c_0}{\beta} \int_{t_{n,i-1}}^{t_{n,i}} \sigma_s^\beta ds)^{p/\beta} \int |y|^p dS^\beta(y)) = 0
$$

hence

$$
\lim_{n\to\infty} \sum_{i=1}^n \Delta_{n,i} |\frac{S_i^\beta}{\Delta_{n,i}^{1/\beta}}|^p = (\frac{c_0}{\beta})^{p/\beta} \int_0^t \sigma_s^p ds \mu_{p,\beta}.
$$

For the first term in (6), we have to look at the different conditions on the Lévy triplet.

Under condition a)-c) the drift term in R_i vanishes and conditionally under σ we can apply Hudson and Mason (1976), which yields $\lim_{n\to\infty} \sum_i |R_i|^p < \infty$ in probability, since the Blumenthal-Getoor index of R_i is $\max(0, \beta - 1) = 0$.

Hence we obtain in probability

$$\sum_{i=1}^{n} \Delta_{n,i}^{1-\frac{p}{\beta}} |R_i|^p \leq \max_i (\Delta_{n,i}^{1-\frac{p}{\beta}}) \sum_{i=1}^{n} |R_i|^p \to 0,$$

as $n \to \infty$ and $p < \beta$.

Under condition d) we can apply Woerner (2003a, Thm. 1) for $1 < p < \beta$, which yields $\lim_{n\to\infty} \sum_i |R_i|^p < \infty$ in probability, since the Blumenthal-Getoor index of R_i is $\beta - 1 \leq 1$. Hence we obtain in probability

$$\sum_{i=1}^{n} \Delta_{n,i}^{1-\frac{p}{\beta}} |R_i|^p \leq \max_i (\Delta_{n,i}^{1-\frac{p}{\beta}}) \sum_{i=1}^{n} |R_i|^p \to 0,$$

as $n \to \infty$ and $1 < p < \beta$. For $\beta - 1 < p \leq 1$ we have to split R_i in the pure jump part $\bar{R}_i$, with Blumenthal-Getoor index $\beta - 1$, and the drift $\mu_i = \int_{t_{n,i-1}}^{t_{n,i}} \int (h(y) - \sigma_s h(y/\sigma_s)) \frac{g(y/\sigma_s)}{\sigma_s} dyds = O(\Delta_{n,i})$. Hence we can again, conditionally under σ, apply Hudson and Mason (1976) and obtain in probability

$$\begin{aligned} &\sum_{i=1}^{n} \Delta_{n,i}^{1-\frac{p}{\beta}} |R_i|^p \\ \leq\ & \max_i (\Delta_{n,i}^{1-\frac{p}{\beta}}) \sum_{i=1}^{n} |\bar{R}_i|^p + \max_i (\Delta_{n,i}^{p-\frac{p}{\beta}}) \sum_{i=1}^{n} \Delta_{n,i}^{1-p} |\mu_i|^p \\ \to\ & 0 \end{aligned}$$

as $n \to \infty$ and $\beta - 1 < p \leq 1$. For $0 < p < \beta - 1$ we have to split R_i in a conditionally stable component $S_i^{\beta-1}$, the drift μ_i and a new remainder $\tilde{R}_i$ which has Blumenthal-Getoor index zero. No we can proceed as for the pervious case and for the second part of (6). This competes the proof of (6).

Next we have to show (5) for $p > 1$. This can be done with the same technique using Minkowski's inequality instead of the triangular inequality, namely

$$\begin{aligned} &P(|(V_p(X, S_n, \Delta^{1-p/\beta}))^{1/p} - (\mu_{p,\beta} (\frac{c_0}{\beta})^{p/\beta} \int_0^t \sigma_s^p ds)^{1/p}| > \gamma) \\ \leq\ & P(|(\sum_{i=1}^{n} \Delta_{n,i} |\frac{R_i}{\Delta_{n,i}^{1/\beta}}|^p)^{1/p}| > \gamma/2) \\ &+ P(|(\sum_{i=1}^{n} \Delta_{n,i} |\frac{S_i^\beta}{\Delta_{n,i}^{1/\beta}}|^p)^{1/p} - ((\frac{c_0}{\beta})^{p/\beta} \mu_{p,\beta} \int_0^t \sigma_s^p ds)^{1/p}| > \gamma/2) < \epsilon \end{aligned}$$

This implies the desired result as for $p \leq 1$, noting that x^p is continuous.

Hence we have completed the first part of the proof and it remains to show that under condition (3) Y is negligible. For simplicity denote $Z_t = \int_0^t \sigma_s dL_s^\beta$ Now for $p \leq 1$ we obtain

$$P(|V_p(X, S_n, \Delta_n^{1-p/\beta}) - \mu_{p,\beta} (\frac{c_0}{\beta})^{p/\beta} \int_0^t \sigma_s^p ds| > \gamma)$$

$$
\begin{aligned}
&\le P(|V_p(X,S_n,\Delta_n^{1-p/\beta}) - V_p(Z,S_n,\Delta_n^{1-p/\beta})| > \gamma/2) \\
&\quad P(|V_p(Z,S_n,\Delta_n^{1-p/\beta}) - \mu_{p,\beta}(\frac{c_0}{\beta})^{p/\beta}\int_0^t \sigma_s^p ds| > \gamma/2) \\
&\le P(V_p(Y,S_n,\Delta_n^{1-p/\beta}) > \gamma/2) \\
&\quad P(|V_p(Z,S_n,\Delta_n^{1-p/\beta}) - \mu_{p,\beta}(\frac{c_0}{\beta})^{p/\beta}\int_0^t \sigma_s^p ds| > \gamma/2) < \epsilon,
\end{aligned}
$$

since $V_p(Y,S_n,\Delta_n^{1-p/\beta}) \xrightarrow{p} 0$ by assumption (3) and $V_p(Z,S_n,\Delta_n^{1-p/\beta}) \xrightarrow{p} \mu_{p,\beta}(\frac{c_0}{\beta})^{p/\beta}\int_0^t \sigma^p(s)ds$ by (6).

For $p > 1$ we again use Minkowski's inequality together with the same technique, which yields

$$
\begin{aligned}
&P(|(V_p(X,S_n,\Delta_n^{1-p/\beta}))^{1/p} - (\mu_{p,\beta}(\frac{c_0}{\beta})^{p/\beta}\int_0^t \sigma_s^p ds)^{1/p}| > \gamma) \\
&\quad\le P((V_p(Y,S_n,\Delta_n^{1-p/\beta}))^{1/p} > \gamma/2) \\
&\qquad P(|(V_p(Z,S_n,\Delta_n^{1-p/\beta}))^{1/p} - (\mu_{p,\beta}(\frac{c_0}{\beta})^{p/\beta}\int_0^t \sigma_s^p ds)^{1/p}| > \gamma/2) < \epsilon.
\end{aligned}
$$

This implies the desired result since x^p is continuous. □

Now we discuss the condition on the process Y. Condition (3) provides lots of flexibility in choosing Y. This can be interpreted as for what kind of mean process our estimate is still valid or how robust our estimate is against misspecification of the model, i.e. what type of noise we can add without affecting the estimate.

- Condition (3) is satisfied when Y is locally Hölder-continuous of the order a, $1 \ge a > 1/\beta$, since this implies that $(Y_{t_{n,j}} - Y_{t_{n,j-1}})/\Delta_{n,j}^a \le C_j < \infty$ and hence

$$
\begin{aligned}
V_p(Y,S_n,\Delta_n^{1-p/\beta}) &= \sum_{j=1}^n \Delta_{n,j}^{1+ap-p/\beta}|\frac{Y_{n,j}-Y_{n,j-1}}{\Delta_{n,j}^a}|^p \\
&\le (\max_j C_j)(\max_j \Delta_{n,j}^{ap-p/\beta})t \to 0
\end{aligned}
$$

  since $ap - p/\beta > 0$ for $a > 1/\beta$. Clearly it only makes sense to take $a \le 1$, since $a > 1$ would imply that Y is constant. We can see that we can even leave the framework of semimartingales and include models or noise where Y is a fractional Brownian motion with Hurst exponent $H \in (1/\beta, 1]$.

  Furthermore, we can see that for $\beta \le 1$ the possible range of a is empty, hence in these cases we cannot have a continuous mean process. However, from a practical point of view this does not matter too much, since normally the time series are treated anyway before, in a way to get rid of the drift component

- Condition (3) is also satisfied if we have finite p-th power variation of Y and take $p < \beta < 2$, which means that the norming sequence tends to zero.

  For jump processes this is also very interesting since we can easily determine for which r we have finite r-th power variation and can then take $p \in [r, \beta)$. From Woerner (2003a) we know that assuming a general semimartingale setting, the r-th power variation is finite if either $1 \leq \beta_Y < r < 2$ and $\langle Y^c \rangle_t = 0$ or $\beta_Y < r \leq 1$, $\langle Y^c \rangle_t = 0$, $B(h) + (x - h) * \nu = 0$ and the jump times of Y_t are previsible. Here β_Y denotes the generalized Blumenthal-Getoor index of Y. Note that for subordinators, i.e. Lévy processes with only positive jumps, or Lévy processes of bounded variation in their usual representation with $h(x) = x$, the condition $B(h)+(x-h)*\nu = 0$ reduces to no drift and the condition of predictable jump times can be dropped for additive processes.

  Hence we can see that we need $\beta_Y < \beta$ and the jumps are negligible for $\beta_Y < p < \beta$.

- Even less restrictive than the previous condition is to assume that

$$V_p(Y, S_n, \Delta_n^\gamma) \to C < \infty,$$

  where $\gamma > 0$ and $1 - (p/\beta) - \gamma > 0$ which implies (3). One example in which this holds is to take

$$Y_t = \int_0^t f_s dL_s,$$

  where L is a Lévy process with Blumenthal-Getoor index $\beta_Y < \beta$ and satisfies the conditions of Theorem 1, f is independent of L and locally Riemann integrable. Then by Theorem 1, for $p < \beta_Y$, $p \neq \beta_Y - 1$

$$V_p(Y, S_n, \Delta_n^{1-p/\beta_Y}) \to \mu_{\beta_Y,p}(\frac{c_{0,Y}}{\beta_Y})^{p/\beta_Y} \int_0^t f_s^p ds.$$

  Here $1 - (p/\beta) - (1 - p/\beta_Y) > 0$ is obviously satisfied since $\beta_Y < \beta$.

  Together with the previous consideration we can see that this type of jump process is negligible for $0 < p < \beta$ and $p \neq \beta_Y - 1$.

- Using the same method as in the proof we can see that Y obviously also satisfies (3), if it is the sum of processes, each of which satisfies (3).

The condition on σ is quite mild, e.g. it is implied by the Cox-Ingersoll-Ross process and it is also satisfied by the jump process model, where σ is an Ornstein-Uhlenbeck type process, as it is shown in Barndorff-Nielsen and Shephard (2003).

Concerning condition (4) we look at some examples which show that it is satisfied for most popular models.

Example: Stable process
For β-stable processes (4) is clearly satisfies with $c_0 = \beta$ and $f(x) = 0$. Hence we obtain for $p < \beta$

$$V_p(X, S_n, \Delta_n^{1-p/\beta}) \xrightarrow{p} \mu_{p,\beta} \int_0^t \sigma_s^p ds,$$

For the special case $Y = 0$ Barndorff-Nielsen and Shephard (2002) have already obtained this result using a different method based on a time change argument.

Example: Generalized hyperbolic Lévy motion
The generalized hyperbolic Lévy motion includes many processes used in finance. The density at $t = 1$ is given by

$$\begin{aligned} d_{GH(\lambda,\alpha,\beta,\delta,\mu)}(x) &= a(\lambda,\alpha,\beta,\delta,\mu)(\delta^2 + (x-\mu)^2)^{(\lambda-1/2)/2} e^{\beta(x-\mu)} \\ &\quad \times K_{\lambda-1/2}(\alpha\sqrt{\delta^2 + (x-\mu)^2}), \end{aligned}$$

where

$$a(\lambda,\alpha,\beta,\delta,\mu) = \frac{(\alpha^2 - \beta^2)^{\lambda/2}}{\sqrt{2\pi}\alpha^{\lambda-1/2}\delta^\lambda K_\lambda(\delta\sqrt{\alpha^2 - \beta^2})}$$

and K_λ denotes the modified Bessel function of the third kind with index λ. We have the following parameter dependence, $\alpha > 0$ determines the sharp, $0 \leq |\beta| < \alpha$ the skewness, $\mu \in I\!R$ is a location parameter and $\delta > 0$ determines the scaling. Unfortunately in general the density of the process is only known for $t = 1$, however we can give a formula for the Lévy triplet

$$\left(\mu, 0, \frac{e^{\beta x}}{|x|}\left(\int_0^\infty \frac{e^{-\sqrt{2y+\alpha^2}|x|}}{\pi^2 y (J_{|\lambda|}^2(\delta\sqrt{2y}) + Y_{|\lambda|}^2(\delta\sqrt{2y}))} dy + \max\{0,\lambda\} e^{-\alpha|x|}\right)\right),$$

where J_λ denotes the Bessel function of the first order with index λ, Y_λ the Bessel function of the second order with index λ. Furthermore, we have $\alpha, \delta > 0$, and $0 \leq |\beta| < \alpha$. For $\lambda = 1$ we obtain the hyperbolic Lévy motion, for $\lambda = -1/2$ the normal inverse Gaussian distribution and for $\alpha = \beta = 0$ the Student-t distribution. For a discussion of all possible limiting cases see Eberlein and von Hammerstein (2003).

From Raible (2000) we know that the density of the Lévy measure can be expanded in the form

$$g(x) = \frac{\delta}{\pi x^2} + \frac{\lambda + 1/2}{2}\frac{|x|}{x^2} + \frac{\delta\beta}{\pi}\frac{1}{x} + o(\frac{|x|}{x^2}),$$

as $x \to 0$. Furthermore, the first moment exists which is equivalent to $\int |x| 1_{|x|>1}(x) g(x) < \infty$. Hence (4) and b) are satisfied and for $p < 1$ we obtain (5) with $\beta = 1$ and $c_0 = \delta/\pi$. We see that we can consider all possible limiting cases for the parameters, except of $\delta \to 0$ which would mean to change the

Blumenthal-Getoor index to zero.

Example: CGMY process
CGMY processes form a flexible class for modelling in finance, see e.g. Carr et.at. (2000). the density of the Lévy measure is given by

$$C\frac{\exp(-G|x|)}{|x|^{1+Y}}1_{x<0}(x)+C\frac{\exp(-M|x|)}{|x|^{1+Y}}1_{x>0}(x),$$

where $C>0$ describes the overall level of activity, $G,M\geq 0$ control the rate of exponential decay on the right and the left tail and $Y<2$ characterizes the fine structure of the process. Clearly for $2>Y>0$ condition (4) is satisfied by using Taylor expansion for the exponential term. For $G,M>0$ b) is satisfied, whereas for $G=M=0$ we are back to a stable process. Hence we obtain (5) with $\beta=Y$ and $c_0=C$.

4 Distributional Theory

In the previous section we have proved consistency for the power variation estimate of the integrated volatility when the number of observations tends to infinity. To get some idea how this estimate performs in practice, when our sample of observations is finite we need a distributional theory, which enables us to calculate confidence intervals. Unfortunately the distributional theory does not hold in the same generality as the consistency, but we need some stronger conditions on the mean process Y, the volatility process σ and the regularity of the partition ϵ.

Theorem 2 *Let*

$$X_t=Y_t+\int_0^t\sigma_s dL_s^\beta, \tag{7}$$

where Y_t is some stochastic process satisfying for some $p<\beta/2$

$$V_{2p}(Y,S_n,\Delta_n^{1-2p/\beta}) \xrightarrow{p} 0 \tag{8}$$

$$\frac{V_p(Y,S_n,\Delta_n^{1-p/\beta})}{\sqrt{\min_i\Delta_{n,i}}} \xrightarrow{p} 0 \tag{9}$$

$$\frac{V_{2p}(Y,S_n,\Delta_n^{2-2p/\beta})}{\min_i\Delta_{n,i}} \xrightarrow{p} 0. \tag{10}$$

Assume the same conditions on L and σ as in Theorem 1. Furthermore, assume that σ has the property that for $\gamma=p$ and $2p$ as $n\to\infty$

$$\frac{1}{\sqrt{\min_j\Delta_{n,j}}}\sum_{j=1}^n\Delta_{n,j}|\sigma^\gamma(\eta_{n,j})-\sigma^\gamma(\chi_{n,j})|\xrightarrow{p}0 \tag{11}$$

for any $\chi_{n,j}$ *and* $\eta_{n,j}$ *such that*

$$0 \leq \chi_{n,1} \leq \eta_{n,1} \leq t_{n,1} \leq \chi_{n,2} \leq \eta_{n,2} \leq t_{n,2} \cdots \leq \chi_{n,n} \leq \eta_{n,n} \leq t.$$

Then for any $t > 0$*, for any sequence of* ϵ*-balanced partitions* S_n*,* $\epsilon \in [2/3, 1]$ *and* $0 < p < \beta - \beta/(2\epsilon)$*, if* L_t *is* β*-stable, or* $0 < p < \beta - \beta/(2\epsilon)$*,* $p \neq \beta - 1, (\beta-1)/2$ *otherwise, we obtain*

$$\frac{V_p(X, S_n, \Delta_n^{1-p/\beta}) - \mu_{p,\beta}(\frac{c_0}{\beta})^{p/\beta} \int_0^t \sigma_s^p ds}{\sqrt{\mu_{2p,\beta}^{-1} \nu_{p,\beta} V_{2p}(X, S_n, \Delta_n^{2-2p/\beta})}} \xrightarrow{\mathcal{D}} N(0,1), \tag{12}$$

as $n \to \infty$*, where* $\mu_{p,\beta} = E(|U|^p)$ *and* $\nu_{p,\beta} = Var(|U|^p)$ *with* $U \sim S^\beta$ *is a symmetric* β*-stable random variable.*

Proof. We us the same notation as in Theorem 1, namely $Z_t = \int_0^t \sigma_s dL_s^\beta$. The idea of the proof is to show (12) for Z first, by using characteristic functions and then extend it to the general X by applying Slutzki's Lemma. First of all we wish to show

$$\frac{V_p(Z, S_n, \Delta_n^{1-p/\beta}) - \mu_{p,\beta}(\frac{c_0}{\beta})^{p/\beta} \int_0^t \sigma_s^p ds}{\sqrt{\mu_{2p,\beta}^{-1} \nu_{p,\beta} V_{2p}(Z, S_n, \Delta_n^{2-2p/\beta})}} \xrightarrow{\mathcal{D}} N(0,1). \tag{13}$$

Since we assume that σ is independent of L^β, we will in the following always work conditionally under σ. The first step is to prove

$$\frac{V_p(Z, S_n, \Delta_n^{1-p/\beta}) - \mu_{p,\beta}(\frac{c_0}{\beta})^{p/\beta} \int_0^t \sigma_s^p ds}{\sqrt{\nu_{p,\beta}(\frac{c_0}{\beta})^{2p/\beta} \sum_{j=1}^n \Delta_{n,j} \int_{t_{n,j-1}}^{t_{n,j}} \sigma_s^{2p} ds}} \xrightarrow{\mathcal{D}} N(0,1), \tag{14}$$

which is equivalent to show that

$$\frac{V_p(Z, S_n, \Delta_n^{1-p/\beta}) - \sum_{i=1}^n \Delta_{n,i} |\frac{S_i^\beta}{\Delta_{n,i}^{1/\beta}}|^p}{\sqrt{\nu_{p,\beta}(\frac{c_0}{\beta})^{2p/\beta} \sum_{j=1}^n \Delta_{n,j} \int_{t_{n,j-1}}^{t_{n,j}} \sigma_s^{2p} ds}} \quad \xrightarrow{p} \quad 0 \tag{15}$$

$$\frac{\sum_{i=1}^n \Delta_{n,i} |\frac{S_i^\beta}{\Delta_{n,i}^{1/\beta}}|^p - \mu_{p,\beta}(\frac{c_0}{\beta})^{p/\beta} \int_0^t \sigma_s^p ds}{\sqrt{\nu_{p,\beta}(\frac{c_0}{\beta})^{2p/\beta} \sum_{j=1}^n \Delta_{n,j} \int_{t_{n,j-1}}^{t_{n,j}} \sigma_s^{2p} ds}} \quad \xrightarrow{\mathcal{D}} \quad N(0,1). \tag{16}$$

For the proof of (15), we can use the same technique as in Theorem 1, namely for $p \leq 1$ the triangular inequality and for $p > 1$ Minkowski's inequality. For simplicity we only show the first part

$$\begin{aligned} &|\frac{\sum_{i=1}^n \Delta_{n,i} |\frac{S_i^\beta + R_i}{\Delta_{n,i}^{1/\beta}}|^p - \sum_{i=1}^n \Delta_{n,i} |\frac{S_i^\beta}{\Delta_{n,i}^{1/\beta}}|^p}{\sqrt{\nu_{p,\beta}(\frac{c_0}{\beta})^{2p/\beta} \sum_{j=1}^n \Delta_{n,j} \int_{t_{n,j-1}}^{t_{n,j}} \sigma_s^{2p} ds}}| \\ &\quad \leq \frac{(\max_i \Delta_{n,i})^{1/(2\epsilon)}}{\sqrt{\min \Delta_{n,i}}} \frac{(\max_i \Delta_{n,i})^{1-1/(2\epsilon)-p/\beta} \sum_{i=1}^n |R_i|^p}{\sqrt{\nu_{p,\beta}(\frac{c_0}{\beta})^{2p/\beta} \int_0^t \sigma_s^{2p} ds}}. \end{aligned}$$

The first term tends to zero since we have an ϵ-balanced partition. The second term tends to zero using the same arguments for the convergence of the sum as in Theorem 1. Besides $1 - 1/(2\epsilon) - p/\beta > 0$ is equivalent to our condition on p, namely $p < \beta - \beta/(2\epsilon)$. Note that for the equally spaced setting, we take $\epsilon = 1$. Of course the first term is then equal to 1 but the second still tends to zero.

We establish (16) by using the conditional characteristic function together with Taylor expansion and the scaling relation for stable processes as in Theorem 1, we obtain

$$
\begin{aligned}
&\log(E(\exp(iu\frac{\sum_{i=1}^n \Delta_{n,i}|\frac{S_i^\beta}{\Delta_{n,i}^{1/\beta}}|^p - \mu_{p,\beta}(\frac{c_0}{\beta})^{p/\beta}\int_0^t \sigma_s^p ds}{\sqrt{\nu_{p,\beta}(\frac{c_0}{\beta})^{2p/\beta}\sum_{j=1}^n \Delta_{n,j}\int_{t_{n,j-1}}^{t_{n,j}} \sigma_s^{2p} ds}})|\sigma)) \\
= \; & \log(E(\exp(iu\frac{\sum_{i=1}^n \Delta_{n,i}(\frac{c_0}{\beta})^{p/\beta}\sigma_{n,i}^{*p}|S^\beta|^p - \mu_{p,\beta}(\frac{c_0}{\beta})^{p/\beta}\int_0^t \sigma_s^p ds}{\sqrt{\nu_{p,\beta}(\frac{c_0}{\beta})^{2p/\beta}\sum_{j=1}^n \Delta_{n,j}\int_{t_{n,j-1}}^{t_{n,j}} \sigma_s^{2p} ds}})|\sigma)) \\
= \; & iu\frac{\sum_{i=1}^n \Delta_{n,i}(\frac{c_0}{\beta})^{p/\beta}\sigma_{n,i}^{*p}(\log\phi_{|S^\beta|^p})'(0) - \mu_{p,\beta}(\frac{c_0}{\beta})^{p/\beta}\int_0^t \sigma_s^p ds}{\sqrt{\nu_{p,\beta}(\frac{c_0}{\beta})^{2p/\beta}\sum_{j=1}^n \Delta_{n,j}\int_{t_{n,j-1}}^{t_{n,j}} \sigma_s^{2p} ds}} \qquad (17) \\
& -\frac{u^2}{2}\sum_{i=1}^n(\frac{2\Delta_{n,i}^2(\frac{c_0}{\beta})^{2p/\beta}\sigma_{n,i}^{*2p}}{\nu_{p,\beta}(\frac{c_0}{\beta})^{2p/\beta}\sum_{j=1}^n \Delta_{n,j}\int_{t_{n,j-1}}^{t_{n,j}} \sigma_s^{2p} ds} \qquad (18) \\
& \times\int_0^1 (1-s)(-1)(\log\phi_{|S^\beta|^p})''(s\frac{\Delta_{n,i}(\frac{c_0}{\beta})^{p/\beta}\sigma_{n,i}^{*p}}{\sqrt{\nu_{p,\beta}(\frac{c_0}{\beta})^{2p/\beta}\sum_{j=1}^n \Delta_{n,j}\int_{t_{n,j-1}}^{t_{n,j}} \sigma_s^{2p} ds}})ds) \\
& \qquad (19)
\end{aligned}
$$

where $\phi_{|S^\beta|^p}$ denotes the characteristic function of $|S^\beta|^p$ and hence $(\log\phi_{|S^\beta|^p})'(0) = E|S^\beta|^p$. Furthermore, we have used the notation $\int_{t_{n,i-1}}^{t_{n,i}} \sigma_s^\beta ds = \Delta_{n,i}\sigma_{n,i}^{*\beta}$ and denoted by S^β a β-stable random variable.

Now we have to show that as $n \to \infty$ the first summand (17) tends to zero. Applying condition (11) and using the notation $\int_{t_{n,i-1}}^{t_{n,i}} \sigma_s^p ds = \Delta_{n,i}\bar{\sigma}_{n,i}^p$ it tends to zero, since

$$
|\frac{\sum_{j=1}^n \Delta_{n,j}(\sigma_{n,j}^{*p} - \bar{\sigma}_{n,j}^p)\mu_{p,\beta}(\frac{c_0}{\beta})^{p/\beta}}{\sqrt{\nu_{p,\beta}(\frac{c_0}{\beta})^{2p/\beta}\sum_{j=1}^n \Delta_{n,j}\int_{t_{n,j-1}}^{t_{n,j}} \sigma_s^{2p} ds}}| \leq |\frac{\sum_{j=1}^n \Delta_{n,j}(\sigma_{n,j}^{*p} - \bar{\sigma}_{n,j}^p)\mu_{p,\beta}(\frac{c_0}{\beta})^{p/\beta}}{\sqrt{\min_j \Delta_{n,j}}\sqrt{\nu_{p,\beta}(\frac{c_0}{\beta})^{2p/\beta}\int_0^t \sigma_s^{2p} ds}}|.
$$

For the second summand (18) we can show similarly that the argument of ϕ'' tends to zero as $n \to \infty$. Using $-(\log\phi_{|S^\beta|^p})''(0) = Var|S^\beta|^p$ and

$$
\frac{\sum_{j=1}^n \Delta_{n,j}^2\sigma_{n,j}^{*2p}}{\sum_{j=1}^n \Delta_{n,j}^2\bar{\sigma}_{n,j}^{2p}} \to 1, \qquad (20)
$$

we obtain the desired result. (20) holds applying condition (11) and that the partitioning is at least 1/2-balanced:

$$\begin{aligned}|\frac{\sum_{j=1}^n \Delta_{n,j}^2(\sigma_{n,j}^{*2p}-\bar{\sigma}_{n,j}^{2p})}{\sum_{j=1}^n \Delta_{n,j}^2\bar{\sigma}_{n,j}^{2p}}| &\leq \frac{\max_j \Delta_{n,j}}{\sqrt{\min_j \Delta_{n,j}}}|\frac{\sum_{j=1}^n \Delta_{n,j}(\sigma_{n,j}^{*2p}-\bar{\sigma}_{n,j}^{2p})}{\sqrt{\min_j \Delta_{n,j}}\int_0^t \sigma_s^{2p}ds}| \\ &\to 0.\end{aligned}$$

This yields that (18/19) tends to $-u^2/2$ and hence provides (16).

It remains to show that we can repace the denominator in (14) by $\sqrt{\mu_{2p,\beta}^{-1}\nu_{p,\beta}V_{2p}(Z,S_n,\Delta^{2-2p/\beta})}$, hence it is sufficient to show that

$$|\frac{\sum_{j=1}^n \Delta_{n,j}^2(|\frac{S_j^\beta+R_j}{\Delta_{n,j}^{1/\beta}}|^{2p}-\bar{\sigma}_{n,j}^{2p}\mu_{2p,\beta}(\frac{c_0}{\beta})^{2p/\beta})}{\sum_{j=1}^n \Delta_{n,j}^2\bar{\sigma}_{n,j}^{2p}\mu_{2p,\beta}(\frac{c_0}{\beta})^{2p/\beta}}| \quad \xrightarrow{p} \quad 0.$$

As before we have to distinguish the case for $2p \leq 1$ and $2p > 1$. We restrict ourselves to show $2p \leq 1$:

$$\begin{aligned}&|\frac{\sum_{j=1}^n \Delta_{n,j}^2(|\frac{S_j^\beta+R_j}{\Delta_{n,j}^{1/\beta}}|^{2p}-\bar{\sigma}_{n,j}^{2p}\mu_{2p,\beta}(\frac{c_0}{\beta})^{2p/\beta})}{\sum_{j=1}^n \Delta_{n,j}^2\bar{\sigma}_{n,j}^{2p}\mu_{2p,\beta}(\frac{c_0}{\beta})^{2p/\beta}}| \\ &\leq \quad \frac{\sum_{j=1}^n \Delta_{n,j}^{2-2p/\beta}|R_j|^{2p}}{\sum_{j=1}^n \Delta_{n,j}^2\bar{\sigma}_{n,j}^{2p}\mu_{2p,\beta}(\frac{c_0}{\beta})^{2p/\beta}} + \frac{\sum_{j=1}^n \Delta_{n,j}^2\sigma_{n,j}^{*2p}(|S^\beta|^{2p}-\mu_{2p,\beta})}{\sum_{j=1}^n \Delta_{n,j}^2\bar{\sigma}_{n,j}^{2p}\mu_{2p,\beta}} \\ &\quad + \frac{\sum_{j=1}^n \Delta_{n,j}^2(\sigma_{n,j}^{*2p}-\bar{\sigma}_{n,j}^{2p})}{\sum_{j=1}^n \Delta_{n,j}^2\bar{\sigma}_{n,j}^{2p}}.\end{aligned}$$

The first term tends to zero as shown for (15), the third as for (20). For the second term we can use the same method as in Barndorff-Nielsen and Shephard (2002), namely that the convergence of this term holds if, conditional under σ, the standard deviation of the numerator is of smaller order than the denominator, as $n \to \infty$:

$$\frac{\sqrt{\nu_{2p,\beta}}}{\mu_{2p,\beta}}\frac{\sqrt{\sum_{j=1}^n \Delta_{n,j}^4\sigma_{n,j}^{*4p}}}{\sum_{j=1}^n \Delta_{n,j}^2\sigma_{n,j}^{*2p}} \leq \frac{(\max_j \Delta_{n,j})^{3/2}}{\min_j \Delta_{n,j}}\frac{\sqrt{\sum_{j=1}^n \Delta_{n,j}\sigma_{n,j}^{*4p}}}{\sum_{j=1}^n \Delta_{n,j}\sigma_{n,j}^{*2p}},$$

where the first term tends to zero since the partition is at least 2/3-balanced and the second tends to a constant. This completes the proof of (13).

Now we have to add the process Y to prove the general form of (12). We use the following reformulation

$$\frac{V_p(X,S_n,\Delta_n^{1-p/\beta})-\mu_{p,\beta}(\frac{c_0}{\beta})^{p/\beta}\int_0^t \sigma_s^p ds}{\sqrt{\mu_{2p,\beta}^{-1}\nu_{p\beta}V_{2p}(X,S_n,\Delta_n^{2-2p/\beta})}}$$

$$= \frac{V_p(X,S_n,\Delta_n^{1-p/\beta}) - V_p(Z,S_n,\Delta_n^{1-p/\beta})}{\sqrt{\mu_{2p,\beta}^{-1}\nu_{p,\beta}V_{2p}(X,S_n,\Delta_n^{2-2p/\beta})}}$$
$$+\frac{V_p(Z,S_n,\Delta_n^{1-p/\beta}) - \mu_{p,\beta}(\frac{c_0}{\beta})^{p/\beta}\int_0^t \sigma_s^p ds}{\sqrt{\mu_{2p,\beta}^{-1}\nu_{p,\beta}V_{2p}(Z,S_n,\Delta_n^{2-2p/\beta})}} \frac{\sqrt{\mu_{2p,\beta}^{-1}\nu_{p,\beta}V_{2p}(Z,S_n,\Delta_n^{2-2p/\beta})}}{\sqrt{\mu_{2p,\beta}^{-1}\nu_{p\beta}V_{2p}(X,S_n,\Delta_n^{2-2p/\beta})}},$$

which leads to the desired result if we can show

$$\frac{V_p(X,S_n,\Delta_n^{1-p/\beta}) - V_p(Z,S_n,\Delta_n^{1-p/\beta})}{\sqrt{\mu_{2p,\beta}^{-1}\nu_{p,\beta}V_{2p}(X,S_n,\Delta_n^{2-2p/\beta})}} \xrightarrow{p} 0 \tag{21}$$

$$\frac{\sqrt{\mu_{2p,\beta}^{-1}\nu_{p,\beta}V_{2p}(Z,S_n,\Delta_n^{2-2p/\beta})}}{\sqrt{\mu_{2p,\beta}^{-1}\nu_{p,\beta}V_{2p}(X,S_n,\Delta_n^{2-2p/\beta})}} \xrightarrow{p} 1. \tag{22}$$

For the proof of (21) we can use the same technique as in Theorem 1, noting that for $p \leq 1$

$$\frac{|V_p(X,S_n,\Delta_n^{1-p/\beta}) - V_p(Z,S_n,\Delta_n^{1-p/\beta})|}{\sqrt{\mu_{2p,\beta}^{-1}\nu_{p\beta}V_{2p}(X,S_n,\Delta_n^{2-2p/\beta})}} \leq \frac{V_p(Y,S_n,\Delta_n^{1-p/\beta})}{\sqrt{\min_i \Delta_{n,i}}\sqrt{\mu_{2p,\beta}^{-1}\nu_{p,\beta}V_{2p}(X,S_n,\Delta_n^{1-2p/\beta})}},$$

where

$$\frac{V_p(Y,S_n,\Delta_n^{1-p/\beta})}{\sqrt{\min_i \Delta_{n,i}}} \xrightarrow{p} 0,$$

$$\sqrt{\mu_{2p,\beta}^{-1}\nu_{p,\beta}V_{2p}(X,S_n,\Delta_n^{1-2p/\beta})} \xrightarrow{p} \sqrt{\nu_{p,\beta}(\frac{c_0}{\beta})^{p/\beta}\int_0^t \sigma_s^{2p} ds}$$

as $n \to \infty$ by the assumptions together with Theorem 1.

Finally, it remains to prove (22). Equivalently, we can show that

$$\frac{|\mu_{2p,\beta}^{-1}\nu_{p,\beta}V_{2p}(Z,S_n,\Delta_n^{2-2p/\beta}) - \mu_{2p,\beta}^{-1}\nu_{p,\beta}V_{2p}(X,S_n,\Delta_n^{2-2p\beta})|}{\mu_{2p}^{-1}\nu_{p,\beta}V_{2p}(X,S_n,\Delta_n^{2-2p/\beta})} \xrightarrow{p} 0$$

Now we can proceed similarly as for (21)

$$\frac{|\mu_{2p,\beta}^{-1}\nu_{p,\beta}V_{2p}(Z,S_n,\Delta_n^{2-2p/\beta}) - \mu_{2p,\beta}^{-1}\nu_{p\beta}V_{2p}(X,S_n,\Delta_n^{2-2p/\beta})|}{\mu_{2p,\beta}^{-1}\nu_{p,\beta}V_{2p}(X,S_n,\Delta_n^{2-2p/\beta})}$$
$$\leq \frac{|\mu_{2p,\beta}^{-1}\nu_{p,\beta}V_{2p}(Y,S_n,\Delta_n^{2-2p/\beta})|}{\mu_{2p,\beta}^{-1}\nu_{p,\beta}V_{2p}(X,S_n,\Delta_n^{2-2p/\beta})}$$
$$\leq \frac{|\mu_{2p,\beta}^{-1}\nu_{p,\beta}V_{2p}(Y,S_n,\Delta_n^{2-2p/\beta})|}{\min_i \Delta_{n,i}\mu_{2p,\beta}^{-1}\nu_{p,\beta}V_{2p}(X,S_n,\Delta_n^{1-2p/\beta})}$$

Piecing together (13), (21) and (22) leads to the desired result. □

Let us discuss the conditions on Y first. The conditions are stronger than for Theorem 1, especially since we need that (8) is satisfied for $2p$.

In addition, we have two parameters which determine how to choose p, namely the balance coefficient ϵ, describing the regularity of the sampling scheme, and the Hölder coefficient or Blumenthal-Getoor index describing the regularity of the sample paths of Y. It turns out the more regular the sampling scheme the less regularity we need in the sample paths and vice versa.

- Similarly as for condition (3), condition (8) is satisfied if Y is Hölder continuous with exponent a, $1 \geq a > 1/\beta$.

  Condition (9) is satisfied, if $\max_j \Delta_{n,j}^{p(a-1/\beta)} / \min_j \Delta_{n,j}^{1/2} \to 0$, hence if S_n is $\beta/(2p(a\beta-1))$-balanced. Hence we must satisfy at least $1 > \beta/(2p(a\beta-1))$, which is however only for the equally spaced case. Taking into account the range of a, we see that $p > \beta/(2(\beta-1))$, hence at most $p > 1$. If we have an ϵ-balanced partition $\epsilon \in [2/3, 1)$, we need $\epsilon \geq \beta/(2p(a\beta-1))$, hence

  $$p \geq \beta/(2\epsilon(a\beta-1)) \tag{23}$$

  The relation (23) allows us to compute the possible range p, when we know the regularity of the paths and the regularity of the partition.

  Finally (10) holds if (9) is satisfied.

- Similarly to the discussion for Theorem 1, for (8) to hold, we need $2p < \beta \leq 2$ hence $p < \beta/2$. Furthermore, (9) is satisfied if we have finite p-th power variation and $\max_i \Delta_{n,i}^{1-p/\beta} / \min_i \Delta_{n,i}^{1/2} \to 0$, hence if the partition is $\beta/(2(\beta-p))$-balanced. This leads to the range $\beta/2 > p$ in an equally spaced setting, or $\beta - \beta/(2\epsilon) \geq p$ for an ϵ-balanced partition. Hence knowing the regularity of our partitioning we can calculate the possible range of p, e.g. in the most irregular spaces case we have to take $\beta/4 \geq p$. Obviously conditions (8) and (9) imply (10).

Our considerations for continuous mean processes and mean processes with jumps show that for the distributional theory we do not get a result when the mean process is a mixture of both since the range of possible values of p is disjoint. However, in practice this does not seem to be very restrictive since the data is normally treated to get rid of the drift term anyway. Condition (12) ensures that $\sum_i \Delta_{n,i}^{1-p/\beta} | \int_{t_{n,i-1}}^{t_{n,i}} \sigma_s ds|^p$ converges fast enough to $\int \sigma_s^p ds$ as required for (16) to hold. In Barndorff-Nielsen and Shephard (2003) it is shown that (12) is satisfied for Ornstein-Uhlenbeck type processes. Unfortunately, for continuous processes σ it is only clear that condition (12) holds if we have Hölder continuity with exponent $> 1/2$, hence more regularity than for a Brownian motion or a diffusion process, normally used as volatility process.

Example:(Ornstein-Uhlenbeck-type model including leverage)

We can modify the model by Barndorff-Nielsen and Shephard (2001)

$$\begin{aligned} dX_t &= \sigma_t dL_t + \rho d\bar{Z}_{\lambda t} \\ d\sigma_t^2 &= -\lambda \sigma_t^2 dt + dZ_{\lambda t}, \end{aligned}$$

where $\bar{Z}_t = Z_t - E(Z_t)$ and it is assumed that the subordinator Z with Blumenthal-Getoor index γ is independent of the purely discontinuous Lévy process L with Blumenthal-Getoor index β. For the equally spaced setting we obtain the distributional theory for $\gamma < p < 1/\beta$, or $\gamma < p \leq \beta - \beta/(2\epsilon)$ for the ϵ-balanced setting. This implies that we have to choose a Lévy process Z with sufficiently small Blumenthal-Getoor index γ, e.g. a compound Poisson process or a Gamma process.

5 Estimating the scale parameter

Leaving the framework of stochastic volatility models by setting the volatility process σ equal to one, we obtain a Lévy process and with the p-th power variation we can estimate the first order coefficient of the Taylor series expansion of the density of the Lévy measure c_0. This coefficient c_0 is obviously equal to some constant times the scale parameter which enters the Lévy triplet as a constant multiplicative term of the Lévy measure and hence changes the time scale. Rammeh (1997) and Woerner (2003c) provided local asymptotical normality for the special case of the scale parameter of stable processes, but unfortunately it turns out that our power variation estimates are not efficient in this sense.

Corollary 1 *Assume we have a purely discontinuous Lévy process L_t^β with Blumenthal-Getoor index $2 > \beta > 0$. Assume that the Lévy triplet is given in the same form as in Theorem 1. Then for any $t > 0$, we obtain for $0 < p < \beta$, if L_t^β is a β-stable process, or $0 < p < \beta$, $p \neq \beta - 1$*

$$V_p(L, S_n, \Delta_n^{1-p/\beta}) \xrightarrow{p} \mu_{p,\beta}(\frac{c_0}{\beta})^{p/\beta} t, \tag{24}$$

and for any sequence of ϵ-balanced partitions S_n, $\epsilon \in [2/3, 1]$ and $0 < p < \beta - \beta/(2\epsilon)$, if L_t^β is β-stable, or $0 < p < \beta - \beta/(2\epsilon)$, $p \neq \beta - 1, (\beta - 1)/2$ otherwise, we obtain

$$\frac{V_p(L, S_n, \Delta_n^{1-p/\beta}) - \mu_{p,\beta}(\frac{c_0}{\beta})^{p/\beta} t}{\sqrt{\mu_{2p,\beta}^{-1} \nu_{p,\beta} V_{2p}(L, S_n, \Delta_n^{2-2p/\beta})}} \xrightarrow{\mathcal{D}} N(0,1), \tag{25}$$

as $n \to \infty$, where $\mu_p = E(|U|^p)$ and $\nu_p = Var(|U|^p)$ with $U \sim S^\beta$ is a symmetric β-stable random variable.

Example:
With the power variation method we can estimate the scale parameter α of a β-stable process, where $c_0 = \alpha\beta$. For generalized hyperbolic Lévy motions we can

infer the scale parameter δ, since in this case $c_0 = \delta/\pi$. Raible (2000) proposed a different type of estimator for the scale parameter of the generalized hyperbolic Lévy motion based on counting the number of jumps. For the CGMY process we can infer the parameter C, which is a measure for the overall level of activity of the process.

6 Stochastic volatility based on time changed Lévy processes

Carr et.al. (2003) proposed a different construction of a Lévy type stochastic volatility model using a time changed Lévy process. From the perspective of fitting data to the model, their construction seems to be more tractable since the characteristic function is straightforward to calculate.

In our framework of estimating the integrated volatility the two models do not behave very differently, only that for the time changed Lévy process we estimate the p/β-th integrated volatility with the p-th power variation instead of the p-th integrated volatility as for the other model. However, since many commonly used Lévy processes like the generalized hyperbolic Lévy motion have $\beta = 1$ it does not lead to a different result.

Theorem 3 *Let*

$$X_t = Y_t + L^{\beta}_{\int_0^t \sigma_s ds}, \tag{26}$$

Denote by L^{β}_t a purely discontinuous Lévy process with Blumenthal-Getoor index $2 > \beta > 0$. Assume that the Lévy triplet is given by $(0, 0, tg(x))_h$. Assume furthermore that the density of the Lévy measure is Riemann integrable and can be expanded in the following Taylor series expansion as $x \to 0$

$$\begin{aligned} g(x) &= \sum_{i=0}^{n} \frac{c_i x^i + k_i |x|^i}{|x|^{1+\beta}} + o(\frac{|x|^n}{|x|^{1+\beta}}) \\ &= \frac{c_0}{|x|^{1+\beta}} + f(x), \end{aligned}$$

with $\beta - n \leq 0$ and $k_0 = 0$.

Assume that σ and Y satisfy the conditions as in Theorem 1, then for any $t > 0$, we obtain for $0 < p < \beta$, if L_t is stable, or $0 < p < \beta$, $p \neq \beta - 1$,

$$V_p(X, S_n, \Delta_n^{1-p/\beta}) \xrightarrow{p} \mu_{p,\beta}(\frac{c_0}{\beta})^{p/\beta} \int_0^t \sigma_s^{p/\beta} ds, \tag{27}$$

as $n \to \infty$.

Proof. The proof is similar as for Theorem 1. The calculation of the conditional characteristic function is even easier

$$E(\exp(iu(L_{\int_0^{t_i} \sigma_s ds} - L_{\int_0^{t_{i-1}} \sigma_s ds}))|\sigma)$$

$$
\begin{aligned}
&= \exp\Big(\int_{t_{i-1}}^{t_i} \sigma_s ds \int (e^{iux} - 1 - iuh(x))g(x)dx\Big) \\
&= \exp\Big(\int_{t_{i-1}}^{t_i} \sigma_s ds \int (e^{iux} - 1 - iuh(x))\frac{c_0}{|x|^{1+\beta}}dx\Big) \\
&\quad \times \exp\Big(\int_{t_{i-1}}^{t_i} \sigma_s ds \int (e^{iux} - 1 - iuh(x))f(x)dx\Big).
\end{aligned}
$$

Hence we can split $L_{\int_0^{t_i} \sigma_s ds} - L_{\int_0^{t_{i-1}} \sigma_s ds}$ into a part S_i^β which is conditional under σ stable with scale parameter $(c_0/\beta)\int_{t_{n,i-1}}^{t_{n,i}} \sigma_s ds$ and characteristic triplet $(0, 0, c_0 \int_{t_{n,i-1}}^{t_{n,i}} \sigma_s ds/|x|^{1+\beta})_h$ and a remainder R_i which is conditional under σ additive with characteristic triplet $(0, 0, \int_{t_{n,i-1}}^{t_{n,i}} \sigma_s ds f(x))_h$. By construction S_i^β and R_i conditional under σ are sequences of independent random variables. Note that the difference to Theorem 1 is, that in the scale parameter of the stable random variable we only have σ, which results in the appropriate change in the limit distribution due to the scaling relation for stable processes. Since we do not have a drift term in R_i we do not need the conditions a) to d) on the Lévy measure. The remaining part of the proof goes along the same lines as for Theorem 1. $\square$

Theorem 4 *Let*

$$X_t = Y_t + L^\beta_{\int_0^t \sigma_s ds},$$

where Y_t and σ satisfy the same conditions as in Theorem 2 and L satisfies the same conditions as in Theorem 3. Then for any $t > 0$, for any sequence of ϵ-balanced partitions S_n, $\epsilon \in [2/3, 1]$ and $0 < p < \beta - \beta/(2\epsilon)$, if L_t is β-stable, or $0 < p < \beta - \beta/(2\epsilon)$, $p \neq \beta - 1, (\beta - 1)/2$ otherwise, we obtain

$$\frac{V_p(X, S_n, \Delta_n^{1-p/\beta}) - \mu_{p,\beta}(\frac{c_0}{\beta})^{p/\beta}\int_0^t \sigma_s^{p/\beta} ds}{\sqrt{\mu_{2p,\beta}^{-1}\nu_{p,\beta} V_{2p}(X, S_n, \Delta_n^{2-2p/\beta})}} \xrightarrow{\mathcal{D}} N(0,1), \tag{28}$$

as $n \to \infty$.

Proof. Taking into account the remarks in the proof of Theorem 3, the proof is analogously as for Theorem 2. $\square$

Example: (Time change with Ornstein-Uhlenbeck-type process)
We can modify the model by Carr et.al. (2003)

$$
\begin{aligned}
X_t &= \bar{Z}_t + L^\beta_{\int_0^t \sigma_s ds} \\
d\sigma_t^2 &= -\lambda \sigma_t^2 dt + dZ_{\lambda t},
\end{aligned}
$$

where $\bar{Z}_t = Z_t - E(Z_t)$ and it is assumed that the subordinator Z with Blumenthal-Getoor index γ is independent of the purely discontinuous Lévy process L with

Blumenthal-Getoor index β. For the equally spaced setting we obtain the distributional theory for $\gamma < p < 1/\beta$, or $\gamma < p \leq \beta - \beta/(2\epsilon)$ for the ϵ-balanced setting. This implies that we have to choose a Lévy process Z with sufficiently small Blumenthal-Getoor index γ, e.g. a compound Poisson process or a Gamma process.

Acknowledgements. I would like to thank Prof. Dr. L. Rüschendorf for helpful comments and discussions.

References

[1] O. E. Barndorff-Nielsen and N. Shephard. Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics (with discussion). *Journal of the Royal Statistical Society, Series B*, 63:167–241, 2001.

[2] O. E. Barndorff-Nielsen and N. Shephard. Power variation and time change. Technical report, www.levyprocess.org, 2002.

[3] O. E. Barndorff-Nielsen and N. Shephard. Realised power variation and stochastic volatility models. *Bernoulli*, 9:243–265, 2003.

[4] S. M. Berman. Sign-invariant random variables and stochastic processes with sign invariant increments. *Trans. Amer. Math. Soc*, 119:216–243, 1965.

[5] P. Carr, H. Geman, D. B. Madan, and M. Yor. The fine structure of asset returns: an empirical investigation. *Journal of Business*, 75:305–332, 2002.

[6] P. Carr, H. Geman, D. B. Madan, and M. Yor. Stochastic volatility for Lévy processes. *Math. Finance*, 13:345–382, 2003.

[7] E. Eberlein, J. Kallsen, and J. Kristen. Riskmanagement based on stochastic volatility. *Journal of Risk*, 2:19–44, 2003.

[8] E. Eberlein and E. A. von Hammerstein. Generalized hyperbolic and inverse Gaussian distributions: limiting cases and approximation of processes. *Proceedings of the 4th Ascona Conference, Birkhäuser Verlag*, 2003.

[9] S. Howison, A. Rafailidis, and H. Rasmussen. On the pricing and hedging of volatility derivatives. Working paper, OCIAM, University of Oxford, 2003.

[10] W. N. Hudson and J. D. Mason. Variational sums for additive processes. *Proc. Amer. Math. Soc*, 55:395–399, 1976.

[11] D. Hull and A. White. The pricing of options on assets with stochastic volatilities. *Journal of Finance*, 42:281–300, 1987.

[12] D. Lepingle. La variation d'ordre p des semi-martingales. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 36:295–316, 1976.

[13] S. Raible. *Lévy processes in finance: Theory, numerics, and empirical facts.* PhD thesis, University of Freiburg, 2000.

[14] H. Rammeh. *Thèse de l'Université.* PhD thesis, University of Paris 6, 1997.

[15] K. Sato. *Lévy Processes and Infinitely Divisible Distributions.* Cambridge University Press, Cambridge, 1999.

[16] L. Scott. Option pricing when the variance changes randomly: theory, estimation and an application. *Journal of Financial and Quantitative Analysis*, 22:419–438, 1987.

[17] A. N. Shiryaev. *Essentials of Stochastic Finance: Facts, Models and Theory.* World Scientific, Singapore, 1999.

[18] E. Stein and C. Stein. Stock price distributions with stochastic volatility: an analytic approach. *Review of Financial Studies*, 4:727–752, 1991.

[19] J. H. C. Woerner. Variational sums and power variation: a unifying approach to model selection and estimation in semimartingale models. *Statistics & Decisions*, 21:47–68, 2003a.

[20] J. H. C. Woerner. Estimation of integrated volatility in stochastic volatility models. 2003-MF-05, Working Paper Series in Mathematical Finance, University of Oxford, www.finance.ox.ac.uk, 2003b.

[21] J. H. C. Woerner. Local asymptotic normality for the scale parameter of stable processes. *Statistics & Probability Letters*, 63:61–65, 2003c.