An Asymptotic Analysis of an American Call Option with Small Volatility

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Summary. In this paper we present an asymptotic analysis of an American call option where the diffusion term (volatility) is small compared to the drift terms (interest rate and continuous dividend yield). We show that in the limit where diffusion is negligible, relative to drift, then, at leading order, the American call's behaviour is the same as a perpetual American call option (except in a boundary layer about the option's expiry date).

1 Introduction

The free boundary formulation for the American call option is analysed using asymptotic analysis. We make the key assumption that $\epsilon^2 = \frac{\sigma^2}{|r-q|} \ll 1$ is a small parameter and use this in our expansion. We do not reduce the problem to the heat equation.

Previous analyses include [2] and [1]. The first uses the Green's function for the heat equation to convert the boundary value problem to an integral equation, which is then solved asymptotically for times close to expiry. In [1] various asymptotic limits of the heat equation form of the problem are investigated.

In Section 2 we present an asymptotic analysis of the European call option, which we then extend in Section 3 to the American call option.

1.1 Problem Formulation

The Black–Scholes equation for the value, $V(S,t)$, of a European call option is,

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0, \tag{1}$$

with the final (or payoff) condition at the option's expiry date T

$$V(S,T) = \max\,(S-E, 0), \tag{2}$$

where S is the price of the underlying risky asset, σ is the volatility of S, r is the constant risk-free interest rate, $q \neq r$ is the constant, continuous dividend yield on S and t denotes time.

An American option can be exercised at any time up to and including the expiry date T. The option value problem can be formulated, after [3], as a free boundary problem where $S^*(t)$ represents the location of the free boundary at time t. The problem for the American option is the same as that defined by (1) and (2), with the additional (free boundary) conditions that

$$V(S^*(t),t) = S^*(t) - E \tag{3}$$

$$\frac{\partial}{\partial S} V(S^*(t),t) = 1. \tag{4}$$

1.2 Non-dimensionalization

We assume $r \neq q$ throughout and introduce the following transformations,

$$V = E\bar{V}\ , \quad S = E\bar{S} \quad \text{and} \quad \tau = (T-t)|r-q|\ , \tag{5}$$

and parameters,

$$\epsilon^2 = \frac{\sigma^2}{|r-q|} \quad \text{and} \quad k = \frac{r}{|r-q|}, \tag{6}$$

to write the European problem (1)–(2) in non-dimensional form as

$$\bar{V}_\tau = \tfrac{1}{2}\epsilon^2 \bar{S}^2 \bar{V}_{\bar{S}\bar{S}} + \bar{S}\bar{V}_{\bar{S}} - k\bar{V}$$
$$\bar{V}(\bar{S},0) = \max(\bar{S}-1,0),$$

where we have used subscripts to indicate partial derivatives, that is $\bar{V}_\tau$ means $\partial\bar{V}/\partial\tau$. The American problem will be non-dimensionalized using the same transformations and parameters later in this paper.

2 European Call Option

We use the transformation

$$\bar{V} = e^{-k\tau} U$$

and expand the solution using

$$U \sim U_0 + \epsilon^2 U_1 + \ldots$$

At leading order, we obtain the first order hyperbolic equation

$$U_{0,\tau} - \bar{S} U_{0,S} = 0$$

with Cauchy data $U_0(\bar{S},0) = \max(\bar{S}-1,0)$. The method of characteristics implies that

$$U_0 = \max(\bar{S}e^\tau - 1, 0)\ .$$

Note that U_0's first derivative with respect to $\bar{S}$ is not continuous along the characteristic $\bar{S}e^\tau$, and hence that its second $\bar{S}$ derivative involves a delta-function along this characteristic. Specifically, in an $O(\epsilon)$ region about this characteristic, the second $\bar{S}$ partial derivative of $\epsilon^2 U_0$ is $O(1)$ rather than $O(\epsilon^2)$.

Writing this in terms of $\bar{V}$ gives

$$V_0 = e^{-k\tau}\max(\bar{S}e^\tau - 1, 0). \tag{7}$$

The assumption that the second derivative is small is not valid along the discounted strike value, therefore we look for an inner region here.

2.1 Inner Solution

We transform to an inner variable, keeping our time-like τ, but use a space-like inner variable x, defined by

$$\bar{S} = e^{-\tau} + \epsilon x \ .$$

If we then use an expansion of the form

$$U \sim \epsilon U_0 + \epsilon^2 U_1 + \ldots$$

equations for the leading and first order equations reduce to

$$\begin{aligned} U_{0,\bar{\tau}} &= U_{0,\zeta\zeta} & U_0(\zeta,0) &= \max(\zeta,0) \\ U_{1,\bar{\tau}} &= U_{1,\zeta\zeta} + 2\zeta U_{0,\zeta\zeta} & U_1(\zeta,0) &= 0, \end{aligned}$$

where ζ and $\bar{\tau}$ are defined by

$$\zeta = e^{\tau} x \quad \text{and} \quad \bar{\tau} = \tfrac{1}{2}\tau \ .$$

Solving the equations for U_0 and U_1 and expressing the solution in terms of the original non-dimensional variables we find that

$$\begin{aligned} \bar{V} \sim & \left(\bar{S}e^{\tau(1-k)} - e^{-k\tau}\right) \mathrm{N}\left[\frac{1}{\epsilon\sqrt{\tau}}(\bar{S}e^{\tau} - 1)\right] \\ & + \frac{\epsilon}{2}\sqrt{\frac{\tau}{2\pi}}\left(e^{-k\tau} + \bar{S}e^{(1-k)\tau}\right)\exp\left(-\tfrac{(\bar{S}e^{\tau}-1)^2}{2\epsilon^2\tau}\right) \ , \end{aligned} \tag{8}$$

where N[.] is the cumulative normal distribution function. The first term matches the outer solution automatically and the second term is always exponentially small away from the inner region, so we can leave it in the solution always and recover (7) in the outer region.

Verifying that this solution coincides with a suitable Taylor expansion of the exact Black–Scholes is a worthwhile exercise. This also enables us to find an expansion valid when $r = q$.

3 American Call Option

3.1 Case $r > q$

When we non-dimensionalize the American call option problem formulated in (1)–(4), using the transformations (5) and parameters (6) we get,

$$\begin{aligned} \bar{V}_\tau &= \tfrac{1}{2}\epsilon^2\bar{S}^2\bar{V}_{\bar{S}\bar{S}} + \bar{S}\bar{V}_{\bar{S}} - k\bar{V} \\ \bar{V}(\bar{S},0) &= \max(\bar{S}-1,0) \\ \bar{V}(S^*,\tau) &= S^* - 1 \\ \bar{V}_{\bar{S}}(S^*,\tau) &= 1. \end{aligned}$$

In the limit $\epsilon \ll 1$, when we expand in a regular asymptotic expansion in powers of ϵ and consider the leading order (first order hyperbolic) term, we find that there are two distinct regions, dependent on the boundary conditions involved. Let us denote the boundary between these regions by $\hat{S}(\tau)$. We call $0 < \bar{S} < \hat{S}$ the *lower* region and denote the option value by $\bar{V}^{\text{lower}}(\bar{S},\tau)$, the solution found is the same as in the European case for $r > q$, as given in (8).

We call $\hat{S} < \bar{S} < S^*$ the *upper* region, where the option value is given by $\bar{V}^{\text{upper}}(\bar{S},\tau)$. In this region the problem given after the asymptotic expansion is solved using conditions (3) and (4) from the free boundary rather than the terminal condition (2) used in the lower region.

Asymptotic Expansion

We make an asymptotic expansion, assuming that $0 < \epsilon^2 \ll 1$. We look for the generally valid outer solution. Let us expand as before

$$\bar{V}^{\text{upper}} \sim \bar{V}_0^{\text{upper}} + \epsilon^2 \bar{V}_1^{\text{upper}} + \dots ,$$

and also expand the free boundary as

$$S^* \sim S_0^* + \epsilon^2 S_1^* + \dots .$$

To leading order we have the first order hyperbolic equation

$$\bar{V}_{0,\tau}^{\text{upper}} - \bar{S}\bar{V}_{0,\bar{S}}^{\text{upper}} = -k\bar{V}_0^{\text{upper}},$$

which we solve using non-dimensionalized boundary conditions on the free boundary,

$$\bar{V}_0^{\text{upper}}(S^*,\tau) = S^* - 1 \quad \text{and} \quad \bar{V}_{0,\bar{S}}^{\text{upper}}(S^*,\tau) = 1 .$$

The first term in the expansion for the free boundary is time independent,

$$S_0^* = \frac{k-1}{k} , \tag{9}$$

and the value of the option is approximated by

$$\bar{V}_0^{\text{upper}}(\bar{S},\tau) = \frac{1}{k-1}\left(\frac{\bar{S}}{S_0^*}\right)^k \qquad \text{for} \qquad \hat{S} < \bar{S} < S_0^* ,$$

where the critical characteristic dividing the two regions is given by $\hat{S} = S_0^* e^{-\tau}$. Note that there is no time dependence in the value of the option in the upper region.

$\bar{V}_0^{\text{lower}}$ and $\bar{V}_0^{\text{upper}}$, and their first derivatives with respect to $\bar{S}$, are equal when evaluated on the critical characteristic $\hat{S}$ where the two regions meet.

If we analyse the ϵ^2 terms in the asymptotic expansions we find that the $\bar{V}^{\text{upper}}$ is independent of τ. In the original variables we have

$$V^{\text{upper}}(S,t) \sim E\left[\frac{r}{q} - 1 - \tfrac{1}{2}\sigma^2 \frac{Er}{(r-q)^2}\log\left(\frac{qS}{rE}\right)\right]\left(\frac{qS}{rE}\right)^{\frac{r}{r-q}} , \tag{10}$$

and the location of the free boundary is given by

$$S^* \sim \frac{Er}{q}\left(1 + \frac{\sigma^2}{2(r-q)}\right) . \tag{11}$$

3.2 Case $r < q$

We non-dimensionalize the American call option problem formulated in (1)–(4) using same transformations and non-dimensional parameters as in the European case and the American $r > q$ case. Making an asymptotic expansion in powers of ϵ we find that there is only one region within which the leading order solution is $V_0 \equiv 0$ and the free boundary is equal to the strike of the option $S_0^* = 1$. If we use the inner variable $y = (\bar{S} - 1)/\epsilon^2$ and also expand the boundary conditions as Taylor series we can find the asymptotic solution to $O(\epsilon^2)$. We state, in dimensional variables, the value of the option,

$$V \sim \frac{\sigma^2 E}{2(q-r)} \exp\left(\frac{2(q-r)}{\sigma^2 E}(S-E) - 1\right) \,, \tag{12}$$

and the free boundary

$$S^* \sim E\left(1 + \frac{\sigma^2}{2(q-r)}\right) \,. \tag{13}$$

3.3 Perpetual American Call Option

The perpetual American call option has no expiry date, so has no time dependence in the problem formulation, or problem solution. Let $V_\infty(S)$ indicate the value of the perpetual option. In dimensional variables, we have the ordinary differential equation

$$\tfrac{1}{2}\sigma^2 S^2 \frac{d^2 V_\infty}{dS^2} + (r-q)S\frac{dV_\infty}{dS} - rV_\infty = 0$$

with boundary conditions

$$V_\infty(0) = 0, \quad V_\infty(S^*) = S^* - E \quad \text{and} \quad \frac{dV_\infty}{dS}(S^*) = 1.$$

When $r > q$ and $0 < \epsilon^2 \ll 1$ and we take the limit as $\epsilon^2 \to 0$ we find that the option value is equal to (10) and the boundary matches (11).

When $r < q$, in the limit as $\epsilon^2 \to 0$ we can confirm that $V_\infty(S)$ tends to (12) and the expression for the free boundary tends to (13).

4 Conclusions

We have found time independent asymptotic expansions for the location of the free boundary of the American call option using the small parameter $\epsilon^2 = \frac{\sigma^2}{|r-q|} \ll 1$. We have shown that in the limit as $\epsilon^2 \to 0$ these equations match the limit of the solution to the perpetual American call problem (except in a boundary layer about the option's expiry date).

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