A matched asymptotic expansions approach to continuity corrections for discretely sampled options. Part 1: barrier options.

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Abstract

We discuss the 'continuity correction' that should be applied to relate the prices of discretely sampled barrier options and their continuously-sampled equivalents. Using a matched asymptotic expansions approach we show that the correction of Broadie, Glasserman & Kou (*Mathematical Finance* **7**, 325 (1997)) can be applied in a very wide variety of cases. We calculate the correction to higher order in terms of the expansion parameter (the scaled time between resets) and we show how to apply the correction in jump-diffusion and local volatility models.

1 Introduction

Barrier options are now standard in many markets, especially FX markets. Using variants of the method of images, it is easy to value a variety of contracts in the standard Black–Scholes framework provided that the barrier condition (knock-out, knock-in etc.) is applied continuously in time, referred to here as continuous sampling. However, for a variety of practical and legal reasons (for example, to avoid disputes as to whether the barrier was crossed or not), in many contracts the barrier condition is only applied at a discrete set of times, for example at the close of a trading day or week; we refer to these as reset times. (For references on discrete sampling in practice, see [1].) The discretely sampled option may be cheaper or more expensive to buy, depending on whether the writer or the holder bears the cost of those sample paths that cross the barrier level between reset times and then re-cross it before the next reset time, and hence trigger the barrier with continous sampling but do not trigger it with discrete sampling. For example, a continuously sampled down-and-out call option is cheaper than its discretely sampled equivalent because the asset price may fall below the barrier without triggering knock-out for the latter;

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by out-in parity ('down-and-out + down-and-in = vanilla'), the corresponding down-and in call is more expensive.

With just one reset, a typical barrier contract such as a down-and-out call can be reformulated as a compound option, but with more resets this approach rapidly becomes less feasible; although, remarkably, an exact solution can be found using a combination of the Z transform and the Wiener–Hopf method [5], it is not a simple expression and only applies to specific contracts in constant-parameter Black–Scholes models. It is therefore of interest to develop approximations valid for a large number of reset dates, relating the continuously and discretely sampled prices, especially if the approximation can be written in terms of a simple formula such as the image formula referred to above, or if it can be widely applied. Conversely, such an approximation is also of interest in relation to Monte-Carlo valuation of continuously-sampled options. If one simulates the path price using a finite number of timesteps, then even if the exact asset price path is simulated (for example by geometric Brownian motion) between resets, the simulation also allows the price to cross the barrier and return between time steps. Monte-Carlo pricing therefore suffers a bias due to discrete sampling of the barrier, and this can be corrected for using the an approximation for a large number of resets.

These questions have been considered in two important papers [1, 2] by Broadie, Glasserman & Kou (referred to jointly as BGK below) using a probabilistic approach involving results from renewal theory. They state that the continuity correction for a down-and-out call with barrier B on an asset with price S is

$$V_{\mathrm{d}}(S, t; B) = V_{\mathrm{cont}}(S, t; Be^{-\beta\sigma\sqrt{T/N}}) + O(\sigma^2 T/N),$$

where V_{cont} (resp. V_{d}) is the continuously (resp. discretely) sampled value, N is the number of equally-spaced sample dates, and

$$\beta = -\frac{\zeta(\frac{1}{2})}{\sqrt{2\pi}} \approx 0.5826,$$

where $\zeta(\cdot)$ is Riemann's zeta-function. That is, the barrier is apparently shifted down by an amount $B\sigma\beta\sqrt{T/N}$ (their result, as stated, is only correct to $O(\sigma\sqrt{T/N})$ and so it is potentially misleading to write $e^{-\beta\sigma\sqrt{T/N}}$ rather than $1 - \beta\sigma\sqrt{T/N}$; however, as we show below, the exponential barrier correction is correct to second order in certain cases). Their analysis is applicable only to the constant-parameter Black–Scholes model; however, the 'BGK correction' is widely used in practice in other situations even though this has yet to be formally justified. The paper [2] also discusses the Monte-Carlo issue, and extends the BGK correction to lookback contracts. Further extensions to other barrier contracts, including double barrier contracts, are described by [11, 7]; the former shows that the BGK correction can be applied to all 8 combinations of call/put, up/down, in/out contracts, while the latter confirms these results and extends them to double barrier options; the paper also gives a different approximation formula (which agrees with the BGK correction to $O(\sigma\sqrt{T/N})$)

for the price, but like the BGK correction itself, it contains terms of higher order whose presence is not justified by the accuracy of the approximation.

Our goal in this paper is to reinterpret the BGK correction using the method of matched asymptotic expansions. This gives a very transparent view of how the BGK correction works, and why it is widely applicable; it allows us to extend the correction to a variety of contracts and models; and it enables us to calculate higher order terms in the correction. Furthermore, it is in the nature of the method that we obtain accurate approximations both when the asset price is far from the barrier and when it is near the barrier.

2 Problem formulation

Although our method is very general, let us begin by considering the standard case of a down-and-out call option with barrier B in the usual Black–Scholes model. In the continuously sampled case the option value $V_{\mathrm{cont}}(S, t; B)$ satisfies the Black–Scholes equation

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q)S \frac{\partial V}{\partial S} - rV = 0, \qquad B < S < \infty, \quad 0 < t < T,$$

where σ is the asset price volatility, r is the risk-free rate and q is the dividend yield. At time T the option has a payoff which, for the present purposes, we do not need to specify, and on the barrier $S = B$ we have the knock-out condition $V_{\mathrm{cont}}(B, t; B) = 0$. The calculation of V_{cont} is a simple application of the method of images, using the result that if $V(S, t)$ is a solution of the Black–Scholes equation, then so is $S^{1-2(r-q)/\sigma^2} V(B^2/S, t)$. For the down-and-out call with strike K above the barrier, we have

$$V_{\mathrm{cont}}(S, t; B) = C_{\mathrm{v}}(S, t; K) - (S/B)^{1-2(r-q)/\sigma^2} C_{\mathrm{v}}(B^2/S, t; K)$$

where $C_{\mathrm{v}}(S, t; K)$ is the corresponding vanilla call price with strike K. (When the barrier is above the strike, two more terms must be added to this expression, to handle the jump in the payoff at $S = B$.)

We assume that the discretely sampled option is reset at N equally spaced reset times $t_1, t_2 = t_1 + \Delta T, \ldots T_N = T - \Delta T$, separated by an interval ΔT. (If the interval from the current time until the first reset is also ΔT, we have $\Delta T = T/(N + 1)$, but we do not assume this.) Then the option value $V_{\mathrm{d}}(S, t)$ (we suppress the dependence on the parameter B) satisfies the Black–Scholes equation between reset times, but now for $0 < S < \infty$ rather than $B < S < \infty$, and at a typical reset time t_i its value is updated by setting

$$V_{\mathrm{d}}(S, t_i-) = \begin{cases} V_{\mathrm{d}}(S, t_i+) & S > B \\ 0 & S \leq B. \end{cases}$$

That is, as the Black–Scholes equation is solved backwards from expiry, as we reach each reset time we discard the option values for $S < B$ and replace them

with zero to implement the knock-out condition. Note there is a discontinuity in $V_d(S, t_m-)$, at $S = B$, representing the value to the holder of asset price paths which just pass over the top of the barrier: they have a small but nonzero probability of reaching expiry without knockout on any remaining barrier.

This is the problem that is solved exactly in [5], following a transformation of the Black–Scholes equation to the heat equation and a Z transform in time. We use the method of matched asymptotic expansions in the limit $N \to \infty$. We first make the preliminary scaling

$$t = T - t'/\sigma^2.$$

For the remainder of the paper, time t' is measured back from expiry and scaled with σ^2 (the Black–Scholes equation is not completely non-dimensionalised, as the scaling invariance with respect to S and the linearity make this unnecessary).

The Black–Scholes equation to be solved is then

$$\frac{\partial V}{\partial t'} = \frac{1}{2}S^2\frac{\partial^2 V}{\partial S^2} + \alpha_1 S\frac{\partial V}{\partial S} - \alpha_2 V, \qquad \alpha_1 = (r-q)/\sigma^2, \quad \alpha_2 = r/\sigma^2. \quad (1)$$

Lastly, we define

$$\epsilon^2 = \sigma^2 \Delta T,$$

the scaled reset interval; for large N, ϵ is small, and this is our expansion parameter. (Note that BGK effectively use $\sigma\sqrt{T/(N+1)}$ as their small parameter, which amounts to assuming that the initial time is the first reset date, although of course if the option is to be considered at all we have $S > B$ at that time so knock-out is impossible. Our results are more general in that they allow any initial interval.)

3   Down-and-out call: approximate solution

The general structure of the approximate solution consists of an outer expansion, valid far above the barrier, and an inner expansion near a typical reset date, joined by matching (the regions are indicated in Figure 1; as discussed below, we should in principle include another outer region far below the barrier, but in view of the lack of practical interest in this region we omit it). This will enable us to compute the 'effective boundary conditions' for the outer solution, from which we can compute the continuity correction to the Black–Scholes value. The timescale for the inner region is, by definition of ϵ, fixed at $O(\epsilon^2)$, and in order to achieve a non-trivial balance in the Black–Scholes equation (1) the price-scale in this region must be $O(\epsilon B)$. We therefore define the inner variables (x, τ) near $S = B$ and near a typical reset time t'_i by

$$S = B(1 + \epsilon x), \quad t' = t'_i + \epsilon^2 \tau,$$

which we use throughout. Note that both τ and x are now dimensionless.
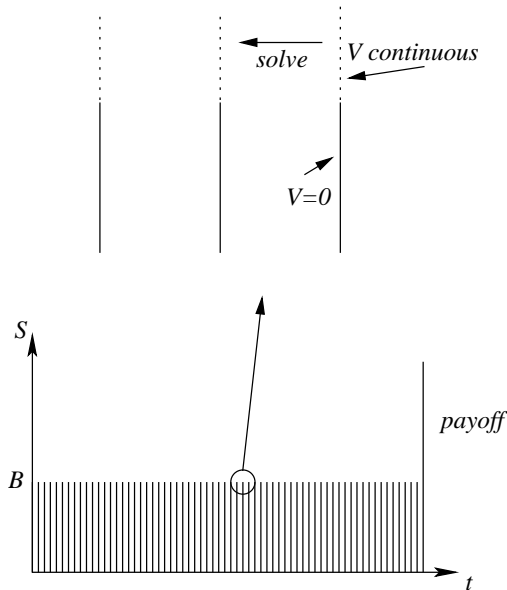
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Figure 1: Schematic of outer and inner regions for discretely sampled barrier options.

3.1 Outer expansion

Away from the barrier level we expect the solution to be close to the Black–Scholes value (for which we have a formula, even if it is only an integral representation). Hence we pose the outer expansion

$$V_{\rm d}(S, t') \sim V_{\rm cont}(S, t') + \epsilon V_1(S, t') + \epsilon^2 V_2(S, t) + O(\epsilon^3),$$

which we expect to be valid for $S/B - 1 \gg O(\epsilon)$, that is, not near the barrier. (The subscript d has been dropped from V_1 and V_2 for clarity: henceforth all option prices are discrete except for $V_{\rm cont}$. Likewise, the dependence on B is suppressed where not necessary.) At this stage we only know $V_{\rm cont}$ and we need to find effective boundary conditions for V_1 and V_2 at $S = B$. Having the matching with the inner expansion near a typical reset date in view, we can find the behaviour of the outer expansion near the barrier, for $x = O(1)$, by a

straightforward Taylor expansion:

$$V_{\mathrm{cont}} + \epsilon V_1 + \epsilon^2 V_2$$

$$\sim V_{\mathrm{cont}}(B,t') + (S-B)\frac{\partial V_{\mathrm{cont}}}{\partial S}(B,t') + \frac{1}{2}(S-B)^2\frac{\partial^2 V_{\mathrm{cont}}}{\partial S^2}(B,t')$$

$$+ \epsilon\left(V_1(B,t') + (S-B)\frac{\partial V_1}{\partial S}(B,t')\right) + \epsilon^2 V_2(B,t') + O(\epsilon^3)$$

$$\sim 0 + \epsilon\left(B\delta_{\mathrm{cont}}(t')x + V_1(B,t')\right)$$

$$+ \epsilon^2\left(\frac{1}{2}B^2\gamma_{\mathrm{cont}}(t')x^2 + Bx\delta_1(t') + V_2(B,t')\right) + O(\epsilon^3) \qquad (2)$$

where

$$\delta_{\mathrm{cont}}(t') = \frac{\partial V_{\mathrm{cont}}}{\partial S}(B,t') = \Delta_{\mathrm{cont}}(B,t'), \quad \gamma_{\mathrm{cont}}(t') = \frac{\partial^2 V_{\mathrm{cont}}}{\partial S^2}(B,t') = \Gamma_{\mathrm{cont}}(B,t')$$

are the Black–Scholes Delta and Gamma at the barrier and, for ease of notation, we also write

$$\frac{\partial V_1}{\partial S}(B,t') = \Delta_1(B,t') = \delta_1(t').$$

Using the scaled Black–Scholes equation (1) (and noting that, at $S=B$, $V_{\mathrm{cont}} = \partial V_{\mathrm{cont}}/\partial t' = 0$), we find that

$$\gamma_{\mathrm{cont}}(t') = -\frac{2\alpha_1}{B}\delta_{\mathrm{cont}}(t').$$

Equation (2) is the three-term inner expansion of the outer expansion, to be matched with the three-term outer expansion of the inner expansion, to which we now turn. Before doing so, we note that (2) contains a term linear in x at $O(\epsilon)$ and one quadratic in x at $O(\epsilon^2)$.

3.2 Inner expansion

As noted above, the inner variables near a typical reset time are defined by $S = B(1+\epsilon x)$, $t' = t'_i + \epsilon^2\tau$, and in view of (2) we set

$$V_{\mathrm{d}}(S,t) = \epsilon v(x,\tau)$$

(again, the subscript d on v is dropped). Then the inner problem is

$$\frac{\partial v}{\partial \tau} = \frac{1}{2}(1+\epsilon x)^2\frac{\partial^2 v}{\partial x^2} + \epsilon\alpha_1(1+\epsilon x)\frac{\partial v}{\partial x} - \epsilon^2\alpha_2 v, \quad -\infty < x < \infty,$$

with $v(x,\tau) \to 0$ as $x \to -\infty$ (this corresponds to small values of S for which knock-out is almost certain). The behaviour as $x \to +\infty$ is determined by matching with (2), giving

$$v(x,\tau) \sim B\delta_{\mathrm{cont}}(t')x + V_1(B,t') + \epsilon\left(\frac{1}{2}B^2\gamma_{\mathrm{cont}}(t')x^2 + B\delta_1(t')x + V_2(B,t')\right) + O(\epsilon^2)$$

as $x \to \infty$. It is the $O(1)$ constants (on the inner time scale) $V_1(B, t')$ and $V_2(B, t')$ that determine the effective boundary conditions for $V_1(S, t')$ and $V_2(S, t')$.

The knock-out condition is imposed in the form

$$v(x, 0+) = v(x, 1+) = v(x, 2+) = \cdots = 0,$$

for $x < 0$, corresponding to the barrier. Note, however, that $v(x, 1-) \neq 0$: these nonzero values are discarded and replaced with 0 as part of the reset process.

The last condition we impose on the inner problem is that

With an error of $O(\epsilon^2)$, $v(x, \tau)$ is periodic in τ with period 1,

by which we mean that $v(x, 0+) = v(x, 1+) = v(x, 2+) = \cdots$ not just for $x < 0$ but for $x > 0$ as well; this then enforces periodicity for other values of τ. The reason for this (to which we return below) is that on the 'fast' timescale τ, the outer solution varies too slowly for its time-dependence to feed into the inner solution (this happens at $O(\epsilon^3)$ on the outer scale, when the derivative $\partial^2 V_{\text{cont}} / \partial S \partial t$ contributes to the matching). Hence quantities such as $\delta(t')$ are effectively constant on this timescale. One might be tempted to think of this as a multiple-scale effect, but is more accurate to say that the diffusion on the inner timescale has such rapid spatial decay that the oscillations induced by the periodic resetting of the barrier have an exponentially small influence on the outer solution.

We now pose a regular expansion

$$v(x, \tau) \sim v_1(x, \tau) + \epsilon v_2(x, \tau) + O(\epsilon^2),$$

noting that because v was scaled with ϵ, only two terms are necessary (the 'missing' term v_0 vanishes identically because V_{cont} vanishes on $S = B$). Hence the problem for v_1 is

$$\frac{\partial v_1}{\partial \tau} - \frac{1}{2} \frac{\partial^2 v_1}{\partial x^2} = 0, \quad -\infty < x < \infty,$$

with

$$v_1(x, \tau) \sim B \delta_{\text{cont}}(t') x + V_1(B, t') \quad \text{as } x \to \infty$$

where, as noted above, $\delta_{\text{cont}}(t')$ and $V_1(B, t')$ are treated as constants. We also have decay to 0 as $x \to -\infty$, and the knock-out condition

$$v_1(x, 0+) = 0, \qquad x < 0,$$

and periodicity condition

$$v_1(x, 0+) = v_1(x, 1+), \qquad -\infty < x < \infty.$$

Similarly, the problem for v_2 is

$$\frac{\partial v_2}{\partial \tau} - \frac{1}{2} \frac{\partial^2 v_2}{\partial x^2} = x \frac{\partial^2 v_1}{\partial x^2} + \alpha_1 \frac{\partial v_1}{\partial x}, \quad -\infty < x < \infty,$$
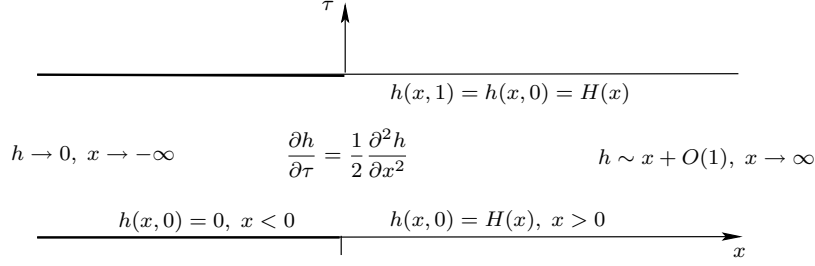
Figure 2: Boundary value problem for the Spitzer function $h(x, \tau)$.

with

$$v_2(x, \tau) \sim \frac{1}{2}B^2\gamma_{\text{cont}}(t')x^2 + B\delta_1(t')x + V_2(B, t') \quad \text{as } x \to \infty,$$

and with the same knock-out and periodicity conditions as for v_1.

It turns out that both v_1 and v_2 can be determined from a function considered by Spitzer [15, 16] in the context of renewal theory, and we therefore digress to describe such of its properties as are necessary.

3.3 The Spitzer function

Spitzer [15, 16] (see also [4, 13, 14]) considered an equivalent of the following problem: find a function $h(x, \tau)$, which we term the Spitzer function, satisfying the following (see Figure 2):

$$\frac{\partial h}{\partial \tau} = \frac{1}{2}\frac{\partial^2 h}{\partial x^2}, \qquad -\infty < x < \infty, \quad 0 < \tau < 1,$$

with

$$h(x, 0) = 0, \quad x < 0, \qquad h(x, 1) = h(x, 0) = H(x), \quad \text{say}, \quad x > 0,$$

and

$$h(x, \tau) \sim x + O(1) \quad \text{as} \quad x \to +\infty.$$

With these conditions, when $h(x, 1-)$ is replaced by $h(x, 1+) = 0$ for $x < 0$, $h(x, \tau)$ is periodic in the sense described above; in diffusion terms, there is a flux of unity from $x = +\infty$ which is exactly sufficient to replace the amount lost by replacing the values $h(x, 1-)$ for $x < 0$ with $h(x, 1+) = 0$. It is apparent that $v_1(x, \tau)$ above is proportional to $h(x, \tau)$. However, as we need further properties of h we present them separately.

Spitzer's results

Using the Green's function for the heat equation we have the equivalent integral equation

$$H(x) = \int_0^\infty k(x - y)H(y)\, dy \tag{3}$$

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for $H(x)$, where $k(x) = e^{-x^2/2}/\sqrt{2\pi}$ is the heat kernel, that is, the standard Normal density. Spitzer used the iterative scheme[1]

$$F_{n+1}(x) = \int_0^\infty k(x-y)F_n(y)\,dy, \qquad F_0(x) = 1.$$

to show the following:

- $\sqrt{n\pi/2}\,F_n(x) \to H(x)$ (and so $F_n(x) \to 0$) as $n \to \infty$.

- $H(0+) = 1/\sqrt{2}$, so H has a jump at $x = 0$. (This is necessary, since if $H(0+) = 0$, by the maximum principle we would have $h(0, 1-) > 0$ and we could not achieve $h(x, 1-) = H(x)$ for $x = 0+$.)

- The Laplace transform of $H(x)$ is

$$\overline{H}(s) = \frac{1}{s\sqrt{2}} \exp\left[-\frac{1}{2\pi} \int_{-\infty}^\infty \frac{s}{s^2 + \xi^2} \log(1 - e^{-\xi^2/2})\,d\xi \right]$$

  (this is established by using the Wiener–Hopf method on the Wiener–Hopf equation (3); here $e^{-\xi^2/2}$ is the characteristic function of the kernel k).

- Crucially for us,

$$\lim_{x \to \infty} [H(x) - (x + \beta)] = 0 \quad \text{where} \quad \beta = -\frac{\zeta(\frac{1}{2})}{\sqrt{2\pi}} \approx 0.5826.$$

  This can be derived from analysis of $\overline{H}(s)$ as $s \to 0$ and we present this analysis below.

- $h(x, \tau)$, and so $H(x)$, is the only such periodic solution.

The principal conclusion as far as we are concerned is that *the $O(1)$ constant in the asymptotic behaviour of $h(x, \tau)$ is determined uniquely by the coefficient of x (here, 1) in the expansion of $h(x, \tau)$ as $x \to \infty$.*

The initial Spitzer function and its asymptote are plotted in Figure 3, and the difference between the two is plotted in Figure 4.

3.4 Further properties of the Spitzer function

We shall need further properties of the Spitzer function, which we establish in this section. We show the following:

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[1]It might seem more natural to take $F_0(x) = x$ in view of the behaviour at large x, but Spitzer interpreted the sequence $F_n(x)$ as the distribution functions of the of random variables

$$Z_0 = 0, \quad Z_1 = X_1^+, \quad Z_2 = (X_2 + X_1^+)^+, \dots,$$

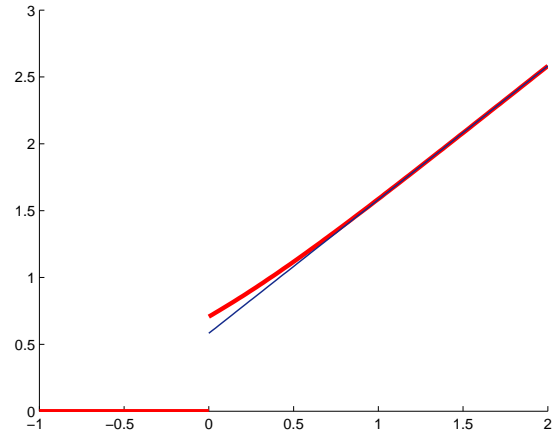where $X^+ = \max(X, 0)$ and X_i are iid $N(0, 1)$.

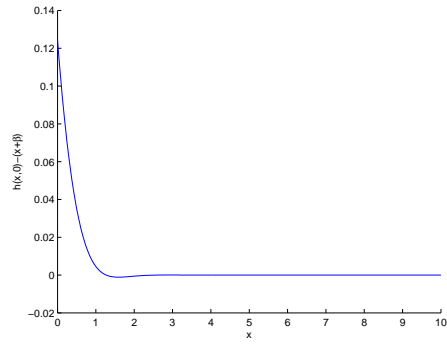Figure 3: The lniial Spitzer function $h(x,0) = H(x)$ and its asymptote.

Figure 4: The difference between the initial Spitzer function $h(x,0) = H(x)$ and its asymptote.

1. As $x \to \infty$, $H(x) \sim x + \beta + o(1)$ (the Spitzer result), where

$$\beta = -\frac{1}{\pi} \int_0^\infty \log\left(\frac{1 - e^{-\xi^2/2}}{\xi^2/2}\right) \frac{d\xi}{\xi^2} \approx 0.5826. \qquad (4)$$

The connection between this integral and the zeta function is given in [3].

2. The area between $H(x)$ and its asymptote is

$$\beta_1 = \int_0^\infty H(x) - (x + \beta)\, dx = \beta^2/2 - 1/8 \approx 0.0447.$$

3. The first moment of the difference from the asymptote is

$$\beta_2 = \int_0^\infty x\left(H(x) - (x + \beta)\right)\, dx = -\frac{1}{6}\beta^3 + \frac{1}{8}\beta - \frac{1}{24} - \beta_2^* \approx 0.0122,$$

where

$$\beta_2^* = \frac{1}{\pi} \int_0^\infty \log\left(\frac{(1 - e^{-\xi^2/2})(1 + \xi^2/4)}{\xi^2/2}\right) \frac{d\xi}{\xi^4}.$$

4. At the origin $x = 0$, we have

$$H(0+) = \frac{1}{\sqrt{2}} \approx 0.7071, \quad H'(0+) = \frac{1}{2\sqrt{\pi}} \sum_1^\infty \frac{1}{n^{\frac{3}{2}}} = \frac{1}{2\sqrt{\pi}} \zeta\left(\frac{3}{2}\right) \approx 0.7369,$$

$$H''(0+) = \frac{1}{\sqrt{2}} \left(H'(0+)\right)^2 \approx 0.3840;$$

here $\zeta(\cdot)$ is again the Riemann zeta function.

In order to do this, we analyse the Laplace transform of $H(x)$,

$$\overline{H}(s) = \frac{1}{s\sqrt{2}} \exp\left(-\frac{1}{2\pi} \int_{-\infty}^\infty \frac{s}{s^2 + \xi^2} \log(1 - e^{-\xi^2/2})\, d\xi\right) = \frac{1}{s\sqrt{2}} \exp\left(-\frac{1}{\pi} I(s)\right),$$

where

$$I(s) = \int_0^\infty \frac{s}{s^2 + \xi^2} \log(1 - e^{-\xi^2/2})\, d\xi,$$

as $s \to 0$ and $s \to \infty$. Because $H(x) \sim x$ at infinity, this function has a double pole at $s = 0$; hence we expect

$$\overline{H}(s) \sim \frac{1}{s^2} + \frac{\beta}{s} + \beta_1 - \beta_2 s + (O(s^2), \quad s \to 0$$

(after subtracting $x + \beta$ to get the first two terms, a Taylor series in s yields the third and fourth terms).

We first need to analyse $I(s)$ for small s. The principal difficulty in so doing is the singularity of $s/(s^2+\xi^2)$ at $\xi = s = 0$. This is dealt with by subtracting the

small-ξ behaviour of the logarithmic term and adding it back, the latter giving integrals that can be evaluated explicitly, in such a way that what remains is integrable at $\xi = 0$ even when $s = 0$, so that $s/(s^2 + \xi^2)$ can safely be expanded in powers of s/ξ. Specifically, we recall the standard results that

$$\int_0^\infty \frac{\log \xi}{s^2 + \xi^2}\, d\xi = \frac{\pi}{2s}\log s, \qquad \int_0^\infty \frac{\log(1 + \xi^2/4)}{\xi^2(s^2 + \xi^2)}\, d\xi = \frac{\pi}{2s^3}\left(s - 2\log(1 + s/2)\right),$$

and then write

$$I(s) = \int_0^\infty \frac{s}{s^2 + \xi^2}\left(\log\left(\frac{1 - e^{-\xi^2/2}}{\xi^2/2}\right) + \log(\xi^2/2)\right) d\xi$$

$$= I_1(s) + \pi\left(\log s - \frac{1}{2}\log 2\right),$$

where we can safely let $s \to 0$ in $I_1(s)$, to show that it has the behaviour $I_1(s) \sim -\pi\beta s + I_2(s)$, where β is defined by the integral (4). Here

$$I_2(s) = s\int_0^\infty \left(\frac{1}{s^2 + \xi^2} - \frac{1}{\xi^2}\right)\log\left(\frac{1 - e^{-\xi^2/2}}{\xi^2/2}\right) d\xi$$

$$= -s^3 \int_0^\infty \frac{1}{\xi^2(s^2 + \xi^2)}\left(\log\left(\frac{(1 - e^{-\xi^2/2})(1 + \xi^2/4)}{\xi^2/2}\right) - \log(1 + \xi^2/4)\right) d\xi$$

so that the argument of the first logarithm on the right is $O(\xi^4)$ as $\xi \to 0$. We can now expand for small s and integrate in ξ, giving

$$\frac{I(s)}{\pi} \sim \log s - \frac{1}{2}\log 2 - \beta s + \frac{1}{8}s^2 - s^3\left(\frac{1}{24} + \beta_2^*\right) + O(s^4),$$

and the required expansion follows immediately.

The expansion for large s is more straightforward. Simply integrating by parts in the definition of the Laplace transform of H, we have that, as $s \to \infty$,

$$\overline{H}(s) \sim \frac{H(0+)}{s} + \frac{H'(0+)}{s^2} + \frac{H''(0+)}{s^3} + \cdots,$$

so we just need to evaluate $I(s)$ as $s \to \infty$ by a regular expansion, giving

$$I(s) \sim \frac{1}{s}\int_0^\infty \log(1 - e^{-\xi^2/2})\, d\xi$$

$$= -\sum_{n=1}^\infty \int_0^\infty \frac{1}{n}e^{-n\xi^2/2}\, d\xi$$

$$= -\sqrt{\frac{\pi}{2}}\sum_{n=1}^\infty n^{-\frac{3}{2}}$$

$$= -\sqrt{\frac{\pi}{2}}\,\zeta(3/2).$$

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Then we find that

$$H'(0+) = \frac{1}{2\sqrt{\pi}} \sum_{n=1}^{\infty} n^{-\frac{3}{2}}, \qquad H''(0+) = \frac{1}{\sqrt{2}} \left(H'(0+)\right)^2.$$

It will also be useful to define an integral of the Spitzer function, namely

$$h^{(1)}(x,\tau) = \int_{-\infty}^{x} h(\xi,\tau)\,d\xi,$$

which is a solution of the heat equation with associated initial values denoted $H^{(1)}(x)$. Using the properties of $H(x)$ above, we have $H^{(1)}(x) = 0$ for $x < 0$ and, as $x \to \infty$,

$$H^{(1)}(x) \sim \frac{1}{2}(x+\beta)^2 - \frac{1}{8}, \qquad h^{(1)}(x,\tau) \sim \frac{1}{2}(x+\beta)^2 + \frac{1}{2}\tau - \frac{1}{8},$$

with an error of $o(1)$. Note that $h^{(1)}$ is not periodic, but instead increases by $\frac{1}{2}$ over a period.

3.5   The inner solution and matching

We can now write down the solution to the inner problem. The leading term $v_1(x,\tau)$ is simply a multiple of the Spitzer function:

$$v_1(x,\tau) = B\delta_{\text{cont}}(t')h(x,\tau). \tag{5}$$

Bearing in mind that $h(x,\tau) \sim x+\beta$ as $x \to \infty$, and as from one-term matching $v_1(x,\tau) \sim B\delta_{\text{cont}}(t')x + V_1(B,t')$ as $x \to \infty$, we have the first effective boundary condition

$$V_1(B,t') = \beta B\delta_{\text{cont}}(t').$$

As we show below, this is equivalent to the BGK correction truncated at $O(\epsilon)$.

A particular solution for $v_2(x,\tau)$ is the function

$$\frac{1}{2}\left(xv_1 - x^2\frac{\partial v_1}{\partial x}\right) - \alpha_1 x v_1,$$

which has the behaviour $B\delta_{\text{cont}}(t')\left(-\alpha_1 x^2 + \beta x(\frac{1}{2} - \alpha_1) + o(1)\right)$ at infinity. Recalling that, from the scaled Black–Scholes equation, $\gamma_{\text{cont}} = -2\alpha_1\delta_{\text{cont}}/B$, we see that this particular solution already has the correct coefficient of x^2 for matching. Hence what is left after subtracting it from v_2 must grow at most linearly at infinity, and must therefore be a multiple of $h(x,\tau)$, as this is the only periodic function with this behaviour. Matching with $B\delta_1 x$, we see that

$$v_2(x,\tau) = \frac{1}{2}\left(xv_1 - x^2\frac{\partial v_1}{\partial x}\right) - \alpha_1 x v_1 + B\left(\delta_1 - \beta(\tfrac{1}{2} - \alpha_1)\delta_{\text{cont}}\right)h(x,\tau), \tag{6}$$

and so the $O(1)$ matching gives the second effective boundary condition

$$V_2(B,t') = \beta B\left(\delta_1(t') - \beta(\tfrac{1}{2} - \alpha_1)\delta_{\text{cont}}(t')\right),$$

which can also be written in the form

$$V_2(B,t') = \beta B \left(\delta_1(t') - \tfrac{1}{2}\beta\delta_{\text{cont}}(t') - \tfrac{1}{2}\beta B\gamma_{\text{cont}}(t') \right).$$

Note that v_2 is much larger than v_1 as $x \to -\infty$; hence the inner expansion becomes invalid in this limit and a second outer expansion is needed. This is best approached using the techniques of ray theory as exploited in [10] in the analysis of American options, but we do not discuss it further here.

3.6 The composite expansion

The outer and inner expansions found above are valid for $(S-B)/B \gg O(\epsilon)$ and $0 < S < B(1 + O(\epsilon))$ respectively. It is possible to test for the size of $S - B$ and then choose which expansion to use, with a preference for the inner expansion in marginal cases. However, it may be more convenient to write a composite expansion, of the form 'inner + outer − common', which is uniformly valid. In our case the outer expansion is

$$\left(V_{\text{cont}}(S,t') + \epsilon V_1(S,t') + \epsilon^2 V_2(S,t')\right) \mathcal{H}(S-B),$$

where $\mathcal{H}(\cdot)$ is the Heaviside function, inserted here to ensure that the outer expansion vanishes for $S < B$; the inner expansion is

$$\epsilon v_1(x,\tau) + \epsilon^2 v_2(x,\tau) = \epsilon v_1\left(\frac{S-B}{\epsilon B}, \frac{t'-t_i'}{\epsilon^2}\right) + \epsilon^2 v_2\left(\frac{S-B}{\epsilon B}, \frac{t'-t_i'}{\epsilon^2}\right),$$

where $t_i' = \lfloor t' \rfloor$ is the reset date immediately before t' (this is in scaled time; in calendar time, it is the reset date immediately after t). Lastly the common expansion is the outer limit of the inner solution or the inner limit of the outer solution, namely

$$\left(\epsilon\left(B\delta_{\text{cont}}(t')x + V_1(B,t')\right) + \epsilon^2\left(\frac{1}{2}B^2\gamma_{\text{cont}}x^2 + B\delta_1(t')x + V_2(B,t')\right)\right)\mathcal{H}(S-B)$$

$$= \left((S-B)\frac{\partial V_{\text{cont}}}{\partial S}(B,t') + \epsilon V_1(B,t') + \frac{1}{2}(S-B)^2\frac{\partial^2 V_{\text{cont}}}{\partial S^2}(B,t')\right.$$

$$\left. +\epsilon(S-B)\frac{\partial V_1}{\partial S}(B,t') + \epsilon^2 V_2(B,t')\right)\mathcal{H}(S-B),$$

where quantities such as $V_1(B,t')$ are calculated from the solution of the outer problem and h is calculated once and for all (a reasonable approximation is given below).

With hedging in mind, we should comment on the degree of smoothness to be expected from the composite expansion, noting that discontinuities are to be expected at $S = B$. In general, we have the following:

- The composite expansion V_{comp} has a jump of $O(\epsilon^3)$ at $S = B$;

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- The composite Delta $\Delta_{\text{comp}} = \partial V_{\text{comp}}/\partial S$ has a jump of $O(\epsilon^2)$ at $S = B$, because the term $\epsilon^2 \partial V_2/\partial S\big|_{S=B}$ is not included in it (it occurs at the next level in the expansion);

- The composite Gamma Γ_{comp} has a jump of $O(\epsilon)$ at $S = B$, because the term $\epsilon^2 \partial^2 V_1/\partial S^2\big|_{S=B}$ is not included.

- Each time we differentiate the approximation near the barrier, the error becomes worse by a factor of $1/\epsilon$, from the inner solution (where the argument is $x = (S/B - 1)/\epsilon$).

These features are illustrated in the numerical examples below, and in Figure 5, which shows on the left the inner and outer expansions plotted separately, and on the right the composite expansion, showing the transition from inner to outer at $S = B$.
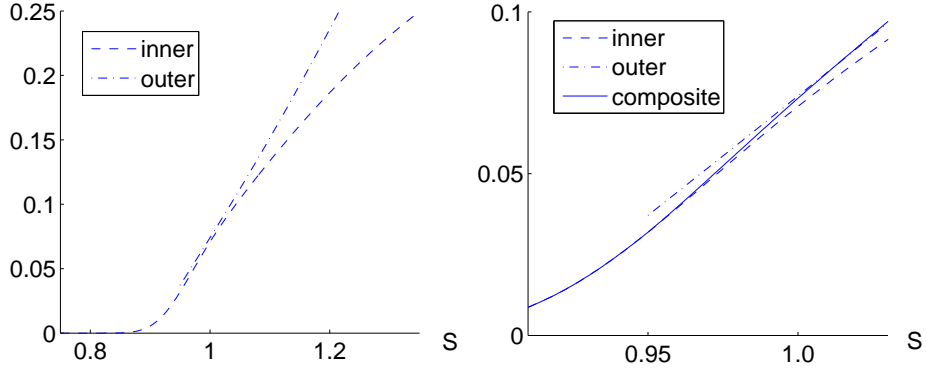

Figure 5: Left: inner and outer expansions plotted separately. Right: Detail near $S = B$ showing the composite expansion (solid line). Parameters as in Figure 11; $B = 0.95$.

The composite expansion just constructed is uniformly valid, but as we have seen it has the undesirable feature of discontinous behaviour at $S = B$. A simple remedy for this is to remove the Heaviside functions in the outer and common expansions, so that, for example, the first term in the outer expansion is the continuously sampled option value with its image (for all $0 < S < \infty$). Although this expansion is not valid for $(S - B)/B \ll 1$ (which is in any case of little practical interest), it is valid in the inner region and the upper outer region above the barrier and, crucially, it can be differentiated to recover the Greeks. We return to this point in the discussion of the numerical illustrations below.

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It is an important feature of the method that the precise form of the payoff is not important for the procedure above. However, the expansion is only valid provided the periodicity assumption (up to $O(\epsilon^2)$) holds, and this means that it is only valid after any initial transient to slowly modulated periodicity. Indeed, Spitzer's iteration shows that if we start an inner problem with initial data that is not a multiple of $H(x)$, the difference between the solution so generated and the Spitzer solution tends to zero after a number of resets that is large compared with unity, but still small compared with $O(1/\epsilon^2)$ (which is the number of resets in an $O(1)$ fraction of the option life). The magnitude of the contribution made by this initial transient to the outer solution depends on the initial discrepancy; thus, if the payoff does not vanish near $S = B$ (as would be the case for a down-and-out call with $K < B$, we may expect the approximation to perform less well.

Finally, note that there is no need for the initial period of the option, before the first reset time, to be equal to the reset interval ΔT: if it is not, we simply apply the outer expansion unchanged, and the inner expansion with the appropriate value of τ (if this is greater than 1, we continue to solve the diffusion equation for $h(x, \tau)$, without resetting the values on the negative x axis to zero). Only if this initial interval is much larger than ΔT (as would be the case for a forward-start barrier option with discrete sampling) do we need to construct a separate expansion for the initial period; apart from remarking that this expansion is straightforward and entails a regular expansion away from, and an inner expansion near, the barrier, we do not pursue it here (a similar expansion for a vanilla call near expiry is constructed as an example in [8]).

3.6.1 An ad hoc approximation to $h(x, \tau)$

It is not especially easy to calculate $h(x, \tau)$ (we used a version of Spitzer's iteration in which we set the value of h at a large value of x to be equal to its asymptote $x + \beta$). Hence it may be useful to have an explicit formula that is close to $h(x, \tau)$. A reasonable approximation can be found using exponentials to approximate $H(x)$: one way is to choose coefficients a_1, a_2, a_3 and a_4 such that the function

$$\left(a_1 e^{-a_2 x} + a_3 e^{-a_4 x^2/2}\right)\mathcal{H}(x)$$

has the same value and first two derivatives at $x = 0+$ as $H(x) - (x+\beta)$, and such that the integral of x times this function is the same as that for $H(x) - (x + \beta)$. (The conditions at the origin ensure that the jump discontinuity in $H(x)$ is well approximated, while the weighted integral contributes to the accuracy of the approximation for larger x.) It is found numerically that the parameter sets $(0.216, 1.22, -0.091, 0.69)$ and $(0.084, 3.2, 0.041, 11)$ both satisfy all these constraints and their maximum initial error, relative to $H(x)$, is less than 1% (see Figure 6, which shows the absolute errors at $\tau = 0$ and $\tau = 1-$; the second
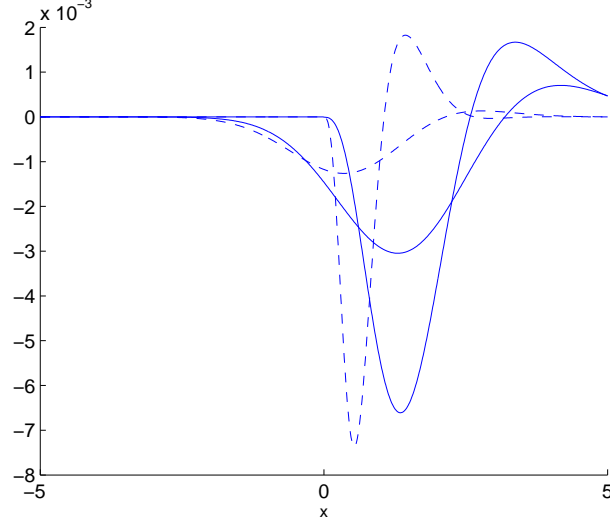
Figure 6: Approximations to $H(x)$ and the corresponding $h(x, 1-)$. Approximation 1 (solid lines) has the parameter set $(0.216, 1.22, -0.091, 0.69)$, approximation 2 (dashed lines) has the set $(0.084, 3.2, 0.041, 11)$.

approximation is probably better). Then for $0 \leq \tau < 1$ we have

$$h(x, \tau) \approx xN(x/\sqrt{\tau}) + \sqrt{\tau}n(x/\sqrt{\tau}) + \beta N(x/\sqrt{\tau})$$

$$+ a_1 e^{-a_1 x + a_1 \tau^2/2} N\left(\frac{x - a_1 \tau}{\sqrt{t}}\right) + a_3 \frac{e^{-a_4 x^2/(1 + 2a_4 \tau)}}{\sqrt{1 + 2a_4 \tau}} N\left(x\sqrt{\frac{1 + 2a_4 \tau}{\tau}}\right).$$

For programming purposes, note that $N(x) = \frac{1}{2}\mathrm{erfc}(-x/\sqrt{2})$.

3.7 Calculation of V_1 and V_2

We now give formulae for V_1 and V_2 in terms of the 'vanilla' contract value V_{v}, which is the solution of the Black–Scholes equation with the same payoff as the barrier option, but without a barrier. That is, its payoff is that of the barrier option for $S > B$, and zero for $S < B$. In general, the value of the continuously sampled down-and-out option is then given by

$$V_{\mathrm{cont}}(S, t) = V_{\mathrm{v}}(S, t) - (S/B)^{1-2\alpha_1} V_{\mathrm{v}}(B^2/S, t), \tag{7}$$

which is the image result stated earlier; recall that $\alpha_1 = (r - q)/\sigma^2$. For a down-and-out call option with $K > B$, we have $V_{\mathrm{v}}(S, t) = C_{\mathrm{v}}(S, t; K)$, the standard Black–Scholes call value of an option with strike K; when $B > K$, we have $V_{\mathrm{v}}(S, t) = C_{\mathrm{v}}(S, t; B) + (B - K)C_{\mathrm{dig}}(S, t; B)$, the latter term being the value of a digital call paying $B - K$ if $S_T > B$, added to compensate for the jump in the payoff at $S = B$.

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We shall write Δ_v and Γ_v for the Delta and Gamma of the vanilla contract, and we return to the use of calendar time t (retaining the same notation for $V(S,t)$ as for $V(S,t')$). We shall use the result that if $V_v(S,t)$ is a solution of the Black–Scholes equation, so are $S\Delta_v$ and $S^2\Gamma_v$ (obviously this also holds for the scaled Black–Scholes equation (1)). It follows that, by the image principle,

$$\left(\frac{S}{B}\right)^{1-2\alpha_1}\frac{B^2}{S}\Delta_v(B^2/S,t), \qquad \left(\frac{S}{B}\right)^{1-2\alpha_1}\frac{B^4}{S^2}\Gamma_v(B^2/S,t)$$

also satisfy the Black–Scholes equation. We also note that, because $V_v(S,T) = 0$ for $S < B$, its image $(S/B)^{1-2\alpha_1}V_v(B^2/S,t)$ vanishes at $t = T$ for $S > B$, and so do all its derivatives.

Consider first V_1, which satisfies the Black–Scholes equation, has zero pay-off at $t = T$ for $S > B$, and has the barrier boundary condition $V_1(B,t) = \beta B\delta_{\text{cont}}(t)$. That is, on $S = B$,

$$
\begin{aligned}
V_1(B,t) &= \beta B\Delta_{\text{cont}}(B,t) \\
&= \beta S\Delta_{\text{cont}}(S,t)|_{S=B} \\
&= \beta\left(2S\Delta_v(S,t) - (1-2\alpha_1)V_v(S,t)\right)|_{S=B} \\
&= \beta\left(\frac{S}{B}\right)^{1-2\alpha_1}\left(2\frac{B^2}{S}\Delta_v(B^2/S,t) - (1-2\alpha_1)V_v(B^2/S,t)\right)\Bigg|_{S=B},
\end{aligned}
$$

where the third line follows from differentiation of (7). Using the results just stated, we have immediately that, for $S > B$,

$$V_1(S,t) = \beta\left(\frac{S}{B}\right)^{1-2\alpha_1}\left(2\frac{B^2}{S}\Delta_v\left(\frac{B^2}{S},t\right) - (1-2\alpha_1)V_v\left(\frac{B^2}{S},t\right)\right). \qquad (8)$$

In a similar way, we have the boundary condition

$$
\begin{aligned}
V_2(B,t) &= \beta\left(B\Delta_1(B,t) - \beta(\tfrac{1}{2}-\alpha_1)B\Delta_{\text{cont}}(B,t)\right) \\
&= -\beta^2\left(2B^2\Gamma_v + 4\alpha_1 B\Delta_v + \frac{1}{2}(1-2\alpha_1)^2 V_v\right)\Bigg|_{S=B},
\end{aligned}
$$

and so

$$
\begin{aligned}
V_2(S,t) = -\beta^2\left(\frac{S}{B}\right)^{1-2\alpha_1} \\
\times\left(2\frac{B^4}{S^2}\Gamma_v\left(\frac{B^2}{S},t\right) + 4\alpha_1\frac{S}{B}\Delta_v\left(\frac{B^2}{S},t\right) + \frac{1}{2}(1-2\alpha_1)^2 V_v\left(\frac{B^2}{S},t\right)\right). \qquad (9)
\end{aligned}
$$

3.8 Numerical illustration

We give three illustrations of the approximation. In all three cases, the parameters $\sigma = 0.3$, $r = 0.06$, $q = 0.02$, $K = 1$ are fixed, and we use the composite approximation. Where comparison is made with BGK, their convention on the

number of resets is used for their approximation, and ours for our approximation.

We begin by showing how the error of the approximation $V_{\text{comp}}(S, t)$, relative to a numerically computed solution $V_{\text{num}}(S, t)$, varies with the number of resets. (We used an explicit scheme for $0 < S < 2.5$, with a step size of 0.001 in the S direction; this is adequate given that we do not attempt to compute high-order derivatives.) In Figure 7 we plot the relative error $(V_{\text{comp}} - V_{\text{num}})/V_{\text{num}}$ against S for 1 to 32 resets. The approximation performs well even when there is only a single reset, and as expected its accuracy improves with the number of resets (the degradation for values of S less than B is due to the breakdown of the inner expansion for large negative x discussed earlier, magnified by the fact that we are dividing by a very small numerical value).

We have also plotted the BGK approximation for 8 resets. It is a remarkably good approximation in any region except the immediate vicinity of the barrier (indeed, marginally better than ours), and this feature is also evident in the two illustrations that follow.
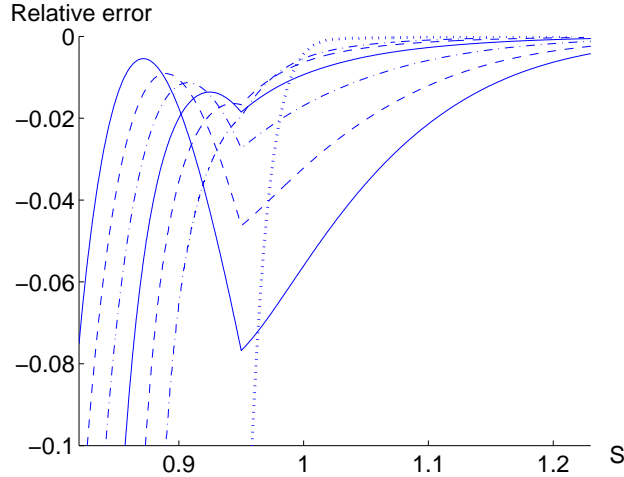

Figure 7: Variation of relative error with number N of resets. Lifetime is $T = 0.25$ and the initial period is the same as the reset interval; $B = 0.95$. At $S = 1$, working from the lowest curver we have $N = 1$ ($\epsilon = 0.1061$, solid curve), 2 ($\epsilon = 0.0866$, dashed), 4 ($\epsilon = 0.0671$, dot-dash), 8 ($\epsilon = 0.0500$, solid), 16 ($\epsilon = 0.0364$, dashed), 32 ($\epsilon = 0.0261$, dot-dash). Also shown (dots) is the BGK approximation for $N = 8$ (their $m = 9$).

Notice the discontinuity in the gradient of the error at $S = B$. This is the $O(\epsilon^2)$ error discussed in Section 3.6, and it is magnified because we have divided by the numerically computed option value, which is small. It follows that we cannot rely on the composite expansion to calculate the delta; if we were to do so, we would find a result like that of Figure 8, in which the jump

in the Delta at $S = B$ is found to be almost exactly equal to $\epsilon^2 \, \partial V_2 / \partial S|_{S=B}$, as predicted in Section 3.6. Instead, it is better to use the second form of the composite expansion proposed in Section 3.6, which gives the very acceptable results illustrated in Figure 9. This figure also illustrates the non-uniformity of the 'continuous' composite expansion as S falls far below B.
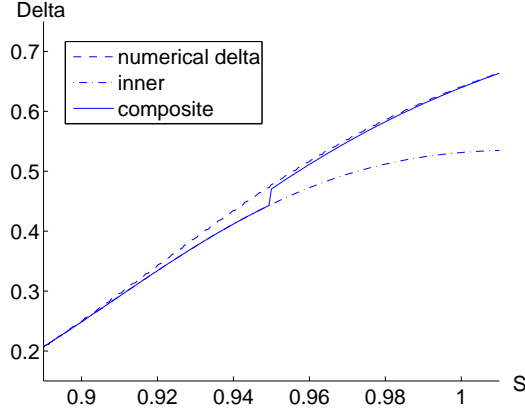

Figure 8: Calculated Delta with 4 resets. Lifetime is $T = 0.25$ and the initial period is the same as the reset interval; $B = 0.95$. The 'discontinuous' composite expansion is used.

For our second illustration, we show the effect of varying B, taking values below, at, and above the strike. Figure 10 shows three cases, and it is clear that when $B > K$ the approximation, although good, is less accurate than when $B < K$. Again, the BGK approximation is excellent in its range of validity.

For our last example, we show the effect of having a shorter initial interval, so that another assumption of the BGK analysis is not satisfied. The results, shown in Figure 11 (note that absolute values, rather than relative errors, are shown). Only the region near $S = B$ is shown; again, the approximation is excellent.

3.9   The connection with the BGK approximation

We now show how the outer expansion constructed above is related to the BGK approximation (BGK have no equivalent of the inner expansion). For technical reasons, we only consider the case when the payoff vanishes at the barrier.

As we shall go to higher order than $O(\sigma\sqrt{T/N})$, first let us consider the niceties of the definition of ϵ which, as discussed earlier, amount to deciding whether the start date is a reset date or not. If the start date is not a reset date, then we have N resets separated by ΔT, together with a starting interval. In the simple case that the starting interval is also equal to ΔT, then our definition of ϵ, namely $\sigma\sqrt{\Delta T}$, is equal to $\sigma\sqrt{T/(N+1)}$. If the starting date
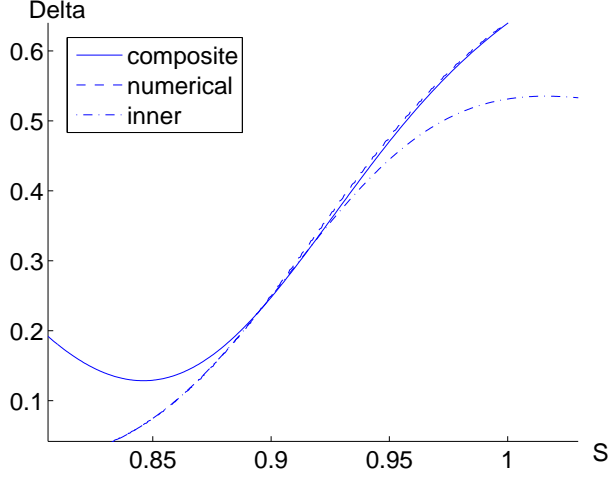
Figure 9: Calculated Delta with 4 resets. Lifetime is $T = 0.25$ and the initial period is the same as the reset interval; $B = 0.95$. The 'continuous' composite expansion is used.

is a reset date, then we have $T = N\Delta T$ where N is the number of reset dates including the start date, and then $\epsilon = \sigma\sqrt{T/N}$, i.e. the BGK definition. The difference between the two definitions is $O(\sigma\sqrt{T}/N^{\frac{3}{2}})$, which is beyond the accuracy of our approximation (although it would not be were we to go to one more term). We therefore stick with the definition $\epsilon = \sigma\sqrt{\Delta T}$, although in numerical comparisons we have used the BGK definition of ϵ in computing their approximation.

The BGK correction states that the discretely sampled option option should be valued as if the barrier is situated at $Be^{-\epsilon\beta}$ and is sampled continuously. That is, their value is $V_{\mathrm{BGK}} = V_{\mathrm{cont}}(S, t; Be^{-\epsilon\beta})$. Another way of stating this is that V_{BGK} satisfies the Black–Scholes equation with the same payoff as V_{d} and with the boundary condition

$$V_{\mathrm{BGK}}(B, t; Be^{-\epsilon\beta}) = 0.$$

If we write
$$V_{\mathrm{BGK}} \sim V_{\mathrm{BGK0}} + \epsilon V_{\mathrm{BGK1}} + \epsilon^2 V_{\mathrm{BGK2}} + \cdots,$$

and we write $e^{-\epsilon\beta} \sim 1 - \epsilon\beta + \frac{1}{2}\epsilon^2\beta^2 - \cdots$, then by a Taylor expansion in B, we
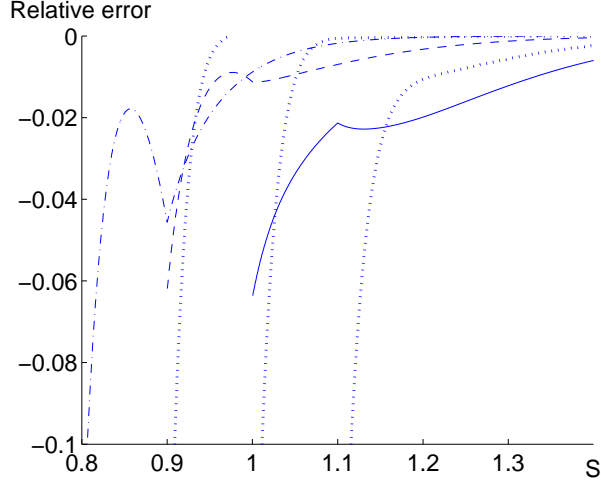
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Figure 10: Variation of relative error with barrier level B. Lifetime is $T = 0.5$ with 5 resets, and the initial period is the same as the reset interval; $\epsilon = 0.0612$. Barrier levels are $B = 0.9$ (dash-dot), $B = 1$ (dashed) and $B = 1.1$ (solid); the relevant BGK errors are shown as dotted curves.

can linearise the boundary condition onto $S = B$, to obtain

$$
V_{\text{BGK0}}(B,t) + \epsilon \left. \left(-\beta B \frac{\partial V_{\text{BGK0}}}{\partial B} + V_{\text{BGK1}} \right) \right|_{S=B}
$$
$$
+ \epsilon^2 \left. \left(\frac{1}{2}\beta^2 B^2 \frac{\partial^2 V_{\text{BGK0}}}{\partial B^2} + \frac{1}{2}\beta^2 B \frac{\partial V_{\text{BGK0}}}{\partial B} - \beta B \frac{\partial V_{\text{BGK1}}}{\partial B} + V_{\text{BGK2}} \right) \right|_{S=B}
$$
$$
+ O(\epsilon^3) = 0.
$$

This tells us that the effective boundary conditions for $V_{\text{BGK0,1,2}}$ on $S = B$ are

$$
V_{\text{BGK0}}(B,t) = 0,
$$
$$
V_{\text{BGK1}}(B,t) = \beta B \frac{\partial V_{\text{BGK0}}}{\partial B},
$$
$$
V_{\text{BGK2}}(B,t) = -\frac{1}{2}\beta^2 \left(B \frac{\partial V_{\text{BGK0}}}{\partial B} + B^2 \frac{\partial^2 V_{\text{BGK0}}}{\partial B^2} \right) + B \frac{\partial V_{\text{BGK1}}}{\partial B},
$$

where the partial derivatives on the right are evaluated at $S = B$. Clearly, then,

$$
V_{\text{BGK0}}(S,t) = V_{\text{cont0}}(S,t; B),
$$

and we now relate the higher order corrections, which must vanish at expiry, to our corrections $V_1(S,t)$ and $V_2(S,t)$.

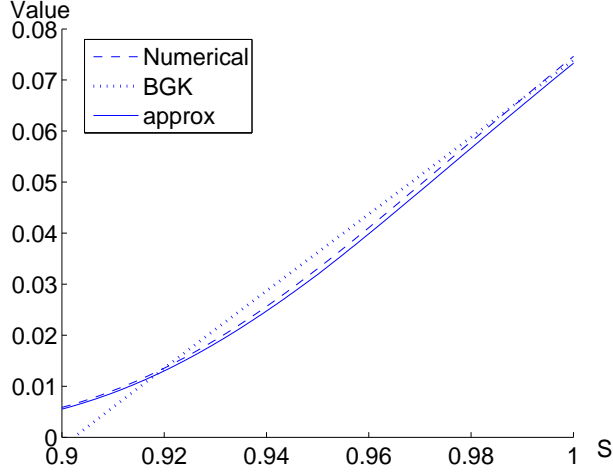First note that, provided that the payoff vanishes at $S = B$ (as is the case for

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Figure 11: Initial interval T_1 not equal to reset interval. Here $T = 0.52$ with 5 resets, and the initial period is 0.02; $\epsilon = 0.0949$ and $B = 0.95$. The relevant BGK value is shown as the dotted curve.

a down-and-out call with $K > B$),[2] then the function $-\beta B \partial V_{\text{cont}}/\partial B$ satisfies the Black–Scholes equation (by direct differentiation), vanishes at expiry, and has the correct values on $S = B$. Hence,

$$V_{\text{BGK1}}(S, t) = -\beta B \frac{\partial V_{\text{cont}}}{\partial B}.$$

However, because $V_{\text{cont}}(B, t; B) = 0$, we have

$$\frac{\partial V_{\text{cont}}}{\partial S} + \frac{\partial V_{\text{cont}}}{\partial B} = 0$$

on $S = B$. Hence the effective boundary condition for V_{BGK1} is equivalent to ours and $V_{\text{BGK1}}(S, t) = V_1(S, t)$ as we expect.

Now consider V_{BGK2}. Using the now-known $V_{\text{BGK1}} = \beta B \partial V_{\text{cont}}/\partial B$, and the same argument as above, we have that

$$V_{\text{BGK2}}(S, t) = \tfrac{1}{2}\beta^2 \left(B \frac{\partial V_{\text{cont}}}{\partial B} + B^2 \frac{\partial^2 V_{\text{cont}}}{\partial B^2} \right).$$

Remarkably, this is equal to $V_2(S, t)$ (again, when 'the barrier is below the strike'). To see this, note first that

$$\frac{\partial V_1}{\partial S} = -\beta B \frac{\partial^2 V_{\text{cont}}}{\partial S \partial B},$$

---

[2]If it does not, then $\partial V(S, T)/\partial B$ has a delta function at $S = B$; this is why the down-and-out call with $B > K$ has an extra contribution from a digital option. This delta function contributes to $\partial V_{\text{cont}}/\partial B$ and our argument relating the two approximations fails. Beyond noting that the BGK correction may be expected to work less well in this case, we do not pursue this here.

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then that differentiating the barrier condition for V_{cont} a second time yields

$$\frac{\partial^2 V_{\text{cont}}}{\partial S^2} + 2\frac{\partial^2 V_{\text{cont}}}{\partial S \partial B} + \frac{\partial^2 V_{\text{cont}}}{\partial B^2} = 0$$

on $S = B$, from which we deduce that

$$\delta_1(t) = \tfrac{1}{2}\beta B\left(\gamma_{\text{cont}}(t) + \frac{\partial^2 V_{\text{cont}}}{\partial B^2}\right)$$

where δ_1 and γ_{cont} are as above. Lastly we substitute for δ_{cont} and δ_1 into the boundary condition for V_2, namely $V_2 = \beta B\left(\delta_1 - \tfrac{1}{2}\beta\delta_{\text{cont}} - \tfrac{1}{2}\beta B\gamma_{\text{cont}}\right)$, to find that the two conditions are identical, and hence so are the corresponding correction terms. This shows that the BGK correction is, in these cases, an order more accurate than originally claimed; one wonders whether it is accurate to higher order still.

4 Discussion and extensions

4.1 Summary

In summary, we have shown the following: Suppose we have a down-and-out contract with barrier B and specified payoff for $S > B$ (it does not matter what). Suppose also that the same option is sampled discretely with reset time ΔT, and that $\epsilon = \sigma\sqrt{\Delta T}$ is small. Then we can calculate an outer expansion

$$V_{\text{d}}^{\text{outer}} \sim V_{\text{cont}}(S,t;B) + \epsilon V_1(S,t) + \epsilon^2 V_2(S,t) + O(\epsilon^3),$$

valid far above the barrier. Furthermore, we have the effective boundary conditions

$$V_1(B,t) = \beta B\delta_{\text{cont}}(t)$$

and

$$V_2(B,t) = \beta B\left(\delta_1(t) - \beta\left(\frac{1}{2} - \alpha_1\right)\delta_{\text{cont}}(t)\right)$$

for the functions $V_1(S,t)$ and $V_2(S,t)$. There are simple representations of these functions in terms of vanilla option values given in (8) and (9) respectively. We can also calculate an inner expansion of the form

$$V_{\text{d}}^{\text{inner}} \sim \epsilon\left(v_1(x,\tau) + \epsilon v_2(x,\tau) + O(\epsilon^2)\right),$$

valid near the barrier. The functions $v_1(x,\tau)$ and $v_2(x,\tau)$ can be expressed in terms of Spitzer's function $h(x,\tau)$ as given in (5) and (6) respectively. The outer and inner expansion together give a complete description of the solution. For convenience, we have also introduced a composite expansion of the form 'inner+outer-common', which is uniformly valid and is given in Section 3.6.

4.2 Other contracts

The method above can be extended to almost any European style contract (American contracts are discussed in [9]). The important feature to note is that it is the barrier Delta and other Greeks of the continuously sampled option that determine the higher-order corrections, rather than (as implied by the BGK approach) a shift in the barrier (i.e. using the barrer sensitivity of V_{cont}), although in many common cases the end result is the same.

Up-and-out options

The appropriate modifications for up-and-out options are simple. The solution in the inner region is now $-h(-x,\tau)$ rather than $h(x,\tau)$, and this has asymptotic behaviour $x - \beta$ as $x \to -\infty$, thereby giving the first-order effective boundary condition as $V_1(B,t) = -\beta\delta_{\text{cont}}(t)$ with a change of sign from the down-and-out case. The second-order correction proceeds similarly.

Rebates

Some barrier options pay a rebate R on knock-out. Consider, for example, a down-and-out call with rebate R. Its continuously-sampled value is that of the same call without rebate, plus R times a standard American digital call paying \$1 if S falls to B, so we just need to value the latter. The difference from the previous case is that we now take $V_{\text{d}} = R$ on the barrier. The outer solution procedure is as before, as is its expansion near the barrier. However, the inner solution now has the expansion

$$v(x,\tau) \sim v_0(x,\tau) + \epsilon v_1(x,\tau) + \epsilon^2 v_2(x,\tau) + \cdots,$$

and we note immediately that $v_0 = R$ is plainly a solution to the new leading-order inner problem with the appropriate periodicity. We find that $v_1(x,\tau)$ is, exactly as before, equal to $B\delta_{\text{cont}}(t')h(x,\tau)$, and so the first-order effective boundary condition remains as $V_1(B,t') = \beta\delta_{\text{cont}}(t')$. At second order, however, we have

$$\frac{\partial v_2}{\partial \tau} - \frac{1}{2}\frac{\partial^2 v_1}{\partial x^2} = x\frac{\partial^2 v_1}{\partial x^2} + \alpha_1\frac{\partial v_1}{\partial x} - \alpha_2 R, \quad -\infty < x < \infty,$$

with

$$v_2(x,\tau) \sim \frac{1}{2}B^2\gamma_{\text{cont}}(t')x^2 + B\delta_1(t')x + V_2(B,t') \quad \text{as } x \to \infty,$$

where now γ_{cont} contains an extra term due to the rebate, being equal to $-2\alpha_1\delta_{\text{cont}}/B + 2\alpha_2 R/B^2$. A particular solution for $v_2(x,\tau)$ is the function

$$\frac{1}{2}\left(xv_1 - x^2\frac{\partial v_1}{\partial x}\right) - \alpha_1 xv_1 - \alpha_2 R\tau,$$

but this is not periodic in τ. However, using the integrated Spitzer function $h^{(1)}(x,\tau)$ introduced in Section 3.3, we find that

$$\frac{1}{2}\left(xv_1 - x^2\frac{\partial v_1}{\partial x}\right) - \alpha_1 xv_1 + \alpha_2 R\left(-\tau + 2h^{(1)}(x,\tau)\right)$$

does satisfy the periodicity conditions. The behaviour of this particular solution at infinity is

$$B\delta_{\text{cont}}\left(-\alpha_1 x^2 + \beta x(\tfrac{1}{2}-\alpha_1)\right) + \alpha_2 R\left((x+\beta)^2 - \frac{1}{4}\right) + o(1).$$

Again, the quadratic terms match automatically, and the remaining linear matching can be achieved by adding a multiple of $h(x,\tau)$ to find that

$$v_2(x,\tau) = \frac{1}{2}\left(xv_1 - x^2 \frac{\partial v_1}{\partial x}\right) - \alpha_1 xv_1 + B\left(\delta_1 - \beta(\tfrac{1}{2}-\alpha_1)\delta_{\text{cont}}\right)h(x,\tau)$$
$$+ \alpha_2 R\left(-\tau + 2h^{(1)}(x,\tau)\right) - 2\alpha_2\beta Rh(x,\tau),$$

and so the $O(1)$ matching gives the second effective boundary condition

$$V_2(B,t') = \beta B\left(\delta_1(t') - \beta(\tfrac{1}{2}-\alpha_1)\delta_{\text{cont}}(t')\right) - \alpha_2 R\left(\beta^2 + \frac{1}{4}\right).$$

The new contribution (the last term on the right) means that V_2 is decreased by a multiple of the continuously sampled American digital put; its cost reflects the time value of paths spent 'between the barriers'.

Double Barrier Options

We briefly describe the extension to discretely sampled double barrier knock-out options (an approximation for these contracts is discussed in [7]). It is straightforward to see that to $O(\epsilon)$ the approximation is given à la BGK by the value of the continuous barrier option with a shift of the upper barrier by $\beta\epsilon$ and the lower barrier by $-\beta\epsilon$ or, as in our view, determined by the barrier Delta and other Greeks. Note that many double barrier contracts will be subject to increased error due to a payoff discontinuity at one or other barrier.

We consider a knock-out option with knock-out conditions at the lower barrier B_- and the upper barrier B_+. The solution can be either represented as sums of images, which works well when the time to expiry is large, or as a Fourier series, which works well for small time to expiry. The continuously-sampled value in terms of the sums of images is given by

$$V_{\text{cont}}(S,t) = \sum_{n=-\infty}^{\infty} (B_+/B_-)^{n(1-2\alpha_1)}V_{\text{v}}((B_+/B_-)^{-2n}S,t)$$
$$- \sum_{n=-\infty}^{\infty} (S/B_-)^{1-2\alpha_1}(B_+/B_-)^{n(1-2\alpha_1)}V_{\text{v}}((B_+/B_-)^{2n}B_-^2/S,t)$$

where V_{v} is the vanilla contract with the same payoff $P(S)$ (extended by zero outside the barriers), or as the Fourier series

$$V_{\text{cont}}(S,t) = B_- e^{(1-2\alpha_1)y/2-(1-2\alpha_1)^2 t'/4-\alpha_2 t'/2}\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{a}y\right)e^{-n^2\pi^2 t'/a^2}$$

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with $y = \log(S/B_-)$, $a = \log(B_+/B_-)$, and the Fourier coefficients

$$C_n = \frac{2}{aB_-} \int_0^a e^{-(1-2\alpha_1)\xi/2} P(B_- e^\xi) \sin\left(\frac{n\pi}{a}\xi\right) d\xi.$$

For simplicity we work only to $O(\epsilon)$, and then the effective boundary conditions for $V_1(S,t)$ at $S = B_-$ and $S = B_+$ are

$$V_1(B_-, t') = \beta B_- \Delta_{\mathrm{cont}}(B_-, t')$$

and

$$V_1(B_+, t') = -\beta B_+ \Delta_{\mathrm{cont}}(B_+, t'),$$

respectively. It remains to calculate $V_1(S,t)$. This can be achieved by transforming the Black-Scholes equation into the heat equation (with a diffusivity of 1, not $\frac{1}{2}$) via the transformation

$$V_1(S,t) = B_- e^{(1-2\alpha_1)y/2 - (1-2\alpha_1)^2 t'/4 - \alpha_2 t'/2} u_1(y, t')$$

and subsequently applying the Laplace transform in t, with the result that

$$u_1(y,t') = \frac{2}{a} \int_0^{t'} \sum_{k=1}^{\infty} \left\{ u_1(0,s) + (-1)^k u_1(a,s) \right\} \left(\frac{k\pi}{a}\right) \sin\left(\frac{k\pi}{a}y\right) e^{-\frac{k^2\pi^2}{a^2}(t'-s)} \, ds.$$

At this point one can choose between the representations of the boundary conditions as sums of images or as a Fourier series and perform the integration numerically. Using the Fourier series representation one can perform the s-integration explicitly, to find that (recall that $y = \log(S/B_-)$)

$$\begin{aligned}
V_1(S,t) = {} & \beta B_- e^{(1-2\alpha_1)y/2 - (1-2\alpha_1)^2 t'/4 - \alpha_2 t'/2} \\
& \times \sum_{k=1}^{\infty} \frac{2}{a} \left\{ a_k \sin\left(\frac{k\pi}{a}y\right) + b_k \cos\left(\frac{k\pi}{a}y\right) \right\} e^{-\frac{k^2\pi^2}{a^2}t'}
\end{aligned} \qquad (10)$$

with the coefficients a_k and b_k given by

$$a_k = \frac{\pi^2 k^2 C_k}{2a^2} t' + \sum_{\substack{n=1 \\ n\neq k}}^{\infty} \frac{nkC_n}{k^2 - n^2} \left(e^{-\pi^2 a^2(n^2 - k^2)t'/a^2} - 1 \right)$$

and

$$b_k = (-1)^k \frac{\pi^2 k^2 C_k}{2a^2} t' + \sum_{\substack{n=1 \\ n\neq k}}^{\infty} (-1)^n \frac{nkC_n}{k^2 - n^2} \left(e^{-\pi^2 a^2(n^2 - k^2)t'/a^2} - 1 \right)$$

respectively.

4.3 Time-varying reset schedules, local volatility models and jump-diffusion

The solutions described above can be extended to a variety of models including smoothly time-varying reset schedules and local volatility models. If the rest interval is $\Delta T_0 f(t)$, where f is a smoothly varying $O(1)$ function of t, we define $\epsilon^2 = \sigma^2 \Delta T_0$ and proceed as above; the principal difference is that the local time interval between resets is no longer 1 but instead is $f(t)$, and x and τ must be rescaled accordingly in order to apply the formulae given earlier (i.e, τ is scaled with $f(t)$ and x with $\sqrt{f(t)}$). Similar remarks apply to local volatility models, in which $\sigma = \sigma_0 \Sigma(S, t)$; we use σ_0 to define ϵ and then rescale x and τ accordingly; see [6] for more details. Finally, in a standard jump-diffusion model, the contribution from the jump term occurs at the same level as the discounting term $-rV$ in the Black–Scholes equation, so does not contribute to the inner expansion to the order of accuracy considered here, reflecting the very small probability of a jump from the inner region between reset times (see [12] for more details).

5 Conclusion

We have described a matched asymptotic expansions approach to the issue of discretely sampled barrier options. We give a complete description of the value function, with two alternative forms of the composite expansion, of which one is uniformly valid but suffers from discontinuities, while the other is invalid far below the barrier but is smooth. A striking feature of our analysis is the excellence of the BGK approximation, and the reason for this is an interesting question for future research.

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