Knowing Thy Neighbor:

Rational Expectations and Social Interaction in Financial Markets *

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Abstract

In the real world, traders constantly communicate, and learn from each others' actions. Yet, standard noisy rational expectations models assume away such social interaction and let traders interact only through the price system. In this paper we propose a generalized noisy rational expectations model which accommodates social interaction in financial markets. On top of the information conveyed through the price system, each trader infers additional information by observing some of the other traders' portfolios. Whom a trader observes is determined by a directed graph that represents the social network. We define a notion of equilibrium for this generalized framework and investigate the implications of our model in several natural settings, including cyclic interaction schemes (where the social network is a cycle) and hierarchic interaction schemes (where the network is a tree). In particular, we show that social interaction can impair the aggregation of dispersed private information in the price system. Some preliminary results are derived in the case where traders' observations of others' portfolios are perturbed by noise.

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> Investing in speculative assets is a social activity. Investors spend a substantial part of their leisure time discussing investments, reading about investments, or gossiping about others' successes or failures in investing.
>
> Robert J. Shiller[1]

1 Introduction

Interaction among heterogenous agents is an important and pervasive feature of any economic environment. In *standard noisy rational expectations models*,[2] agents interact only through the price system: they make their decisions in *isolation*, using only their private information and the information conveyed by security prices.[3] Yet, in the real world, agents communicate, and learn from each others' actions. That is, the *economic agent* is also a *social agent*.

The general objective of this paper is to introduce social interaction into the modelling of financial markets. In particular, we develop a framework that allows agents to infer information from *social interaction* as well as the *price system*. We propose a rational expectations model in which each agent observes equilibrium security demands (i.e., portfolios) of some of the other agents, whom we refer to as "*uphill neighbors*",[4] on top of her own private signal and security prices. The *uphill neighborhood* of an agent is determined by a given directed graph, which represents the social network. Such a model, of course, encompasses the conventional rational expectations model: once the underlying social network is taken to be a trivial graph without any edges, all agents are isolated, and we return to the standard rational expectations economy.

Our generalized framework presents a new information source on top of the price system: uphill neighbors' portfolios. The information inferred from an uphill neighbor not only refers to the uphill neighbor's private signal but also aggregates signals of agents observed by the uphill neighbor, and

[1] Shiller (1984)

[2] For a critical survey, see Admati (1989).

[3] For a critical assessment of interaction in economic systems, see Kirman (1996).

[4] We use the term "uphill neighbor" rather than "neighbor" to emphasize that observations are not necessarily bilateral. That is, for agents A and B of the economy, all of the following interaction patterns are possible:

- A observes B's portfolio while B does not observe A's, in which case B is an uphill neighbor of A;
- B observes A's portfolio while A does not observe B's, in which case A is an uphill neighbor of B;
- Both A and B observe each other's portfolios, in which case both A and B are uphill neighbors of each other.

signals of those who are observed by agents observed by the uphill neighbor, and so on. That is, the information inferred from an uphill neighbor's portfolio is not uni-dimensional unless the uphill neighbor has no agent to observe. So, in general, we expect multi-dimensional information to be aggregated into the inference from social interaction. We also have a byproduct due to this generalized framework: a new heterogeneity (other than risk aversion and information quality) arises due to the possibility of asymmetric interaction patterns in the given social network.

The main novelty of this paper is that it accommodates both social interaction and price as informational conveyers in a *tractable* model which exploits parametric assumptions of constant absolute risk aversion and normal distribution. These assumptions considerably simplify the analysis, and they facilitate closed-form solutions.

The first result we establish is the non-existence of linear equilibrium price in an economy with a social network dictated by a *cycle*. This result is surprising, because linear pricing rules are sustainable when there is no interaction among agents, i.e., in the case of a standard REE model. Information conveyed through social interaction in a cycle *destructively interferes* with the information conveyed through linear prices. This clearly illustrates that social interaction matters, that is, it cannot be simply assumed away in the analysis of financial markets.

We next consider social networks dictated by acyclic graphs, in particular, *trees*. Trees represent *hierarchic* interaction schemes. We prove the existence of linear equilibrium price for trees under the condition that the random liquidity supply of securities has a sufficiently large variance.

Retaining the tree as the underlying social network, we also investigate the case in which the social interaction is the unique conveyer of information. Such property pertaining to information transmission is obtained in the limit when the variance of liquidity tends to infinity. This leads agents to ignore price as an informational source since the large liquidity noise completely overshadows the information contained in price. For this case, we obtain a limit equilibrium in closed-form. The equilibrium reveals that the weight of each agent's signal in the linear price is greater than the weight of the signal of her successor in the tree. Hence the signal of the agent at the root of the tree is most influential for the determination of price whereas the signals of agents at the terminal nodes are comparatively less significant. In other words, hierarchy in observation leads to hierarchy in influence.

Next we consider a special case of a tree: *star*. In a star, social network imposes an interaction scheme which makes all agents observe one central agent. To illustrate this case, consider a CEO as

the central agent who owns stocks of her own company. The SEC regulations force the CEO to reveal major changes in her portfolio. Thus the SEC effectively dictates star as an interaction scheme. In the case of star, we find a unique linear equilibrium in closed-form. The limit equilibrium, when variance of liquidity is taken to infinity, shows that the effect of central agent's signal on price will increase as the number of agents observing her portfolio increases.

One of the important results in our paper is that social interaction can impair information aggregation in the price system. If there is no social interaction in the market (i.e., in the standard REE framework), each private signal's weight in the linear equilibrium price is proportional to the signal's precision. This reflects efficient aggregation of information, because if the signal of one agent, say A's, is more precise than the signal of another agent, then A's signal's impact on price is proportionately larger. However, if agents interact in a social network dictated by a star, then the weight of the central agent's signal in the linear equilibrium price is genericly greater than the weights of other agents' signals even if the central agent's signal is not more precise than the signals of others. Hence the information aggregation in price is impaired by social interaction in the case of a star. To illustrate the importance of this result, consider the large price swings in the stock market. Many of these swings are actually not preceded by any significant events.[5] When the information aggregation is efficient as in the REE framework, it is hard to account for such large price swings. The result discussed above shows a new way out of this dilemma: the swings may be due to *lumpy information aggregation*, as is the case in the star. DeMarzo, Vayanos and Zwiebel (2003) point out an actual example for lumpy information aggregation and mention its consequences pertaining to stock market: "[...] consider the individuals participating in internet chat rooms on financial investments. Many of these individuals have inaccurate (and sometimes false) information. However, the fact that they have a large audience [...] can give them enough influence to affect market prices. For example, the Wall Street Journal (November 6, 2000) reports: 'Preliminary figures show that market manipulation accounted for 8% of the roughly 500 cases the SEC brought in fiscal 2000, ended September 30, up from 3% in fiscal 1999. Manipulation on the Internet is where the action is, and appears to be replacing brokerage boiler rooms of the past, said SEC enforcement-division director Richard Walker'."

Finally, we study the implications of social interaction on security pricing when agents' observations of uphill neighbors' portfolios are perturbed by noise. In particular, we show that a linear equilibrium price exists in cycles when observations are noisy.

[5] Cutler, Poterba, Summers (1989) document this for the postwar price movements.

There have been some empirical studies highlighting the role of social interaction as a conveyer of information in financial markets. One of the earliest studies is due to Shiller and Pound (1989): their survey questions 131 investors in the stock market. Majority of these investors asserted that their initial interest in their most recent stock purchase was prompted by discussions with their peers. Of course, the evidence here is only suggestive, but the idea of information transmission via social interaction seems very reasonable in light of this survey. Recent empirical studies provide further evidence of information transmission through social channels in financial markets. Hong, Kubik, and Stein (2002a) show that mutual-fund managers are heavily influenced by the decisions of other fund managers working in the same city. In particular, authors observe the following pattern in fund managers' decisions: a fund manager is likely to hold (or buy, or sell) a particular stock in any quarter if other managers from different fund families located in the same city are holding (or buying, or selling) that same stock. The authors interpret this using an epidemic model where investors spread information about stocks directly to one another by word of mouth. In a different empirical work (2002b), the same authors also argue that stock market participation is influenced by social interaction, i.e., "social investors" find market more attractive when more of their peers participate.

It is also worth mentioning the theoretical literatures, to which our paper is related. In the models of social learning theory[6], social interaction takes place through sequential observations of others' actions over time. The memory of past actions may reduce (or completely prevent) social learning in these models. This causes lumpy information aggregation, as is the case in our paper. However, unlike social learning theory, we let agents interact according to a social network in a static economy with a full-blown financial market.

Another closely related paper is due to DeMarzo, Vayanos and Zwiebel (2003). Their paper proposes a boundedly-rational model of opinion formation in social networks.[7] The pivotal assumption of the model is *persuasion bias*, which allows double counting of repeated information. That means if an agent receives information from two other agents, who happen to share information with each other, the agent acts as if the pieces of information received from the two are mutually independent. This assumption leads to *social influence*: "well-connected" agents may have more influence in the overall

[6]The seminal papers in this literature are Banerjee (1992), Bikchandani, Hirshleifer and Welch (1992), and Ellison and Fudenberg (1995). Avery and Zemsky (1998) investigate the implications of social learning theory in the financial markets. Bikchandani and Sharma (2001), and Devenow and Welch (1996) provide extensive literature reviews on social learning in financial markets.

[7]In a companion paper (2001), the authors explore the implications of their original model in financial markets.

formation of opinions in the economy regardless of their information accuracies. We obtain a similar result in our paper, however, we do not dispense with rationality.

Our paper is organized as follows. In Section 2, we recall the standard rational expectations model (à la Hellwig), and obtain the unique rational expectations equilibrium (REE) for the economy consisting of agents homogenous in risk aversion and signal precision. The obtained REE serves as a benchmark for us to assess the implications of social interaction. Section 3 exhibits our generalized REE model which accommodates social interaction in the financial market. We describe the functioning of the model pertaining to information transmission via observations of uphill neighbors' portfolios, and then introduce the equilibrium concept for this new framework. Section 4 and Section 5 investigate the implications of our model in several natural settings, including cyclic interaction schemes (dictated by cycles) and hierarchic interaction schemes (dictated by trees). In Section 4 we show the non-existence of linear equilibrium price for the case in which the interaction scheme is delivered by a cycle. In Section 5, we prove existence of linear equilibrium price for hierarchic interaction schemes provided there is sufficient liquidity risk in the economy. Later we characterize the acquired equilibrium under certain restrictions regarding the pattern of interaction and level of liquidity risk. Section 6 exhibits that social interaction can impair aggregation of dispersed private information in the price system. In Section 7, we sketch the implications of the case in which agents' social inferences are perturbed by some exogenous noise. Section 8 concludes. The proofs are provided in the appendix.

2 Rational Expectations when Only Price Conveys Information

We begin by recalling the standard rational expectations equilibrium (REE) approach to information transmission in the financial markets, i.e., we look into the case when only the price system conveys information (with some exogenous noise) in the economy. By doing so, we believe later it will be easier for the reader to compare the approach of the standard REE framework with that of ours. We describe the rational expectations economy à la Hellwig (1980). The physical environment for the financial market economy is given as follows:

There are $n \geq 2$ agents, indexed by $i = 1, ..., n$. In a two-period economy, trade takes place in the first period and consumption of a single good in the second. A risk-free security (in units of the consumption good) and a risky security are traded. The risky security has a future stochastic payoff $\tilde{X}$, which realizes in the second period. The price and the payoff of the risk-free security are normalized

to 1. We let p be the price of the risky security[8]. Each agent i is endowed with *deterministic* wealth w_{0i} (in units of consumption good). If agent i purchases z_i units of the risky security, i's portfolio yields the random final wealth

$$\tilde{w}_{1i} = z_i \tilde{X} + (w_{0i} - p z_i).$$

We specify agents' preferences by the following assumption:

A1. *All agents have CARA preferences; i.e., for $i = 1, ..., n$, agent i maximizes expected utility of final wealth*

$$\mathrm{E}_i \, u_i(\tilde{w}_{1i}) = \mathrm{E}_i[-exp(-\rho_i \, \tilde{w}_{1i})],$$

where $\rho_i \in (0, \infty)$ denotes the (absolute) risk aversion coefficient for agent i. The expectation operator, E_i, is based on agent i's information I_i.

Note that under this assumption agent i's (risky security) demand becomes independent of her initial wealth w_{0i}. Essentially, the demand of agent i will only depend on the price p and the information set I_i of agent i.

Next, we impose normal distributions for random parameters of the economy:

A2. *Each agent i receives a private random signal $\tilde{s}_i$, which communicates the true stochastic payoff $\tilde{X}$ perturbed by some additive noise $\tilde{\epsilon}_i$:*

$$\tilde{s}_i = \tilde{X} + \tilde{\epsilon}_i.$$

The net supply (i.e., liquidity) of the risky security L is taken to be the realization of a random variable $\tilde{L}$. The random vector[9]

$$(\tilde{X}, \tilde{L}, \tilde{\epsilon}_1, ..., \tilde{\epsilon}_n)$$

is normally distributed with mean

$$(\mu_x, 0, 0, ..., 0),$$

[8]From now on, the terms *price* and *demand* will be exclusively used for the risky security price and demand, respectively, unless otherwise stated.

[9]Throughout the text, we use the following convention: Random variables are denoted with *tilde* (such as $\tilde{y}$), and the realizations of random variables are denoted without *tilde* (such as y).

and nonsingular variance-covariance matrix

$$(\sigma_x{}^2, \sigma_L{}^2, \sigma_{\epsilon_1}{}^2, ..., \sigma_{\epsilon_n}{}^2)\, I_{n+2},$$

where I_{n+2} denotes the $(n+2)$ dimensional identity matrix.[10]

The (risky security) price p is dependent on the realized liquidity L and the private signals $s_1, ..., s_n$. Considering the whole range of realizations for the random variables $\tilde{L}$ and $\tilde{s}_1, ..., \tilde{s}_n$, the realized prices generate a random variable $\tilde{p}$. The following assumption describes the information transmission in the economy. In particular, it imposes the hypothesis that expectations, determined through the observations of private signals and price, are rational.

A3. *For $i = 1, ..., n$, agent i knows the actual joint distribution of the random vector $(\tilde{X}, \tilde{s}_i, \tilde{p})$. Agent i also observes realizations s_i and p. For any information*

$$I_i = (s_i, p),$$

agent i's expectation E_i is based on the actual conditional distribution of $\tilde{X}$ given s_i, p.

For the described economy, the following definition is standard:

DEFINITION. *A **rational expectations equilibrium (REE)** consists of a risky security price function $P(s_1, ..., s_n; L)$ with*

$$P : \mathbb{R}^{n+1} \to \mathbb{R}$$

and risky security demands $\{z_i(p, s_i)\}_{i=1,...,n}$ with

$$z_i : \mathbb{R}^2 \to \mathbb{R}$$

such that for all realizations of $(s_1, ..., s_n; L)$ of $(\tilde{s}_1, ..., \tilde{s}_n; \tilde{L})$

(a) $z_i(p, s_i) \in argmax_{z_i} \mathrm{E}\left[u_i(\tilde{w}_{1i}) \middle| s_i, p = P(s_1, ..., s_n; L)\right], \qquad \forall\, i = 1, ..., n;$

(b) $\sum_{i=1}^{n} z_i(P(s_1, ..., s_n; L), s_i) = L.$

[10] Nonsingularity guarantees

$$\sigma_x{}^2 \neq 0, \qquad \sigma_L{}^2 \neq 0, \qquad \sigma_{\epsilon_i}{}^2 \neq 0, \quad i = 1, ..., n.$$

Hellwig (1980) showed the existence of *linear* REE price under assumptions A1-A3, and also partially characterized the equilibrium. A closed-form solution for linear equilibrium is not available in the full generality of the model. Neither is the uniqueness of linear equilibrium guaranteed. Both of these properties can be established for the linear REE in a large economy: to that end, Hellwig (1980) focuses on the limit linear REE where the number of agents tends to infinity, and Admati (1985) utilizes a model with a continuum of agents[11].

To facilitate the analysis of social interaction later in the paper, we would like to retain the finite-agent economy. For uniqueness and closed-form solution, we rather employ the following assumption:

A4. *For all* $i = 1, ..., n$,

$$\rho_i = \rho, \qquad {\sigma_{\epsilon_i}}^2 = {\sigma_\epsilon}^2.$$

Homogeneity of risk aversion nullifies agents' motive to trade for risk-sharing purposes, however, they will still trade competitively either to speculate on future payoff of the risky security (speculative trading) or to accommodate liquidity shocks (liquidity trading).

Using Hellwig's (1980) results[12] and employing assumption A4, we can easily show the existence of a *unique* linear REE price with *a closed form solution*. Since we will be imposing A4 for all further analysis with social interaction, the following result will essentially be the benchmark for us:

Proposition 1 *Assume A1, A2, A3, and A4. Then there exists a unique linear REE price*

$$\tilde{p} = P(\tilde{s}_1, ..., \tilde{s}_n; \tilde{L}) = \pi_0 + \sum_{i=1}^{n} \pi_i \, \tilde{s}_i - \gamma \, \tilde{L} \tag{2.1}$$

with

$$\begin{aligned}
\pi_i &= \gamma \, q, \qquad {\scriptstyle i=1,...,n,} \\
\gamma &= \frac{1 + \frac{1}{\rho} \frac{n(n-1)q}{((n-1)q^2 {\sigma_\epsilon}^2 + {\sigma_L}^2)}}{n \, \frac{{\sigma_x}^2 + {\sigma_\epsilon}^2}{\rho {\sigma_x}^2 {\sigma_\epsilon}^2} + \frac{1}{\rho} \frac{n(n-1)q^2}{((n-1)q^2 {\sigma_\epsilon}^2 + {\sigma_L}^2)}}, \\
\pi_0 &= \frac{\frac{n \mu_x}{\rho {\sigma_x}^2}}{n \, \frac{{\sigma_x}^2 + {\sigma_\epsilon}^2}{\rho {\sigma_x}^2 {\sigma_\epsilon}^2} + \frac{n(n-1)q^2}{\rho((n-1)q^2 {\sigma_\epsilon}^2 + {\sigma_L}^2)}},
\end{aligned}$$

where

$$q = \sqrt[3]{\frac{{\sigma_L}^2}{2(n-1)\rho {\sigma_\epsilon}^4}} \sqrt[3]{1 + \sqrt{1 + \frac{4}{27} \frac{\rho^2 {\sigma_L}^2 {\sigma_\epsilon}^2}{n-1}}} - \sqrt[3]{-1 + \sqrt{1 + \frac{4}{27} \frac{\rho^2 {\sigma_L}^2 {\sigma_\epsilon}^2}{n-1}}}.$$

[11] Admati's (1985) model is a generalization of Hellwig's (1980) to the multi-security case.

[12] These results are provided in the appendix.

Proposition 1 establishes certain properties for the linear REE. A closed-form solution and uniqueness are, of course, among these properties. On top of these, the proposition depicts the *symmetry* among agents of the economy under assumption A4: without diversity in attitudes towards risk or information quality (signal precisions), we cannot create diversity among agents of an REE economy with regard to their effects in the price formation process. In later sections, we will show that the properties acquired here do not necessarily hold when social interaction enters the picture.

3 Rational Expectations with Social Interaction

After this quick recap of the standard rational expectations model, we propose a more general framework which allows formation of rational expectations in the presence of social interaction.

3.1 The Model of an REE Economy with Interacting Agents

Our model retains the basic features of the model considered in §2 with the notable exception of assumption A3. In particular, we carry over the description of the physical environment, and assumptions A1, A2, A4. Assumption A3 will be replaced by S3, which describes the information transmission in the presence of social interaction.

We begin by specifying a social network for the economy: the underlying social network for interactions is a simple directed graph[13], where the vertices and the directed edges represent the agents, $i = 1, ..., n$, and the directions of risky security demand observations, respectively. The agents, whose demands are observed by i, are called i's *uphill neighbors*, and the set of i's uphill neighbors is denoted by $\mathcal{N}_i$. That is, agent i observes (realized) risky security demand z_j of agent j for all $j \in \mathcal{N}_i$. Note that $\mathcal{N}_i$ may be an empty set.

Let $\delta_i(j)$ denote the (realization of) information that i infers (on top of price p) from the observation of z_j for $j \in \mathcal{N}_i$. In §3.3, we will show what $\delta_i(j)$ stands for in terms of the exogenous parameters of the model. We will also verify that $\delta_i(j)$ is the realization of a *gaussian* distribution $\tilde{\delta}_i(j)$ provided price $\tilde{p}$ is a linear function of $\tilde{s}_1, ..., \tilde{s}_n, \tilde{L}$. However, to that end, we need to define the equilibrium

[13] A graph is called *simple* if multiple edges between the same pair of vertices or edges connecting a vertex to itself are forbidden.

A graph is called *directed* if edges exhibit inherent direction, implying every relationship so represented is asymmetric.

concept, which will be done in §3.2.

Now we can describe the information transmission in this complex environment:

S3. *Agent i knows the actual joint distribution of $\left(\tilde{X}, \tilde{s}_i, \tilde{p}, \{\tilde{\delta}_i(j)\}_{j\in\mathcal{N}_i}\right)$, and moreover she observes the realizations s_i, p, and $\{\delta_i(j)\}_{j\in\mathcal{N}_i}$, that is,*

$$I_i = (s_i, p, \{\delta_i(j)\}_{j\in\mathcal{N}_i}) .$$

We further assume the following for all $j \in \mathcal{N}_i$:

- *agent i knows that j has a CARA utility with the risk aversion coefficient ρ_j,*
- *agent i also knows the actual joint distribution of the random vector $\left(\tilde{X}, \tilde{s}_j, \tilde{p}, \{\tilde{\delta}_j(k)\}_{k\in\mathcal{N}_j}\right)$,*
- *however, i does not observe realizations s_j or $\{\delta_j(k)\}_{k\in\mathcal{N}_j}$.*

Since i knows the distribution of $\left(\tilde{X}, \tilde{s}_j, \tilde{p}, \{\tilde{\delta}_j(k)\}_{k\in\mathcal{N}_j}\right)$ and ρ_j for $j \in \mathcal{N}_i$, she also knows how her uphill neighbor j forms expectations and what *the functional form of j's demand z_j* is. This allows agent i to disentangle price p from z_j, which then substantially simplifies our computations of the correlations between $\tilde{p}$ and $\{\tilde{\delta}_i(j)\}_{j\in\mathcal{N}_i}$. We will clarify functioning of the model in §3.3.

Finally, note that the basic REE model (of §2) is a special case of our model: if the social network is a graph without any edges, then assumption S3 is equivalent to A3.

3.2 The Equilibrium Concept in the Presence of Social Interaction

The appropriate equilibrium concept for the framework of §3.1 is the following:

DEFINITION. *A **rational expectations equilibrium with social interaction (REESI)** consists of a risky security price function $P(s_1, ..., s_n; L)$ with*

$$P : \mathbb{R}^{n+1} \to \mathbb{R},$$

and risky security demands $\{z_i(s_i, p, \{z_j\}_{j\in\mathcal{N}_i})\}_{i=1,...,n}$ with[14]

$$z_i : \mathbb{R}^2 \times \mathbb{R}^{\mathrm{c}(\mathcal{N}_i)} \to \mathbb{R}$$

[14] $\mathrm{c}(A)$ denotes the cardinality of set A.

such that for all realizations $(s_1, ..., s_n; L)$ *of* $(\tilde{s}_1, ..., \tilde{s}_n; \tilde{L})$

(a) $z_i\left(s_i, p, \{z_j\}_{j\in\mathcal{N}_i}\right) \in argmax_{z_i} \mathrm{E}\left[u_i(\tilde{w}_{1i})\middle| s_i, p = P(s_1, ..., s_n; L), \{z_j = z_j\left(s_i, p, \{z_k\}_{k\in\mathcal{N}_j}\right)\}_{j\in\mathcal{N}_i}\right]$, $\forall\, i = 1, ..., n$,

(b) $\sum_{i=1}^{n} z_i\left(s_i, P(s_1, ..., s_n; L), \{z_j\}_{j\in\mathcal{N}_i}\right) = L$.

Naturally, *REESI* reduces to *REE* if agents have no uphill neighbors, i.e., when $\mathcal{N}_i = \emptyset, \quad \forall i = 1, ..., n$.

As in Hellwig (1980) and most of the literature on REE in financial markets, we will focus on a REESI price that is linear in the private signals and the liquidity, meaning that we look for a function of the form

$$p = P(s_1, ..., s_n; L) = \pi_0 + \sum_{i=1}^{n} \pi_i s_i - \gamma L,$$

which can be also equivalently stated in random parameters as follows:

$$\tilde{p} = P(\tilde{s}_1, ..., \tilde{s}_n; \tilde{L}) = \pi_0 + \sum_{i=1}^{n} \pi_i \tilde{s}_i - \gamma \tilde{L}.$$

From now on we will use the terms *equilibrium* and *linear equilibrium price* only to refer to *REESI* and *linear REESI price*, respectively, unless otherwise noted.

3.3 Functioning of the Model

In this section we will clarify how our model functions: in particular, we will show that the inferred information from social interaction is gaussian for linear equilibrium prices. To that end, we need to explain what the inferred information $\delta_i(j)$ explicitly stands for.

Choose an arbitrary agent i with $\mathcal{N}_i \neq \emptyset$. Assume the price $\tilde{p}$ is a linear function of private signals $\{\tilde{s}_i\}_{i=1,...,n}$ and liquidity $\tilde{L}$. This assumption guarantees that $\tilde{p}$ is normally distributed. Also, assume at this point that $\{\tilde{\delta}_i(j)\}_{j\in\mathcal{N}_i}$ are normal random variables (a property that will be verified later). Then the conditional distribution of risky payoff $\tilde{X}$ as assessed by agent i has the mean

$$\mathrm{E}[\tilde{X}|I_i] \equiv \mathrm{E}\left[\tilde{X}\middle| s_i, p, \{\delta_i(j)\}_{j\in\mathcal{N}_i}\right] = a_{0i} + a_{1i}s_i + a_{2i}p + \sum_{j\in\mathcal{N}_i} a_{3i}(j)\ \delta_i(j),$$

and variance

$$\mathrm{var}(\tilde{X}|I_i) \equiv \mathrm{var}\left(\tilde{X}\middle| s_i, p, \{\delta_i(j)\}_{j\in\mathcal{N}_i}\right) = b_i,$$

where the values of coefficients a_{0i}, a_{1i}, a_{2i}, $\{a_{3i}(j)\}_{j\in\mathcal{N}_i}$, b_i depend on the variance-covariance matrix of $\left(\tilde{X}, \tilde{s}_i, \tilde{p}, \{\tilde{\delta}_i(j)\}_{j\in\mathcal{N}_i}\right)$. These coefficients are independent of the realizations $(X, s_i, p, \{\delta_i(j)\}_{j\in\mathcal{N}_i})$ but rather determined just by the joint distribution.

In equilibrium, when agent i determines her (risky security) demand she only uses the demands of her uphill neighbors as an information source to form an expectation on the risky payoff $\tilde{X}$. By definition, $\delta_i(j)$ is the information that i infers (on top of price p) from the observation of her uphill neighbor's demand z_j, and therefore

$$z_i\left(s_i, p, \{z_j\}_{j\in\mathcal{N}_i}\right) \equiv z_i\left(s_i, p, \{\delta_i(j)\}_{j\in\mathcal{N}_i}\right).$$

Actually, the CARA-Gaussian setup dictates the following (risky security) demand for agent i:

$$\begin{aligned} z_i\left(s_i, p, \{\delta_i(j)\}_{j\in\mathcal{N}_i}\right) &= \frac{\mathrm{E}[\tilde{X}|I_i] - p}{\rho\,\mathrm{var}(\tilde{X}|I_i)} \\ &= \frac{a_{0i} + a_{1i}s_i + (a_{2i}-1)p + \sum_{j\in\mathcal{N}_i} a_{3i}(j)\ \delta_i(j)}{\rho_i\, b_i}. \end{aligned}$$

Like agent i, her uphill neighbor j has a demand of the form

$$z_j(s_j, p, \{\delta_j(k)\}_{k\in\mathcal{N}_j}) = \frac{a_{0j} + a_{1j}s_j + (a_{2j}-1)p + \sum_{k\in\mathcal{N}_j} a_{3j}(k)\ \delta_j(k)}{\rho_j\, b_j},$$

where a_{0j}, a_{1j}, a_{2j}, $\{a_{3j}(k)\}_{k\in\mathcal{N}_j}$, b_j depend on the variance-covariance matrix of

$$\left(\tilde{X}, \tilde{s}_j, \tilde{p}, \{\tilde{\delta}_j(k)\}_{k\in\mathcal{N}_j}\right).$$

By assumption S3, i knows the actual joint distribution of the random vector $(\tilde{X}, \tilde{s}_j, \tilde{p}, \{\tilde{\delta}_j(k)\}_{k\in\mathcal{N}_j})$ and risk aversion coefficient ρ_j. Thus all coefficients in demand of agent j are known constants for agent i. On top of that, agent i, like any other agent in the economy, observes the realization p of the price. So having all this knowledge, i can drop $\frac{a_{0j}}{\rho_j b_j}$ and $\frac{a_{2j}-1}{\rho_j b_j}\,p$ from z_j for all $j \in \mathcal{N}_i$. Thus what agent i infers from her uphill neighbor j's demand *on top of her observation of price* p is

$$\delta_i(j) = \frac{a_{1j}}{\rho_j b_j}\, s_j + \sum_{k\in\mathcal{N}_j} \frac{a_{3j}(k)}{\rho_j b_j}\ \delta_j(k). \tag{3.1}$$

Notice that i cannot further drop any other component from z_j since she observes neither the realization s_j nor the realizations $\{\delta_j(k)\}_{k\in\mathcal{N}_j}$. So with respect to the informational contents, the observations $(s_i, p, \{z_j\}_{j\in\mathcal{N}_i})$ and $(s_i, p, \{\delta_i(j)\}_{j\in\mathcal{N}_i})$ do not differ at all. On the other hand, getting rid of the redundant information in z_j substantially simplifies the analysis of the problem. Since there are finitely

many agents in the economy, doing similar substitutions for the social inferences $\{\delta_j(k)\}_{k\in\mathcal{N}_j}$ of agent j and for the social inferences of agent j's uphill neighbors and so on, we eventually obtain the reduced form

$$\delta_i(j) = f_{ij}(s_1, ..., s_n), \tag{3.2}$$

where $f_{ij} : \mathbb{R}^n \to \mathbb{R}$ is a linear function. Of course, for $1 \leq m \leq n$, who is neither an uphill neighbor of i nor an indirect[15] uphill neighbor of i, the corresponding weight for s_m is 0. For others, it is dependent on the recursive formulation given by (3.1). As (3.2) holds for all realizations $\{s_1, ..., s_n\}$ of $\{\tilde{s}_1, ..., \tilde{s}_n\}$, one has

$$\tilde{\delta}_i(j) = f_{ij}(\tilde{s}_1, ..., \tilde{s}_n),$$

and the fact that $\tilde{\delta}_i(j)$ is *gaussian* follows from the linearity of f_{ij} (as well as from the fact that signals $\{\tilde{s}_1, ..., \tilde{s}_n\}$ are jointly normally distributed).

Note that the explicit form of f_{ij} depends on the underlying graph employed for social interaction. Later we will investigate the cases where the underlying graph is non-acyclic and acyclic, and derive explicit form of f_{ij} in each corresponding case (see §9).

The arguments above clarify how our model functions. The CARA-Gaussian nature of the model (hence its tractability) is guaranteed since each agent's information set consists of the realizations of gaussian variables (assuming a linear equilibrium price $\tilde{p}$).

3.4 Applications to Cycles and Trees

Next we begin to explore the implications of our model in financial markets: we analyze several natural settings, including cyclic interaction schemes (where the social network is a cycle) and hierarchic interaction schemes (where the network is a tree). The analysis falls short of a complete one since we are not able to investigate all possible *non-acyclic* and *acyclic* graphs as the underlying social networks.[16]

[15] One will be regarded as an indirect uphill neighbor of i if her demand is observed by some agent $j \in \mathcal{N}_i$ or an uphill neighbor of the agent j or an uphill neighbor of an uphill neighbor of the agent j and so on.

[16] A *cycle* is a sequence of vertices v_1, v_2, ..., v_n such that each pair (v_i, v_{i-1}) are connected by an edge for all $1 \leq i \leq n$, where $i-1$ is considered in modulo n.

A graph is called *non-acyclic* if it contains a cycle.

A graph is called *acyclic* if it contains no cycles.

In particular, our analysis will be restricted to the social networks where each agent i has *at most one uphill neighbor*. Such restriction reflects the *limited interaction capability* of the agents in the economy, but it is imposed primarily for the sake of tractability. *The unique agent (if exists), who is the uphill neighbor of i, will be represented by i^-*. That is, our analysis will only allow

$$\mathcal{N}_i = \emptyset \quad \text{or} \quad \mathcal{N}_i = \{i^-\} \quad \text{for } i = 1, ..., n.$$

From now on, we also drop j from the notation $\delta_i(j)$, and let δ_i denote the (realization of) gaussian information that i infers (on top of price p) from the observation of z_{i^-}.

Despite its limitations, we still believe the analysis of cycles and trees provides a representative and appealing sample for possible social interaction patterns. For instance, the investigation of trees encompasses all possible acyclic graphs if we only consider the family of social networks in which

- each agent has at most one uphill neighbor (a restriction already employed above),
- there are no disjoint subgraphs, i.e, there do not exist two disjoint subsets of agents, say $\mathcal{G}_1$, $\mathcal{G}_2$, such that no agent in $\mathcal{G}_1$ is observed by an an agent in $\mathcal{G}_2$ and no agent in $\mathcal{G}_2$ is observed by an agent in $\mathcal{G}_1$.

4 Non-Existence of Linear Equilibrium in Cyclic Interaction Schemes

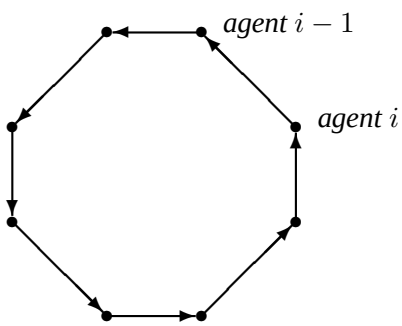

FIGURE I - A CYCLE REPRESENTING INTERACTION SCHEME IN THE MARKET

Arrows indicate the direction of demand observation. For all $i = 1, .., n$, agent i has additional information that comes from demand of agent $i - 1 \pmod n$ on top of s_i and p.

We now explore information transmission in a financial market economy where the underlying

social network posits a cyclic interaction scheme. We are especially drawn to the analysis of the cycle, because one wonders whether the presence of social interaction can substantially affect the price formation process even in such a symmetric environment where all agents in the economy observe and are observed by the same number of agents. We find that social interaction in the given scheme accomplishes more than just altering the price function obtained in the standard REE model (§2.2): it actually eliminates the possibility of linear pricing rules.

Proposition 2 *Assume A1, A2, S3, A4. If the social network is a cycle, there does not exist any linear REESI price in the financial market economy.*

Prior to social interaction the economy can accommodate the conventional linear pricing rule, and after the introduction of the cyclic interaction scheme this rule suddenly cannot be sustained. This is surprising, because social interaction only brings additional information to the economic agents. However, this new information source turns out to be *self-destructive*.

So, what is the underlying intuition? We believe the cause of non-existence is *infinite regress*.[17] Here, due to the nature of the cycle, each agent's social inference essentially contains information from every other agent's social inference, including one's own. This means each agent forms an expectation on the payoff of the risky security by referring to her very own expectation. However, this referral takes place in a way which causes infinite regress, that is, the agent cannot disentangle her own expectation from others expectations' in her social inference. This implies none of the agents can form an expectation, meaning that, the equilibrium cannot exist. The proof, provided in the appendix, formally shows the failure of linear pricing rule in the cyclic interaction schemes.

5 Existence and Characterization in Hierarchic Interaction Schemes

Last section vividly illustrates that non-acyclic graphs are not the most favorable interaction schemes to explore the effects of social interaction, because the tractable path, i.e., the linear equilibrium price, is ruled out even in the very basic interaction patterns such as cycles. Therefore we now turn our focus on

[17] Infinite regress problem also plagues the dynamic REE: in the discrete dynamic framework one has traders forecasting the forecasts of other traders over time, who also forecast the forecasts that others make of their forecasts and so on, which brings an ever increasing state history with time. The reader may refer to Townsend (1983a, 1983b), or Singleton (1987) for more on the issue pertaining to infinite regress.

social networks dictated by acyclic graphs: in particular, to those dictated by trees. As we elaborated on before in §3.4, once we assume that each agent has at most one uphill neighbor and that there are no disjoint subgraphs in the social network, we will have all acyclic graphs necessarily reduced to trees.

Trees have a natural appeal, because they dictate *hierarchic interaction schemes*. They also bring asymmetry into the social interaction pattern, hence heterogeneity into the financial market economy.

Before going into the formal analysis of hierarchic interaction schemes, we introduce some convention and notation for this environment:

- Without loss of generality, agent 1 is taken to be the root in the tree.
- For any agent $i > 1$, i^- denotes the immediate predecessor of i in the tree.
- For $k \geq 1$, i^{-k} denotes the kth predecessor of agent i in the tree.
- We also use i^{-0} to denote agent i.
- Let j be a predecessor of agent i (though not necessarily an immediate predecessor). $\#(i,j)$ denotes the integer that satisfies $i^{-\#(i,j)} = j$, i.e., $\#(i,j)$ gives the order of precedence of agent j according to agent i. Hence one has $\#(i,i) = 0$ and $\#(i,i^-) = 1$.
- For any agent i, $\mathcal{H}(i)$ denotes the set of predecessors of i, i.e.,

  $$\mathcal{H}(i) = \left\{ j \in \{1, ..., n\} : j = i^{-k}, \;\; 1 \leq k \leq \#(i,1) \right\}.$$

  Note that $1 \in \mathcal{H}(i)$ for all $i > 1$.
- $\mathcal{H}^{-1}(i)$ denotes the set of agents that are preceded by agent i, i.e.,

  $$\mathcal{H}^{-1}(i) = \left\{ m \in \{1, ..., n\} : i = m^{-k}, \;\; k \geq 1 \right\}.$$

  Thus $\mathcal{H}^{-1}(1) = \{2, 3, ..., n\}$, and $i \in \mathcal{H}^{-1}(i^-)$ for all $i > 1$.

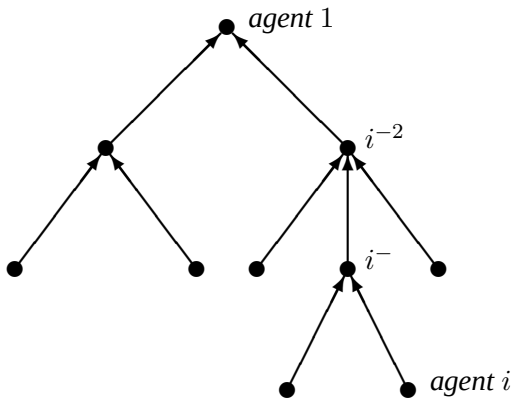

FIGURE III - A TREE REPRESENTING INTERACTION SCHEME IN THE MARKET

Arrows indicate the direction of demand observation. For all $i > 1$, agent i has additional information that comes from observing demand of agent i^- (i.e., her predecessor in the tree) on top of her private signal s_i and price p. Agent 1 only observes s_1 and p.

5.1 Existence of Linear Equilibrium with Sufficiently High Liquidity Risk

The main problem facing us in a framework with social interaction is the presence of *double counting*: agents infer information from both *social interaction* and *price*, and the informational contents of these two sources are not disjoint. In particular, when the interaction scheme is hierarchic, for any given agent i, $i > 1$, both price p and social inference δ_i contain information regarding agent i's predecessors' signals.

Actually, in the standard REE framework a similar situation also shows up: an agent's private signal is part of her information set both as a direct source and as an indirect source through price. However, since the agent is assumed to know the correlation between her private signal and price, she can always avoid *double counting* and compensate for his indirect reaction to s_i through the price by reducing her direct reaction to s_i.[18] In the case of social interaction, *the double counting problem cannot be avoided*. For instance, if agent i is down below the tree, that is if $\#(i,1) > 1$, then agent i will have the set of signals $\{s_{i-k}\}_{k=1,\ldots,\#(i,1)}$ appearing in both price p and her social inference δ_i, however, since she does not know the piecewise correlations of her predecessors' private signals with price p, she cannot efficiently adjust her reaction to the double appearance of the same informational content.

The existence of double counting complicates guessing (on our part) possible price and portfolio

[18] See Hellwig (1980).

reactions shown to news and liquidity shocks. This consequently paralyzes the "guess and verify" argument employed for the existence of equilibrium, because the argument requires us to guess a compact set in the Euclidean space where the price coefficients have a fixed point and then verify that there indeed exists a fixed point in the chosen set.[19] We can avoid such complications in the limit case where the variance of liquidity tends to infinity. In this limit case the precision of price becomes nil, hence agents (endogenously) attach no weight to price as an information source. Then we are back to an economy where a unique vehicle, namely social interaction, conveys information (recall that the unique vehicle is price in the standard REE economy). With a unique conveyer, we are able to predict easily how price and portfolio reactions would be, which consequently allows us to prove existence of a linear equilibrium price in the limit. The limit case is, of course, an approximation for the economy involving large liquidity risk. Using a continuity argument pertaining to the variance of liquidity, we extend the existence result provided there is sufficient noise (i.e., liquidity risk). Formally, we have the following:

Theorem 1 *Assume A1, A2, S3, A4. Suppose the social network is a tree. There is a level of liquidity variance* $\underline{\sigma_L}^2 < \infty$ *such that for all* $\sigma_L{}^2 \geq \underline{\sigma_L}^2$*, a linear REESI price exists with the functional form*

$$\tilde{p} = P(\tilde{s}_1, ..., \tilde{s}_n; \tilde{L}) = \pi_0^h + \sum_{i=1}^{n} \pi_i^h \, \tilde{s}_i - \gamma^h \, \tilde{L}, \qquad \gamma^h \neq 0.$$

Having provided an equilibrium existence result, we now begin to explore how this equilibrium behaves for different restrictions imposed on the exogenous parameters.

5.2 Linear Equilibrium when Only Social Interaction Conveys Information

Characterization of the equilibrium in its full generality proves to be quite a challenging task. Thus we focus on the polar cases. There are two polar cases pertaining to information transmission. One is when price is the unique conveyer of information, and this case is already analyzed in standard REE models. The other is when only social interaction conveys information. As we discussed in §5.1, the latter is obtained in the limit when noise (variance of liquidity) tends to infinity.

[19] This is actually the same proof technique employed by Hellwig (1980) to prove existence of linear REE.

Proposition 3 *Assume A1, A2, S3, A4. Suppose the social network is a tree. If ${\sigma_L}^2 \to \infty$, then*

(a) there is a sequence of linear REESI prices that converges to

$$\tilde{p}^s = \pi_0^s + \sum_{i=1}^{n} \pi_i^s \tilde{s}_i - \gamma^s \tilde{L},$$

where

$$\pi_0^s = \gamma^s \frac{n\mu_x}{\rho {\sigma_x}^2}, \tag{5.1a}$$

$$\pi_i^s = \gamma^s \left(1 + \sum_{m \in H^{-1}(i)} 1\right) \frac{1}{\rho {\sigma_\epsilon}^2}, \qquad i = 1, ..., n, \tag{5.1b}$$

$$\gamma^s = \frac{1}{\frac{n}{\rho {\sigma_x}^2} + \frac{n}{\rho {\sigma_\epsilon}^2} + \frac{1}{\rho {\sigma_\epsilon}^2} \sum_{i=2}^{n} \#(i,1)}; \tag{5.1c}$$

(b) and the corresponding sequence of REESI (risky security) demands of agent i, for $i = 1, ..., n$, converges to

$$z_i^s = \frac{\mu_x}{\rho {\sigma_x}^2} + \frac{1}{\rho {\sigma_\epsilon}^2} \sum_{k=0}^{\#(i,1)} \tilde{s}_{i^{-k}} - \left(\frac{1}{\rho {\sigma_x}^2} + (\#(i,1)+1) \frac{1}{\rho {\sigma_\epsilon}^2}\right) \tilde{p}^s. \tag{5.2}$$

From (5.1b), it follows that

$$\pi_{i^-}^s > \pi_i^s, \qquad i = 2, ..., n,$$

that is, the weight of each agent's signal is larger than that of her successor's signal in the price. In fact, the weight of an agent's signal is proportional to the number of agents she precedes in the tree. So, the higher the agent is in the hierarchy (tree), the higher the weight of her signal in the price. In particular, if we compare agent 1 (the root) and an agent at a terminal node of the tree, we get

$$\frac{\pi_1^s}{\pi_T^s} = n,$$

where π_T^s denotes the price coefficient of an agent in the terminal node. So, as the number of agents in the economy gets larger (i.e., as $n \to \infty$) agent 1's uncertainties exceedingly overshadow uncertainties of agents at the terminal nodes of the tree in the price formation.

As we elaborated in §1, DeMarzo, Vayanos and Zwiebel (2003) obtain a result similar to Proposition 3 employing a boundedly-rational model: they show that the position of an agent in the social network matters in the overall formation of opinions even if all agents' private signals have the same

precision. There is also a similarity between the mentioned implication of Proposition 3 and the two phenomena arising in social learning theory, namely, herd behavior and informational cascades. Herd behavior requires an informed agent to follow the trend in past trades even if the trend is counter to her initial information about the security payoff. In an informational cascade, no new information can reach the market because informed traders begin to act independent from their private information.[20] Our economy is competitive and static, and we are not seeking such extreme behaviors on the agents' part. However, we still have a clear similarity between our model and that of social learning theory: in ours, we see *early predecessors's signals* (i.e., signals of agents closer to the root) gaining a substantial weight in price compared to the signals of agents closer to the terminal nodes even though all agents in the economy have similar risk attitudes and signal precisions. Thus observational learning in the hierarchic scheme allows some of the agents' information (that of early predecessors') be overweighed, which is also a situation necessarily arising in any herd behavior or informational cascade.

5.3 Star Interaction Scheme

In this section we consider a star graph as the social network. Star is a special tree in which all vertices except a root are terminal nodes (see Figure IV). So, in terms of information flow, we have a central agent who is observed by every other agent in the economy.

In the analysis of star, we are motivated by the Security Exchange Commission (SEC) regulations enforcing information disclosure. Empowered by the Securities Exchange Act of 1934 and disclosure rules following this act, SEC can require any publicly traded company to offer full disclosure in events related to company's stocks such as repurchase plans, splits, defaults as well as any major portfolio position taken in company's stocks by directors, officers, and principal stockholders. In situations pertaining to major portfolio movements due to officers and principal stockholders of a publicly traded company, SEC effectively dictates star as an interaction scheme: the CEO (or the principal stockholder) and the minor stockholders of the company can be considered as the root and the terminal nodes of the star, respectively.

[20] Avery and Zemsky (1998)

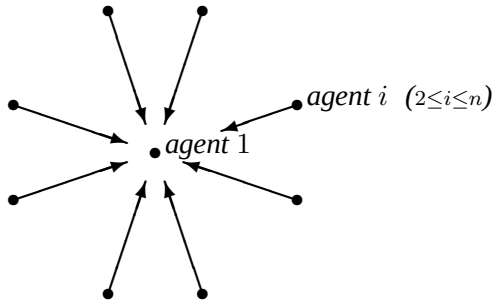

FIGURE IV - A STAR REPRESENTING INTERACTION SCHEME IN THE MARKET

Arrows indicate the direction of demand observation. So, for $2 \leq i \leq n$, agent i has additional information that comes from observation of agent 1's demand on top of her private signal s_i and price p.

Star scheme is the simplest hierarchic interaction scheme. Without relying on highly volatile liquidity supply, we can prove the existence of linear REESI in a star. Moreover, the obtained equilibrium price is unique and has closed form.

Proposition 4 *Assume A1, A2, S3, A4. Suppose the social network is a star in which agent 1 is the root. There exists a unique linear REESI price of the form*

$$\tilde{p} = P(\tilde{s}_1, ..., \tilde{s}_n; \tilde{L}) = \pi_0^* + \sum_{i=1}^{n} \pi_i^* \, \tilde{s}_i - \gamma^* \, \tilde{L} \quad \textit{with non-zero } \gamma^*. \tag{5.3}$$

The linear REESI price has

$$\begin{aligned}
\pi_i^* &= \gamma^* \, q^*, \qquad i{=}2,...,n; \\
\pi_1^* &= \gamma^* \, \frac{n}{\rho {\sigma_\epsilon}^2 + \frac{(n-1)q^* {\sigma_\epsilon}^2}{(n-1)(q^*)^2 {\sigma_\epsilon}^2 + {\sigma_L}^2} + (n-1) \frac{(n-2)q^* {\sigma_\epsilon}^2}{(n-2)(q^*)^2 {\sigma_\epsilon}^2 + {\sigma_L}^2}}, \\
\gamma^* &= \frac{1 + \frac{1}{\rho} \frac{(n-1)q}{(n-1)(q^*)^2 {\sigma_\epsilon}^2 + {\sigma_L}^2} + \frac{1}{\rho} \frac{(n-1)(n-2)q^*}{(n-2)(q^*)^2 {\sigma_\epsilon}^2 + {\sigma_L}^2}}{\frac{n}{\rho {\sigma_x}^2} + \frac{2n-1}{\rho {\sigma_\epsilon}^2} + \frac{1}{\rho} \frac{(n-1)^2 (q^*)^2}{(n-1)(q^*)^2 {\sigma_\epsilon}^2 + {\sigma_L}^2} + \frac{1}{\rho} \frac{(n-1)(n-2)^2 (q^*)^2}{(n-2)(q^*)^2 {\sigma_\epsilon}^2 + {\sigma_L}^2}}, \\
\pi_0^* &= \frac{\gamma^* n \frac{\mu_x}{\rho {\sigma_x}^2}}{1 + \frac{1}{\rho} \frac{(n-1)q^*}{(n-1)(q^*)^2 {\sigma_\epsilon}^2 + {\sigma_L}^2} + \frac{1}{\rho} \frac{(n-1)(n-2)q^*}{(n-2)(q^*)^2 {\sigma_\epsilon}^2 + {\sigma_L}^2}},
\end{aligned}$$

where

$$q^* = \sqrt[3]{\frac{{\sigma_L}^2}{2(n-2)\rho {\sigma_\epsilon}^4}} \left(\sqrt[3]{1 + \sqrt{1 + \frac{4}{27} \frac{\rho^2 {\sigma_L}^2 {\sigma_\epsilon}^2}{n-2}}} - \sqrt[3]{-1 + \sqrt{1 + \frac{4}{27} \frac{\rho^2 {\sigma_L}^2 {\sigma_\epsilon}^2}{n-2}}} \right).$$

One observes from this result that all agents' signals but agent 1's are equally significant in the price formation, i.e, $\pi_i^* = \pi_j^*, \ \forall\, i, j \in \{2, ...n\}$. Of course, this result is anticipated given the fact that agents $2, ..., n$ are completely homogenous with respect to their signal precision, risk aversion, and location in the interaction pattern.

Next we investigate the case where social interaction is the only conveyer of information (i.e., when ${\sigma_L}^2 \to \infty$). Going back to the analogy between the star scheme and the information flow arising after disclosure of information imposed on the CEO by SEC, it is plausible to expect the effect of disclosure to surpass any information revelation through price in the event of a major change in the CEO's portfolio. In our framework, this translates into social interaction being the major vehicle of information transmission, hence we believe the limit case ${\sigma_L}^2 \to \infty$ does not fall short of reality. Formally, we have the following result:

Corollary 1 *Assume A1, A2, S3, A4. Suppose the social network is a star in which agent* 1 *is the root. As* ${\sigma_L}^2 \to \infty$*, the linear REESI price converges to*

$$\tilde{p}^{*s} = \pi_0^{*s} + \sum_{i=1}^{n} \pi_i^{*s} \tilde{s}_i - \gamma^{*s} \tilde{L}, \tag{5.4}$$

with

$$\begin{aligned}
\pi_0^{*s} &= \gamma^{*s} \frac{n\mu_x}{\rho {\sigma_x}^2}, \\
\pi_1^{*s} &= \gamma^{*s} \frac{n}{\rho {\sigma_\epsilon}^2}, \\
\pi_i^{*s} &= \gamma^{*s} \frac{1}{\rho {\sigma_\epsilon}^2}, \qquad i = 2, ..., n, \\
\gamma^{*s} &= \frac{1}{\frac{n}{\rho {\sigma_x}^2} + \frac{2n-1}{\rho {\sigma_\epsilon}^2}};
\end{aligned}$$

Following Corollary 1, for $i = 2, ..., n$, the weight of agent i's signal becomes completely insignificant compared to the weight of agent 1' signal as the number of agents in the economy continues to grow without bound:

$$\lim_{n \to \infty} \frac{\pi_1^{*s}}{\pi_i^{*s}} = \infty, \qquad \forall\, i \geq 2.$$

Note that the significance of agent 1's signal is only due to agent 1's location in the interaction pattern since all agents are otherwise homogenous in terms of signal precision and risk aversion.

In summary, star scheme provides us an example where the linear equilibrium price is vividly different than that where the agents are making their portfolio decisions in isolation (the standard REE model). Comparison of the results in this section and Proposition 1 clearly illustrates this.

6 Social Interaction and Information Aggregation

We now explore the effect of social interaction on the aggregation of dispersed private information through the price system. To facilitate the formal analysis, we define *sufficient statistic* following Huang and Litzenberger (1988):

DEFINITION. *Let $\tilde{\tau}$, $\tilde{\theta}$ be random vectors, and τ, θ be their corresponding realizations. Let $\tilde{z}$ be a random variable with realization z. Also, let $f(\tau, z|\theta)$ be the joint density function of $\tilde{\tau}$ and $\tilde{z}$ conditional on $\tilde{\theta}$. The random variable $\tilde{z}$ is a* ***sufficient statistic*** *for the joint density f if there exist functions g_1 and g_2 such that for all realizations τ, z and θ*

$$f(\tau, z|\theta) = g_1(\tau, z)\, g_2(z, \theta).$$

Using Bayes' rule, one can verify that if $\tilde{z}$ is a sufficient statistic for $f(\tau, z|\theta)$, the joint density of $\tilde{\theta}$ conditional on $\tilde{\tau}$ and $\tilde{z}$ is independent of $\tilde{\tau}$.

Applying this to our model, if $\tilde{z}$ is a sufficient statistic for the joint density of $(\tilde{s}_1, ..., \tilde{s}_n, \tilde{z})$ conditional on $\tilde{X}$, then the conditional density of $\tilde{X}$ given $(\tilde{s}_1, ..., \tilde{s}_n)$ and $\tilde{z}$ is independent of $(\tilde{s}_1, ..., \tilde{s}_n)$. In particular,

$$\mathrm{E}[\tilde{X}|\tilde{s}_1, ...\tilde{s}_n, \tilde{z}] = \mathrm{E}[\tilde{X}|\tilde{z}], \tag{6.1}$$

$$\mathrm{var}(\tilde{X}|\tilde{s}_1, ...\tilde{s}_n, \tilde{z}) = \mathrm{var}(\tilde{X}|\tilde{z}). \tag{6.2}$$

This means that agents, who have access to the knowledge of $\tilde{z}$, know all the relevant information in the economy. Since we would like to analyze the performance of price $\tilde{p}$ as an aggregator of diverse information, we shall question whether price is a sufficient statistic for the conditional density of $(\tilde{s}_1, ..., \tilde{s}_n, \tilde{p})$ given $\tilde{X}$. If it is, then (6.1) and (6.2) hold for $\tilde{z} = \tilde{p}$. However, price can not be a sufficient statistic in our framework due to the existence of random liquidity supply (noise). In other words, price $\tilde{p}$ can not be *fully revealing*. This is a crucial feature of price in our model; because if price were fully revealing, social interaction would not convey any *additional information*.

Given that price itself can not be a sufficient statistic, we next question whether the informational content of price relevant to risky payoff $\tilde{X}$ can be one. If it is, then (6.1) and (6.2) will hold with $\tilde{z}$ being the informational content of $\tilde{p}$ relevant to $\tilde{X}$. With a linear price function, we can easily identify the informational content of price relevant to the risky payoff:

Let price $\tilde{p}$ be a linear function of the information vector $(\tilde{s}_1, ..., \tilde{s}_n)$ and liquidity supply (noise) $\tilde{L}$ such that

$$\tilde{p} = P(\tilde{s}_1, ..., \tilde{s}_n; \tilde{L}) = \pi_0 + \sum_{i=1}^{n} \pi_i \tilde{s}_i - \gamma \tilde{L}. \tag{6.3}$$

Then the *informational content of $\tilde{p}$ relevant to $\tilde{X}$* is given by $\sum_{i=1}^{n} \pi_i \tilde{s}_i$.

Accordingly, we define efficiency of price pertaining to information aggregation as follows:

DEFINITION. *Let price $\tilde{p}$ be of the form (6.3). We say that* ***price efficiently aggregates information*** $(\tilde{\mathbf{s}}_\mathbf{1}, ..., \tilde{\mathbf{s}}_\mathbf{n})$ *if the informational content of $\tilde{p}$ relevant to $\tilde{X}$, $\sum_{i=1}^{n} \pi_i \tilde{s}_i$, is a sufficient statistic for the joint density of $(\tilde{s}_1, ..., \tilde{s}_n, \sum_{i=1}^{n} \pi_i \tilde{s}_i)$ conditional on $\tilde{X}$, so that*

$$\begin{aligned} \mathrm{E}\left[\tilde{X} \middle| \tilde{s}_1, ...\tilde{s}_n, \sum_{i=1}^{n} \pi_i \tilde{s}_i\right] &= \mathrm{E}\left[\tilde{X} \middle| \sum_{i=1}^{n} \pi_i \tilde{s}_i\right], \\ \mathrm{var}\left(\tilde{X} \middle| \tilde{s}_1, ...\tilde{s}_n, \sum_{i=1}^{n} \pi_i \tilde{s}_i\right) &= \mathrm{var}\left(\tilde{X} \middle| \sum_{i=1}^{n} \pi_i \tilde{s}_i\right). \end{aligned}$$

Under assumptions A2 and A4 (i.e., joint normality of signals and i.i.d. error terms), we can verify that

$$\tilde{\mathcal{S}} \equiv \frac{1}{{\sigma_x}^2 + {\sigma_\epsilon}^2} \sum_{i=1}^{n} \tilde{s}_i$$

is a sufficient statistic for the joint density of $(\tilde{s}_1, ..., \tilde{s}_n, \tilde{\mathcal{S}})$ conditional on $\tilde{X}$. Intuitively, this follows, because the precision weighted sum of signals is more informative than any individual $\tilde{s}_i$. Taking a glance back at Proposition 1, we notice that the informational content of the REE price relevant to $\tilde{X}$ carries equal weights for all signals, i.e., it is equal to $\tilde{\mathcal{S}}$ multiplied by a constant. This means the informational content of the REE price relevant to $\tilde{X}$ is informationally equivalent to $\tilde{\mathcal{S}}$. So we have the following result:

Proposition 5 *Assume A1, A2, A3, A4. Then the linear REE price efficiently aggregates information* $(\tilde{s}_1, ..., \tilde{s}_n)$.

The presence of social interaction significantly affects the performance of price as an aggregator of information. In an economy with interacting agents, one of the determinants for the weights of signals in the linear equilibrium price is the interaction pattern itself. In particular, asymmetry in the interaction pattern can cause some agents' signals be overweighted in the equilibrium price even though all signals have the same precision level. As a consequence, the information aggregation through the price system can be impaired by social interaction. We see this happening in the star interaction scheme:

Proposition 6 *Assume A1, A2, S3, A4. Suppose the social network is a star in which agent* 1 *is the root. Then the linear REESI price does not efficiently aggregate information* $(\tilde{s}_1, ..., \tilde{s}_n)$ *for the generic exogenous parameters* ${\sigma_\epsilon}^2$, ${\sigma_L}^2$ *and* n.

Propositions 5 and 6 suggest that using REE price to assess real world markets may lead to false implications. One problem attributed to the informationally efficient REE is its inability to explain large price swings in the stock market. Cutler, Poterba and Summers (1989) document that there were no significant events prior to many large price swings for the postwar S&P 500 index. Therefore, if information is aggregated efficiently in the price, it is hard to account for such large price swings. Proposition 6 shows a new way out of this dilemma: the swings might be due to *lumpy aggregation of information through price*, as is the case in the star scheme. To illustrate the case, we can consider a minor change in a CEO's demand triggering a large price swing since the impact of her demand on price is amplified by the stock holders observing her action.

A similar phenomenon (i.e., lumpy information aggregation) is also observed in the models of social learning theory, where agents interact through the sequential observations of actions. The memory of past actions may reduce (or completely prevent) social learning in these models. Our paper extends the scope of social learning theory and some of its implications to a static economy with a social network and a full-blown financial market.

7 What If Social Inference Is Noisy?

In the model of §3 it was assumed that agents perfectly observe their uphill neighbors' portfolios. This is a stretch from the real world experience given that most of the time traders have uncertain information on others' portfolios. In this section we derive some preliminary results in the case where agents' observations of others' demands are noisy in a cyclic interaction scheme (cycle). In particular, we show that linear pricing rule can be recovered in cycles in the presence of noisy social inference.

7.1 A Model of Noisy Social Inference in Cycles

To introduce noise into social inference within *a cyclic interaction scheme (cycle)*, we alter the model given in §3.1 as follows (retaining all else, namely A1 and A4, as they were):

For $i = 1, ..., n$, agent i is able to observe

$$z_{i-1} + \eta_{i-1},$$

where z_{i-1} is the (realized) risky security demand of agent $i-1$ (modulo n), and η_{i-1} is some additive noise, which is taken to be the realization of a random variable $\tilde{\eta}_{i-1}$.

Let Δ_i denote the (realization of) information that i infers (on top of price p) from the observation of $z_{i-1} + \eta_{i-1}$. Each agent i knows the actual joint distributions of the random vectors $(\tilde{X}, \tilde{s}_i, \tilde{p}, \tilde{\Delta}_i)$ and $(\tilde{X}, \tilde{s}_{i-1}, \tilde{p}, \tilde{\Delta}_{i-1})$ as well as the risk aversion coefficient ρ_{i-1} of $i-1$. Agent i only knows the realizations s_i, p, and Δ_i, i.e.,

$$I_i = (s_i, p, \Delta_i).$$

Also, the random vector $(\tilde{X}, \tilde{\epsilon}_1, ..., \tilde{\epsilon}_n, \tilde{\eta}_1, ..., \tilde{\eta}_n, \tilde{L})$ is normally distributed with mean

$$(\mu_x, 0, ..., 0, 0, ..., 0, 0),$$

and nonsingular variance-covariance matrix

$$({\sigma_x}^2, {\sigma_\epsilon}^2, ..., {\sigma_\epsilon}^2, {\sigma_\eta}^2, ..., {\sigma_\eta}^2, {\sigma_L}^2)\, I_{2n+2},$$

where I_{2n+2} denotes the $(2n+2)$ dimensional identity matrix.

For the described economy with noisy social inference, we have the following equilibrium definition:

DEFINITION. *A **noisy REESI** for a cycle consists of a risky security price function* $P(s_1, ..., s_n; \eta_1, ..., \eta_n; L)$ *with*

$$P : \mathbb{R}^{2n+1} \to \mathbb{R}$$

and risky security demands $\{z_i\left(s_i, p, z_{i-1} + \eta_{i-1}\right)\}_{i \equiv 1,...,n \pmod n}$ *with*

$$z_i : \mathbb{R}^3 \to \mathbb{R}$$

such that for all realizations $(s_1, ..., s_n; \eta_1, ..., \eta_n; L)$ of $(\tilde{s}_1, ..., \tilde{s}_n; \tilde{\eta}_1, ..., \tilde{\eta}_n; L)$

(a) $z_i\left(s_i, p, z_{i-1} + \eta_{i-1}\right) \in argmax_{z_i} \mathrm{E}\left[u_i(\tilde{w}_{1i}) \middle| s_i, p = P(s_1, ..., s_n; \eta_1, ..., \eta_n; L), z_{i-1} + \eta_{i-1}\right]$, $\forall i \equiv 1, ..., n \pmod n$

(b) $\sum_{i=1}^{n} z_i(s_i, P(s_1, ..., s_n; \eta_1, ..., \eta_n; L), z_{i-1} + \eta_{i-1}) = L$.

In equilibrium, when agent i determines her (risky security) demand she only uses the the noisy observation of her uphill neighbor's demand as an information source to form an expectation on the risky payoff $\tilde{X}$. By definition, Δ_i is the information that i infers (on top of price p) from the observation of $z_{i-1} + \eta_{i-1}$, and therefore

$$z_i\left(s_i, p, z_{i-1} + \eta_{i-1}\right) \equiv z_i\left(s_i, p, \Delta_i\right).$$

7.2 Linear Pricing Rule Can Be Recovered in Cycles

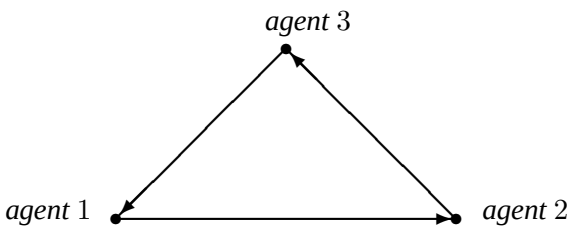

FIGURE VI - INTERACTION IN A TRIANGULAR SCHEME

Arrows indicate the direction of noisy demand observation. So each agent i observes demands of agents $i-1$ (modulo 3) on top of her own private signal s_i and price p.

We unfortunately have not been able to provide a general existence result for noisy REESI in cyclic interaction schemes. However, we have an example where noisy REESI exists for a particular choice

of exogenous parameters. The underlying reason driving existence in this extended framework is the leverage provided by noise in social inference. When we let the variance ${\sigma_\eta}^2$ of noise tend to ∞, we effectively eliminate the informational role played by social inference since any information aggregated in it becomes overshadowed by the infinitely volatile noise component. In this limit case, we are back to the standard REE framework where price is the only conveyer of information. From Proposition 1, the existence of linear equilibrium is already known in the limit case, hence by the help of continuity one hopes to retain this existence result when ${\sigma_\eta}^2$ is sufficiently large. Our hope is justified by the following remark:

Remark 1 *Assume the economy is as described in §8.1. Let $n = 3$ and take*

$$\rho = {\sigma_x}^2 = {\sigma_\epsilon}^2 = {\sigma_L}^2 = {\sigma_\eta}^2 = 1.$$

Then there exist multiple linear noisy REESI in the given economy. The linear noisy REESI prices

$$\tilde{p} = P(\tilde{s}_1, \tilde{s}_3, \tilde{s}_3; \tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3; \tilde{L}) = \Pi_0 + \sum_{i=1}^{3} \Pi_i \, \tilde{s}_i + \sum_{i=1}^{3} \Lambda_i \, \tilde{\eta}_i - \Gamma \, \tilde{L} \qquad \textit{with nonzero } \Gamma \tag{7.1}$$

and corresponding noisy REESI (risky security) demands

$$z_i(\tilde{s}_i, \tilde{p}, \tilde{\Delta}_i) = \theta_{0i} + \theta_{1i} \, \tilde{s}_i + \left(\theta_{2i} - \frac{1}{v_i}\right) \tilde{p} + \theta_{3i} \, \tilde{\Delta}_i, \qquad i = 1, ..., n \tag{7.2}$$

satisfy

$$\Pi_i = \Pi, \quad \Lambda_i = \Lambda; \quad \theta_{0i} = \theta_0, \theta_{1i} = \theta_1, \quad \theta_{2i} = \theta_2, \quad \theta_{3i} = \theta_3, \quad v_i = v, \qquad \forall \, i = 1, 2, 3,$$

where $(\Pi, \Lambda; \theta_0, \theta_1, \theta_2, \theta_3, v) \in \mathbb{R}^7$.

In particular, the linear noisy REESI coefficients $(\Pi_0, \Pi, \Lambda, \Gamma; \theta_0, \theta_1, \theta_2, \theta_3, v)$ *are elements of the following set:*

$$\Big\{ \quad (-218.076\mu_x, \quad 2.57761, \quad 18.3982, \quad -5.11641 \quad ; \quad 14.2076\mu_x, \quad 0.194071, \quad -0.0562592, \quad 1.38522, \quad -8.23662),$$

$$(0.032146\mu_x, \quad 0.0108026, \quad -0.0771057, \quad 0.0214426 \quad ; \quad 0.499722\mu_x, \quad -0.194071, \quad -13.424, \quad 1.38522, \quad 0.471385),$$

$$(154.584\mu_x, \quad -1.70977, \quad -11.8818, \quad 2.84085 \quad ; \quad 18.1382\mu_x, \quad 0.189114, \quad 0.103538, \quad 1.31422, \quad 4.52747),$$

$$(0.0326261\mu_x, \quad 0.0132595, \quad -0.0921447, \quad 0.0220311 \quad ; \quad 0.493638\mu_x, \quad -0.189114, \quad -13.351, \quad 1.31422, \quad 0.562074),$$

$$
\begin{aligned}
&(0.781218\mu_x, \quad 0.153613, \quad 0.0460308, \quad 0.220617 \quad ; \quad 1.18036\mu_x, \quad 0.57609, \quad 2.73627, \quad 0.172628, \quad 0.23545),\\
&(0.0558877\mu_x, \quad -0.083763, \quad 0.0251, \quad 0.120299 \quad ; \quad 0.154858\mu_x, \quad -0.57609, \quad -5.01806, \quad 0.172628, \quad -0.445),\\
&(1.50689\mu_x, \quad 0.296297, \quad -0.152274, \quad 0.266456 \quad ; \quad 1.8851\mu_x, \quad 2.59494, \quad 3.23153, \quad -1.3336, \quad 0.223089),\\
&(-0.135359\mu_x, \quad -0.23639, \quad -0.121486, \quad 0.212583 \quad ; \quad -0.212245\mu_x, \quad -2.59494, \quad -4.05052, \quad -1.3336, \quad -0.402819),\\
&(-0.608863\mu_x, \quad 0.018692, \quad -0.0664429, \quad 0.0997985 \quad ; \quad -2.03364\mu_x, \quad 0.560385, \quad 4.28061, \quad -1.99196, \quad 0.131222),\\
&(0.0207993\mu_x, \quad -0.0031277, \quad -0.0111178, \quad 0.0166991 \quad ; \quad 0.415177\mu_x, \quad -0.560385, \quad -25.5818, \quad -1.99196, \quad -0.177914)\Big\}.
\end{aligned}
$$

The remark illustrates that linear pricing rule can be recovered in cyclic interaction schemes once we let social inference be noisy. This is encouraging for further research on social networks which are dictated by non-acyclic graphs. The remark also shows that erratic equilibria may arise in the presence of noisy social inference: e.g., the price may plummet after arrival of good news about the security traded since in some equilibria the price coefficient Π, attached to agents' signals, takes negative value. Also, the fact that there are multiple equilibria, distinguishes the analysis of financial markets with social interaction from that in the standard REE framework (recall that we had a unique REE in Proposition 1).

8 Concluding Remarks

This paper studies security pricing when both price system and social interaction convey information in a rational expectations economy. The substance of all the results presented here is simply that social interaction matters in the pricing of securities. Presence of social interaction provides results and intuitions that cannot be obtained by studying the standard rational expectations models.

Two implications of our model are especially worth mentioning: one pertains to the relationship between hierarchy in observation and hierarchy in influence, the other concerns *lumpy information aggregation*. When social interaction is the only conveyer of information, the weight of each agent's signal in price is higher than that of her successor's signal in the hierarchic interaction schemes. That is, the hierarchy in observation leads to a hierarchy in influence. By *lumpy information aggregation*, we refer to the fact that in the presence of social interaction some traders' signals may be overweighted compared to the others' even if all signals share the same precision level. Such overweighting is not observed in the standard REE models. Since there were virtually no significant events prior to many

large price swings in the stock market (as documented by Cutler, Poterba and Summers (1989)), it is hard to account for large swings in the context of standard REE models, where signals are adequately weighted by their precisions in the equilibrium price. Lumpy information aggregation brought by social interaction shows us a new way out of this dilemma.

This paper is just a step toward an understanding of the role of social interaction in the functioning of financial markets. We shall mention several ways to extend our analysis. One is allowing for heterogeneity in risk aversion and signal precision among the traders. Naturally, one expects to see traders with less risk aversion and more precise signals to have more influence in the price formation process. A richer class of interaction patterns may also bring out new results: especially the assumption that each trader has at most one uphill neighbor is a significant limitation. Another extension pertains to the competitive behavior in our model. How imperfectly competitive markets would operate in the presence of social interaction remains to be seen. Also, most of our analysis is concerned with the case in which traders perfectly observe demands of their uphill neighbors. Adding noise to these observations makes the analysis more realistic. In this paper, we have already seen that noise in social inference can allow for the existence of linear equilibrium price in cyclic interaction schemes. Therefore we believe studying noisy social inference presents a promising direction for future research.

9 Appendix: Proofs

THE REE Á LA HELLWIG

For the proof of Proposition 1, we use the following lemmas from Hellwig (1980):

Lemma 1 *Assume A1, A2, and A3. Then there exists a linear REE price*

$$\tilde{p} = P(\tilde{s}_1, ..., \tilde{s}_n; \tilde{L}) = \pi_0 + \sum_{i=1}^{n} \pi_i \, \tilde{s}_i - \gamma \, \tilde{L} \quad \textit{with non-zero } \gamma, \tag{9.1}$$

and REE price coefficients satisfy the following:

$$\pi_i = \frac{\gamma}{\rho_i {\sigma_i}^2} \frac{\sum_{k=1}^n {\pi_k}^2 {\sigma_k}^2 + \gamma^2 {\sigma_L}^2 - \pi_i \pi {\sigma_i}^2}{\sum_{k=1}^n {\pi_k}^2 {\sigma_k}^2 + \gamma^2 {\sigma_L}^2 - {\pi_i}^2 {\sigma_i}^2}, \quad {\scriptstyle i=1,\ldots,n,} \tag{9.2a}$$

$$\frac{1}{\gamma} = \sum_{i=1}^n \frac{{\sigma_x}^2 + \sigma_i^2}{\rho_i {\sigma_x}^2 {\sigma_i}^2} + \sum_{i=1}^n \frac{(\pi - \pi_i)^2 - (\pi - \pi_i)}{\rho_i \left(\sum_{k=1}^n {\pi_k}^2 {\sigma_k}^2 + \gamma^2 {\sigma_L}^2 - {\pi_i}^2 {\sigma_i}^2\right)}, \tag{9.2b}$$

$$\pi_0 = \frac{\mu_x}{{\sigma_x}^2} \sum_{i=1}^n \frac{1}{\rho_i} - \gamma \pi_0 \sum_{i=1}^n \frac{\pi - \pi_i}{\rho_i \left(\sum_{k=1}^n {\pi_k}^2 {\sigma_k}^2 + \gamma^2 {\sigma_L}^2 - {\pi_i}^2 {\sigma_i}^2\right)}, \tag{9.2c}$$

where $\pi \equiv \sum_{k=1}^n \pi_k$.

Proof. See Proposition 3.3 and Eqs. (7a)-(7c) in Hellwig (1980).

Lemma 2 *Assume A1, A2, and A3. Let* $\pi_0, \{\pi_i\}_{i=1,\ldots,n}, \gamma$ *be the coefficients of an REE price function.*
(a) If $\rho_i \geq \rho_j$ *and* ${\sigma_i}^2 \geq {\sigma_j}^2$*, then* $\pi_i \leq \pi_j$.
(b) If $\rho_i \geq \rho_j$ *and* ${\sigma_i}^2 = {\sigma_j}^2$*, then* $\rho_i \pi_i \geq \rho_j \pi_j$.
(c) If $\rho_i = \rho_j$ *and* ${\sigma_i}^2 \geq {\sigma_j}^2$*, then* ${\sigma_i}^2 \pi_i \leq {\sigma_j}^2 \pi_j$.
If one of the inequalities in the premises of statements (a)-(c) is strict, the inequality in the corresponding conclusion is also strict.

Proof. See Proposition 4.1 in Hellwig (1980).

Proof of Proposition 1. The existence of a linear REE price follows from Lemma 1. We also have $\pi_i = \pi_j, \ \forall\, i, j$ from Lemma 2. Define a new variable $q \equiv \frac{\pi_i}{\gamma}$. Rewriting (9.2a), one obtains

$$(n-1){\sigma_\epsilon}^2 q^3 + {\sigma_L}^2 q - \frac{{\sigma_L}^2}{\rho {\sigma_\epsilon}^2} = 0. \tag{9.3a}$$

Let $f(x) \equiv (n-1){\sigma_\epsilon}^2 x^3 + {\sigma_L}^2 x - \frac{{\sigma_L}^2}{\rho {\sigma_\epsilon}^2}$. f is a strictly increasing continuous function, and it takes both positive and negative values. Therefore f must take value 0 for a unique $x \in \mathbb{R}$. This implies the existence of a *unique solution* q for equation (9.3a).

Rewriting (9.2b) and (9.2c), one gets

$$\gamma = \frac{1 + \frac{n(n-1)q}{\rho((n-1)q^2 {\sigma_\epsilon}^2 + {\sigma_L}^2)}}{n \frac{{\sigma_x}^2 + {\sigma_\epsilon}^2}{\rho {\sigma_x}^2 {\sigma_\epsilon}^2} + \frac{n(n-1)q^2}{\rho((n-1)q^2 {\sigma_\epsilon}^2 + {\sigma_L}^2)}}, \tag{9.3b}$$

$$\pi_0 = \frac{\frac{n \mu_x}{\rho {\sigma_x}^2}}{n \frac{{\sigma_x}^2 + {\sigma_\epsilon}^2}{\rho {\sigma_x}^2 {\sigma_\epsilon}^2} + \frac{n(n-1)q^2}{\rho((n-1)q^2 {\sigma_\epsilon}^2 + {\sigma_L}^2)}}. \tag{9.3c}$$

Since there exists a unique q satisfying equation (9.3a) (which is equivalent to (9.2a)), γ and π_0 are also uniquely given by the equations above. Hence uniqueness of price coefficients $\pi_i = \gamma\, q$, $i = 1, \ldots, n$, follow.

To derive the closed-form solution, notice that (9.3a) is a cubic polynomial in q without a quadratic component. Thus by *Cardano's Formula* (see, e.g., Artin (1991)), the unique (real) solution of (9.3a) is given by

$$q=\sqrt[3]{\frac{{\sigma_L}^2}{2(n-1)\rho{\sigma_\epsilon}^4}+\sqrt{\frac{{\sigma_L}^4}{4(n-1)^2\rho^2{\sigma_\epsilon}^8}+\frac{{\sigma_L}^6}{27(n-1)^3{\sigma_\epsilon}^6}}}-\sqrt[3]{-\frac{{\sigma_L}^2}{2(n-1)\rho{\sigma_\epsilon}^4}+\sqrt{\frac{{\sigma_L}^4}{4(n-1)^2\rho^2{\sigma_\epsilon}^8}+\frac{{\sigma_L}^6}{27(n-1)^3{\sigma_\epsilon}^6}}}.$$

Rewriting q, using equations (9.3b), (9.3c), and the fact that $\pi_i = \gamma\, q$, one obtains the desired result. □

THE CYCLIC INTERACTION SCHEMES (CYCLES)

Proof of Proposition 2: Without loss of generality, we relabel the agents in the given cycle so that for $i = 1, ..., n$, agent i observes demand of agent $i-1$, i.e.,

$$i^- = i - 1 \pmod n.$$

Suppose there exists a linear REESI price, which is given by

$$\tilde{p} = P(\tilde{s}_1, ..., \tilde{s}_n; \tilde{L}) = \mathring{\pi}_0 + \sum_{i=1}^{n} \mathring{\pi}_i\, \tilde{s}_i - \mathring{\gamma}\, \tilde{L}. \tag{9.4}$$

Recall that δ_i is the (realized) gaussian information that agent i derives by observing demand of $i-1$, for $i = 1, ..., n$ (from now on we consider the index $i-1$ modulo n). Each agent's information set consists of (the realizations of) her own private signal, price, and the information due to the observation of uphill neighbor's demand, i.e, $I_i = (s_i, p, \delta_i)$ for each $i = 1, ..., n$. Let $\mathring{V}_i$ denote the variance-covariance matrix of $(\tilde{s}_i, \tilde{p}, \tilde{\delta}_i)$, and $\mathring{W}_i$ be the covariance matrix of $\tilde{X}$ and $(\tilde{s}_i, \tilde{p}, \tilde{\delta}_i)$. The expectations of the agents are of the form

$$\begin{aligned} \mathrm{E}[\tilde{X}|I_i] &= \mathring{a}_{0i} + \mathring{a}_{1i}s_i + \mathring{a}_{2i}p + \mathring{a}_{3i}\delta_i, \\ \mathrm{var}(\tilde{X}|I_i) &= \mathring{b}_i, \end{aligned}$$

where the values of the coefficients $\mathring{a}_{0i}, \mathring{a}_{1i}, \mathring{a}_{2i}, \mathring{a}_{3i}, \mathring{b}_i$ depend on $\mathring{V}_i$ and $\mathring{W}_i$. Due to the homogeneity of agents, identical signals and total symmetry in the interaction pattern, we necessarily have $\mathring{a}_0, \mathring{a}_1, \mathring{a}_2, \mathring{a}_3, \mathring{b}$ such that

$$\mathring{a}_{0i} = \mathring{a}_0, \quad \mathring{a}_{1i} = \mathring{a}_1, \quad \mathring{a}_{2i} = \mathring{a}_2, \quad \mathring{a}_{3i} = \mathring{a}_3, \quad \mathring{b}_i = \mathring{b}; \quad \forall i = 1, ..., n.$$

So given the CARA-Gaussian setup, demands of agents will be given by the following functions:

$$\begin{aligned} z_i(s_i, p, \delta_i) &= \frac{\mathrm{E}[\tilde{X}|s_i, p, \delta_i] - p}{\rho \mathrm{var}(\tilde{X}|s_i, p, \delta_i)} \\ &= \frac{\mathring{a}_0 + \mathring{a}_1 s_i + (\mathring{a}_2 - 1)p + \mathring{a}_3\delta_i}{\rho\mathring{b}}, \qquad i = 1, ..., n. \end{aligned}$$

Now we can derive gaussian informations inferred from demand observations explicitly:

Claim 1 (Inference from Uphill Neighbor's Demand) *For each $i = 1, ..., n$, the gaussian information, which agent i infers (on top of price $\tilde{p}$) from demand of agent $i-1$, is*[21]

$$\tilde{\delta}_i = \frac{\frac{\mathring{a}_1}{\rho \mathring{b}}}{1 - \left(\frac{\mathring{a}_3}{\rho \mathring{b}}\right)^n} \sum_{k=1}^{n} \left(\frac{\mathring{a}_3}{\rho \mathring{b}}\right)^{k-1} \tilde{s}_{i-k}. \tag{9.5}$$

Proof. Following (3.1), the additional information i derives (on top of p) from the observation of agent $i-1$'s demand is explicitly of the form

$$\delta_i = \frac{\mathring{a}_1 s_{i-1} + \mathring{a}_3 \delta_{i-1}}{\rho \mathring{b}}. \tag{9.6}$$

Using the very same relation to substitute for $\{\delta_{i-k}\}_{k=1,...,n}$ in (9.6) above, delivers us

$$\delta_i = \frac{\mathring{a}_1}{\rho \mathring{b}} \sum_{k=1}^{n} \left(\frac{\mathring{a}_3}{\rho \mathring{b}}\right)^{k-1} s_{i-k} + \left(\frac{\mathring{a}_3}{\rho \mathring{b}}\right)^{n} \delta_{i-n} \tag{9.7}$$

Since $i - n \equiv i \pmod{n}$, this equation yields

$$\delta_i = \frac{\frac{\mathring{a}_1}{\rho \mathring{b}}}{1 - \left(\frac{\mathring{a}_3}{\rho \mathring{b}}\right)^n} \sum_{k=1}^{n} \left(\frac{\mathring{a}_3}{\rho \mathring{b}}\right)^{k-1} s_{i-k}.$$

The equation above holds for all realizations $\{s_k\}_{k=1,...,n}$ of $\{\tilde{s}_k\}_{k=1,...,n}$, hence (9.5) is acquired. The fact that $\tilde{\delta}_i$ is normally distributed follows from the linear formulation. □ [*end of claim*]

At this point, also note that weights of each agent in price are exactly equal due to the strict homogeneity in the market, i.e., for all $i = 1, ..., n$ there exists some $\mathring{\pi}$ dependent on demand coefficients $\mathring{a}_0, \mathring{a}_1, \mathring{a}_2, \mathring{a}_3, \mathring{b}$ such that

$$\mathring{\pi}_i = \mathring{\pi}.$$

Actually one can verify that

$$\mathring{\pi} = \mathring{\gamma} \frac{\mathring{a}_1}{\rho \mathring{b}} \left(1 + \frac{\frac{\mathring{a}_3}{\rho \mathring{b}}}{1 - \frac{\mathring{a}_3}{\rho \mathring{b}}}\right), \tag{9.8a}$$

$$\mathring{\gamma} = \frac{\rho \mathring{b}}{n(1 - \mathring{a}_2)}, \tag{9.8b}$$

$$\mathring{\pi}_0 = \mathring{\gamma} \frac{n \mathring{a}_0}{\rho \mathring{b}}, \tag{9.8c}$$

using market clearing condition

$$\sum_{i=1}^{n} z_i(s_i, p, \delta_i) = L,$$

and solving for p.

[21] Once again, consider indices $i - k$ modulo n.

Now we can write down the variance-covariance matrix $\mathring{V}_i$ of $(\tilde{s}_i, \tilde{p}, \tilde{\delta}_i)$:

$$\mathring{V}_i = \mathring{V} = \begin{bmatrix} {\sigma_x}^2+{\sigma_\epsilon}^2 & n\mathring{\pi}\,{\sigma_x}^2+\mathring{\pi}\,{\sigma_\epsilon}^2 & \frac{\frac{\mathring{a}_1}{\rho\mathring{b}}}{1-\frac{\mathring{a}_3}{\rho\mathring{b}}}\,{\sigma_x}^2+\frac{\frac{\mathring{a}_1}{\rho\mathring{b}}\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^{n-1}}{1-\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^{n}}\,{\sigma_\epsilon}^2 \\ n\mathring{\pi}\,{\sigma_x}^2+\mathring{\pi}\,{\sigma_\epsilon}^2 & n^2\mathring{\pi}^2\,{\sigma_x}^2+n\mathring{\pi}^2\,{\sigma_\epsilon}^2+\mathring{\gamma}^2\,{\sigma_L}^2 & n\mathring{\pi}\,\frac{\frac{\mathring{a}_1}{\rho\mathring{b}}}{1-\frac{\mathring{a}_3}{\rho\mathring{b}}}\,{\sigma_x}^2+\mathring{\pi}\,\frac{\frac{\mathring{a}_1}{\rho\mathring{b}}}{1-\frac{\mathring{a}_3}{\rho\mathring{b}}}\,{\sigma_\epsilon}^2 \\ \frac{\frac{\mathring{a}_1}{\rho\mathring{b}}}{1-\frac{\mathring{a}_3}{\rho\mathring{b}}}\,{\sigma_x}^2+\frac{\frac{\mathring{a}_1}{\rho\mathring{b}}\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^{n-1}}{1-\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^{n}}\,{\sigma_\epsilon}^2 & n\mathring{\pi}\,\frac{\frac{\mathring{a}_1}{\rho\mathring{b}}}{1-\frac{\mathring{a}_3}{\rho\mathring{b}}}\,{\sigma_x}^2+\mathring{\pi}\,\frac{\frac{\mathring{a}_1}{\rho\mathring{b}}}{1-\frac{\mathring{a}_3}{\rho\mathring{b}}}\,{\sigma_\epsilon}^2 & \frac{\left(\frac{\mathring{a}_1}{\rho\mathring{b}}\right)^2}{\left(1-\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^2}\,{\sigma_x}^2+\frac{\left(\frac{\mathring{a}_1}{\rho\mathring{b}}\right)^2}{\left(1-\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^{n}\right)^2}\,\frac{1-\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^{2n}}{1-\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^2}\,{\sigma_\epsilon}^2 \end{bmatrix}.$$

On the other hand, the covariance matrix of $\tilde{X}$ and $(\tilde{s}_i, \tilde{p}, \tilde{\delta}_i)$ is of the form

$$\mathring{W}_i \;=\; \mathring{W} \;=\; {\sigma_x}^2 \begin{bmatrix} 1 \\ n\mathring{\pi} \\ \frac{\frac{\mathring{a}_1}{\rho\mathring{b}}}{1-\frac{\mathring{a}_3}{\rho\mathring{b}}} \end{bmatrix}.$$

Normal distribution theory dictates

$$[\mathring{a}_1 \quad \mathring{a}_2 \quad \mathring{a}_3]\,\mathring{V} \;=\; \mathring{W}',$$

which further implies

$$\mathring{a}_1\,({\sigma_x}^2+{\sigma_\epsilon}^2)+\mathring{a}_2\,(n\mathring{\pi}{\sigma_x}^2+\mathring{\pi}{\sigma_\epsilon}^2)+\mathring{a}_3\left(\frac{\frac{\mathring{a}_1}{\rho\mathring{b}}}{1-\frac{\mathring{a}_3}{\rho\mathring{b}}}\,{\sigma_x}^2+\frac{\frac{\mathring{a}_1}{\rho\mathring{b}}\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^{n-1}}{1-\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^{n}}\,{\sigma_\epsilon}^2\right) \;=\; {\sigma_x}^2, \tag{9.9a}$$

$$\mathring{a}_1\,(n\mathring{\pi}{\sigma_x}^2+\mathring{\pi}{\sigma_\epsilon}^2)+\mathring{a}_2\,(n^2\mathring{\pi}^2{\sigma_x}^2+n\mathring{\pi}^2{\sigma_\epsilon}^2+\mathring{\gamma}^2{\sigma_L}^2)+\mathring{a}_3\left(n\mathring{\pi}\,\frac{\frac{\mathring{a}_1}{\rho\mathring{b}}}{1-\frac{\mathring{a}_3}{\rho\mathring{b}}}\,{\sigma_x}^2+\mathring{\pi}\,\frac{\frac{\mathring{a}_1}{\rho\mathring{b}}}{1-\frac{\mathring{a}_3}{\rho\mathring{b}}}\,{\sigma_\epsilon}^2\right) \;=\; n\mathring{\pi}{\sigma_x}^2, \tag{9.9b}$$

$$\begin{aligned}\mathring{a}_1\left(\frac{\frac{\mathring{a}_1}{\rho\mathring{b}}}{1-\frac{\mathring{a}_3}{\rho\mathring{b}}}\,{\sigma_x}^2+\frac{\frac{\mathring{a}_1}{\rho\mathring{b}}\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^{n-1}}{1-\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^{n}}\,{\sigma_\epsilon}^2\right)+\mathring{a}_2\left(n\mathring{\pi}\,\frac{\frac{\mathring{a}_1}{\rho\mathring{b}}}{1-\frac{\mathring{a}_3}{\rho\mathring{b}}}\,{\sigma_x}^2+\mathring{\pi}\,\frac{\frac{\mathring{a}_1}{\rho\mathring{b}}}{1-\frac{\mathring{a}_3}{\rho\mathring{b}}}\,{\sigma_\epsilon}^2\right)+& \\ \mathring{a}_3\left(\frac{\left(\frac{\mathring{a}_1}{\rho\mathring{b}}\right)^2}{\left(1-\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^2}\,{\sigma_x}^2+\frac{\left(\frac{\mathring{a}_1}{\rho\mathring{b}}\right)^2}{\left(1-\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^{n}\right)^2}\,\frac{1-\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^{2n}}{1-\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^2}\,{\sigma_\epsilon}^2\right) &= \frac{\frac{\mathring{a}_1}{\rho\mathring{b}}}{1-\frac{\mathring{a}_3}{\rho\mathring{b}}}{\sigma_x}^2.\end{aligned} \tag{9.9c}$$

Multiplying (9.9a) with $-\frac{\frac{\mathring{a}_1}{\rho\mathring{b}}}{1-\frac{\mathring{a}_3}{\rho\mathring{b}}}$, adding the result to (9.9c), and dividing the derived equation by $\rho\mathring{b}$, one acquires

$$-\left(\frac{\mathring{a}_1}{\rho\mathring{b}}\right)^2\frac{1-\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^{n-1}}{\left(1-\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^{n}\right)\left(1-\frac{\mathring{a}_3}{\rho\mathring{b}}\right)}{\sigma_\epsilon}^2+\frac{\mathring{a}_3}{\rho\mathring{b}}\left(\frac{\mathring{a}_1}{\rho\mathring{b}}\right)^2\left(\frac{1-\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^{n-1}}{\left(1-\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^{n}\right)\left(1-\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^{2}\right)}\right){\sigma_\epsilon}^2 = 0.$$

Simplifying the equation above, we get

$$\left(\frac{\mathring{a}_1}{\rho\mathring{b}}\right)^2\frac{1-\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^{n-1}}{\left(1-\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^{n}\right)\left(1-\left(\frac{\mathring{a}_3}{\rho\mathring{b}}\right)^{2}\right)} = 0.$$

The only solution to this equation is $\frac{\mathring{a}_1}{\rho\mathring{b}} = 0$, which in turn implies

$$\mathring{a}_1 = 0.$$

Then (9.9a) reduces to

$$\mathring{a}_2(n\mathring{\pi}{\sigma_x}^2 + \mathring{\pi}{\sigma_\epsilon}^2) = {\sigma_x}^2, \tag{9.10}$$

while plugging $\mathring{a}_1 = 0$ in (9.8a) results with

$$\mathring{\pi} = 0. \tag{9.11}$$

Eqs. (9.10) and (9.11) together imply

$${\sigma_x}^2 = 0,$$

which violates A2. Thus there cannot exist any linear REESI price. □

THE HIERARCHIC INTERACTION SCHEMES (TREES)

Lemma 3 *Assume A1, A2, S3, A4. Let the social network be dictated by a tree. There exists a linear REESI price within the class of functions of the form*

$$\tilde{p} = P(\tilde{s}_1, ..., \tilde{s}_n; \tilde{L}) = \pi_0^h + \sum_{i=1}^{n} \pi_i^h \, \tilde{s}_i - \gamma^h \, \tilde{L} \quad \textit{with non-zero } \gamma^h \tag{9.12}$$

if and only if the following system of equations has a solution in $\left(\{\pi_i^h\}_{i=1,...,n}, \gamma^h, \pi_0^h\right)$*:*

$$\begin{aligned}
\pi_i^h &= \gamma^h \left(1 + \sum_{m \in \mathcal{H}^{-1}(i)} \prod_{j=0}^{\#(m,i)-1} \theta_{3,m^{-j}}^h\right) \theta_{1i}^h, \qquad i = 1, ..., n \\
\gamma^h &= \left(\sum_{i=1}^{n} \frac{1}{\rho b_i^h} - \theta_{2i}^h\right)^{-1} \neq 0, \\
\pi_0^h &= \gamma^h \sum_{i=1}^{n} \theta_{0i}^h,
\end{aligned}$$

where $\left(\{\theta_{0i}^h, \theta_{1i}^h, \theta_{2i}^h, b_i^h\}_{i=1,...,n}; \{\theta_{3i}^h\}_{i=2,...,n}\right)$ *constitute the equilibrium demand coefficients of*

$$\begin{aligned}
z_1(s_1, p) &= \theta_{01}^h + \theta_{11}^h + \left(\theta_{21}^h - \frac{1}{\rho b_1^h}\right) p, \\
z_i(s_i, p, \delta_{i^-}^i) &= \theta_{0i}^h + \theta_{1i}^h + \left(\theta_{2i}^h - \frac{1}{\rho b_i^h}\right) p + \theta_{3i}^h \delta_{i^-}^i, \qquad i > 1
\end{aligned}$$

such that

$$\begin{aligned}
\theta_{01}^h &= \mu_x \left(\frac{1}{\rho b_1^h} - \theta_{11}^h - \theta_{21}^h \pi^h\right) - \theta_{21}^h \pi_0^h, \\
\theta_{11}^h &= \frac{1}{\rho {\sigma_\epsilon}^2} \frac{\left(\sum_{k=1}^{n} (\pi_k^h)^2 - \pi^h \pi_1^h\right) {\sigma_\epsilon}^2 + (\gamma^h)^2 {\sigma_L}^2}{\sum_{k \neq 1} (\pi_k^h)^2 {\sigma_\epsilon}^2 + (\gamma^h)^2 {\sigma_L}^2}, \\
\theta_{21}^h &= \frac{\pi^h - \pi_1^h}{\rho \left(\sum_{k \neq 1} (\pi_k^h)^2 {\sigma_\epsilon}^2 + (\gamma^h)^2 {\sigma_L}^2\right)}, \\
b_1^h &= \frac{1}{\rho} \left(\frac{1}{\rho {\sigma_x}^2} + \theta_{11}^h + \theta_{21}^h \pi^h\right)^{-1};
\end{aligned}$$

and for $i > 1$

$$\theta_{0i}^h = \mu_x\Big(\tfrac{1}{\rho b_i^h} - \theta_{1i}^h - \theta_{2i}^h \pi^h - \theta_{3i}^h\Big[\theta_{1,i^-}^h + \textstyle\sum_{k=2}^{\#(i,1)}\Big(\prod_{j=1}^{k-1}\theta_{3,i^-j}^h\Big)\theta_{1,i^-k}^h\Big]\Big) - \theta_{2i}^h \pi_0^h,$$

$$\theta_{1i}^h = \tfrac{1}{\rho\beta_i}\Bigg[\Big(\Big(\textstyle\sum_{k=1}^n(\pi_k^h)^2 - \pi^h\pi_i^h\Big){\sigma_\epsilon}^2 + (\gamma^h)^2{\sigma_L}^2\Big)\Big(\Big(\theta_{1,i^-}^h\Big)^2 + \textstyle\sum_{k=2}^{\#(i,1)}\Big(\prod_{j=1}^{k-1}\theta_{3,i^-j}^h\Big)^2\Big(\theta_{1,i^-k}\Big)^2\Big) +$$

$$\Big(\theta_{1,i^-}^h(\pi_i^h - \pi_{i^-}^h) + \textstyle\sum_{k=2}^{\#(i,1)}\Big(\prod_{j=1}^{k-1}\theta_{3,i^-j}^h\Big)\theta_{1,i^-k}^h(\pi_i^h - \pi_{i^-k}^h)\Big)\Big(\theta_{1,i^-}^h\pi_{i^-}^h + \textstyle\sum_{k=2}^{\#(i,1)}\Big(\prod_{j=1}^{k-1}\theta_{3,i^-j}^h\Big)\theta_{1,i^-k}^h\pi_{i^-k}^h\Big){\sigma_\epsilon}^2\Bigg],$$

$$\theta_{2i}^h = \tfrac{1}{\rho\beta_i}\Bigg[(\pi^h - \pi_i^h)\Big(\Big(\theta_{1,i^-}^h\Big)^2 + \textstyle\sum_{k=2}^{\#(i,1)}\Big(\prod_{j=1}^{k-1}\theta_{3,i^-j}^h\Big)^2\Big(\theta_{1,i^-k}^h\Big)^2\Big){\sigma_\epsilon}^2 -$$

$$\Big(\theta_{1,i^-}^h + \textstyle\sum_{k=2}^{\#(i,1)}\Big(\prod_{j=1}^{k-1}\theta_{3,i^-j}^h\Big)\theta_{1,i^-k}^h\Big)\Big(\theta_{1,i^-}^h\pi_{i^-}^h + \textstyle\sum_{k=2}^{\#(i,1)}\Big(\prod_{j=1}^{k-1}\theta_{3,i^-j}^h\Big)\theta_{1,i^-k}^h\pi_{i^-k}^h\Big){\sigma_\epsilon}^2\Bigg],$$

$$\theta_{3i}^h = \tfrac{1}{\rho\beta_i}\Bigg[\Big(\textstyle\sum_{k\neq i}(\pi_k^h)^2{\sigma_\epsilon}^2 + (\gamma^h)^2{\sigma_L}^2\Big)\Big(\theta_{1,i^-}^h + \textstyle\sum_{k=2}^{\#(i,1)}\Big(\prod_{j=1}^{k-1}\theta_{3,i^-j}^h\Big)\theta_{1,i^-k}^h\Big) -$$

$$(\pi^h - \pi_i^h)\Big(\theta_{1,i^-}^h\pi_{i^-}^h + \textstyle\sum_{k=2}^{\#(i,1)}\Big(\prod_{j=1}^{k-1}\theta_{3,i^-j}^h\Big)\theta_{1,i^-k}^h\pi_{i^-k}^h\Big){\sigma_\epsilon}^2\Bigg],$$

$$b_i^h = \tfrac{1}{\rho}\Big(\tfrac{1}{\rho{\sigma_x}^2} + \theta_{1i}^h + \theta_{2i}^h\pi^h + \theta_{3i}^h\Big[\theta_{1,i^-}^h + \textstyle\sum_{k=2}^{\#(i,1)}\Big(\prod_{j=1}^{k-1}\theta_{3,i^-j}^h\Big)\theta_{1,i^-k}^h\Big]\Big)^{-1},$$

$$\beta_i = {\sigma_\epsilon}^2\Bigg[\Big(\textstyle\sum_{k\neq i}(\pi_k^h)^2{\sigma_\epsilon}^2 + (\gamma^h)^2{\sigma_L}^2\Big)\Big(\Big(\theta_{1,i^-}^h\Big)^2 + \textstyle\sum_{k=2}^{\#(i,1)}\Big(\prod_{j=1}^{k-1}\theta_{3,i^-j}^h\Big)^2\Big(\theta_{1,i^-k}^h\Big)^2\Big) -$$

$$\Big(\theta_{1,i^-}^h\pi_{i^-}^h + \textstyle\sum_{k=2}^{\#(i,1)}\Big(\prod_{j=1}^{k-1}\theta_{3,i^-j}^h\Big)\theta_{1,i^-k}^h\pi_{i^-k}^h\Big)^2{\sigma_\epsilon}^2\Bigg],$$

with $\pi^h = \sum_{i=1}^n \pi_i^h$.

Proof: Recall that δ_i is the (realized) gaussian information that agent $i > 1$ infers from demand of agent i^- (i.e, the immediate uphill neighbor preceding her on the tree). Assuming risky security price $\tilde{p}$ is given by (9.12), the tuple $(\tilde{s}_1, \tilde{p})$ and the triples $(\tilde{s}_i, \tilde{p}, \tilde{\delta}_i)_{i=2,..,n}$ are normally distributed. We let V_1^h be the variance-covariance matrix of $(\tilde{s}_1, \tilde{p})$, and W_1^h be the covariance matrix of $\tilde{X}$ and $(\tilde{s}_1, \tilde{p})$. For $i = 2, ..., n$, let V_i^h denote the variance-covariance matrix of $(\tilde{s}_i, \tilde{p}, \tilde{\delta}_i)$, and W_i^h be the covariance matrix of $\tilde{X}$ and $(\tilde{s}_i, \tilde{p}, \tilde{\delta}_i)$.

By normal distribution theory we have

$$\begin{aligned} \mathrm{E}[\tilde{X}|s_1, p] &= a_{01}^h + a_{11}^h s_1 + a_{21}^h p, \\ \mathrm{var}(\tilde{X}|s_1, p) &= b_1^h; \end{aligned}$$

and for $i = 2, ..., n$

$$\begin{aligned} \mathrm{E}[\tilde{X}|s_i, p, \delta_i] &= a_{0i}^h + a_{1i}^h s_i + a_{2i}^h p + a_{3i}^h \delta_i, \\ \mathrm{var}(\tilde{X}|s_i, p, \delta_i) &= b_i^h, \end{aligned}$$

where a_{0i}^h, a_{1i}^h, a_{2i}^h, a_{3i}^h, and b_i^h depend on the variance-covariance matrix V_i^h and W_i^h for all i.

Given the CARA-Gaussian setup, the demands of agents are of the form

$$\begin{aligned} z_1(s_1,p) &= \frac{\mathrm{E}[\tilde{X}|s_1,p]-p}{\rho \mathrm{var}(\tilde{X}|s_1,p)} \\ &= \frac{a_{01}^h + a_{11}^h s_1 + (a_{21}^h - 1)p}{\rho b_1^h}, \end{aligned} \tag{9.13a}$$

and for $i = 2, ..., n$

$$\begin{aligned} z_i(s_i,p,\delta_i) &= \frac{\mathrm{E}[\tilde{X}|s_i,p,\delta_i]-p}{\rho \mathrm{var}(\tilde{X}|s_i,p,\delta_i)} \\ &= \frac{a_{0i}^h + a_{1i}^h s_i + (a_{2i}^h - 1)p + a_{3i}\delta_i}{\rho b_i^h}. \end{aligned} \tag{9.13b}$$

Now we can derive the gaussian informations $\tilde{\delta}_i$ explicitly:

Claim 2 (Inference from Uphill Neighbor's Demand) *For each $i = 2, ..., n$, the gaussian information, inferred from demand of agent i^-, is*[22]

$$\tilde{\delta}_i = \frac{a_{1,i^-}^h}{\rho b_{i^-}^h}\,\tilde{s}_{i^-} + \sum_{k=2}^{\#(i,1)} \left(\prod_{j=1}^{k-1} \frac{a_{3,i^{-j}}^h}{\rho b_{i^{-j}}^h} \right) \frac{a_{1,i^{-k}}^h}{\rho b_{i^{-k}}^h}\,\tilde{s}_{i^{-k}}. \tag{9.14}$$

Proof. Following (3.1), for all i such that $i^- = 1$, the additional information i infers from z_1 (on top of p) is

$$\delta_i = \frac{a_{11}^h}{\rho b_1^h}\,s_1. \tag{9.15a}$$

On the other hand, it also follows from (3.1) that for all i such that $i^- > 1$, the information agent i infers from demand of agent i^- is given by

$$\delta_i = \frac{a_{1,i^-}^h s_{i^-} + a_{3,i^-}^h \delta_{i^-}}{\rho b_{i^-}^h}.$$

Now using the very same equation to substitute for all $\delta_{i^{-k+1}}$ with $2 \le k \le \#(i,1)$, and also using equation (9.15a), one acquires

$$\delta_i = \frac{a_{1,i^-}^h}{\rho b_{i^-}^h}\,s_{i^-} + \sum_{k=2}^{\#(i,1)} \left(\prod_{j=1}^{k-1} \frac{a_{3,i^{-j}}^h}{\rho b_{i^{-j}}^h} \right) \frac{a_{1,i^{-k}}^h}{\rho b_{i^{-k}}^h}\,s_{i^{-k}}. \tag{9.15b}$$

Since (9.15a) and (9.15b) hold for all realizations $\{s_j\}_{j=1,...,n}$ of $\{\tilde{s}_j\}_{j=1,...,n}$, we get the desired result. Normal distribution for the inferred information follows from the linear functional form. □ [*end of claim*]

[22] Recall that $\#(i,1)$ is the integer that satisfies $i^{-\#(i,1)} = 1$. Thus if $\#(i,1) = 1$, then (9.14) simply reduces to

$$\tilde{\delta}_i = \frac{a_{1,i^-}^h}{\rho b_{i^-}^h}\,\tilde{s}_{i^-}.$$

Having this explicit expression for the inferred information $\tilde{\delta}_i$, we can write down the distribution matrices. To ease the notation, we define

$$\pi^h = \sum_{k=1}^{n} \pi_k^h.$$

The variance-covariance matrix of $(\tilde{s}_1, \tilde{p})$ is

$$V_1^h = \begin{bmatrix} {\sigma_x}^2 + {\sigma_\epsilon}^2 & \pi^h {\sigma_x}^2 + \pi_1^h {\sigma_\epsilon}^2 \\ \pi^h {\sigma_x}^2 + \pi_1^h {\sigma_\epsilon}^2 & (\pi^h)^2 {\sigma_x}^2 + \sum_{k=1}^{n} (\pi_k^h)^2 {\sigma_k}^2 + (\gamma^h)^2 {\sigma_L}^2 \end{bmatrix},$$

and the covariance matrix of $\tilde{X}$ and $(\tilde{s}_1, \tilde{p})$ is

$$W_1^h = {\sigma_x}^2 \begin{bmatrix} 1 \\ \pi^h \end{bmatrix}.$$

For $i > 1$, the variance-covariance matrix of $(\tilde{s}_i, \tilde{p}, \tilde{\delta}_i)$ is

$$V_i^h = \begin{bmatrix} {\sigma_x}^2 + {\sigma_\epsilon}^2 & \pi^h {\sigma_x}^2 + \pi_i^h {\sigma_\epsilon}^2 & \mathrm{cov}(\tilde{s}_i, \tilde{\delta}_{i-}^i) \\ \pi^h {\sigma_x}^2 + \pi_i^h {\sigma_\epsilon}^2 & (\pi^h)^2 {\sigma_x}^2 + \sum_{k=1}^{n} (\pi_k^h)^2 {\sigma_k}^2 + (\gamma^h)^2 {\sigma_L}^2 & \mathrm{cov}(\tilde{p}, \tilde{\delta}_{i-}^i) \\ \mathrm{cov}(\tilde{s}_i, \tilde{\delta}_{i-}^i) & \mathrm{cov}(\tilde{p}, \tilde{\delta}_{i-}^i) & \mathrm{var}(\tilde{\delta}_{i-}^i) \end{bmatrix},$$

where

$$\begin{aligned}
\mathrm{cov}(\tilde{s}_i, \tilde{\delta}_i) &= \left(\frac{a_{1,i-}^h}{\rho b_{i-}^h} + \sum_{k=2}^{\#(i,1)} \left(\prod_{j=1}^{k-1} \frac{a_{3,i-j}^h}{\rho b_{i-j}^h} \right) \frac{a_{1,i-k}^h}{\rho b_{i-k}^h} \right) {\sigma_x}^2, \\
\mathrm{cov}(\tilde{p}, \tilde{\delta}_i) &= \left(\frac{a_{1,i-}^h}{\rho b_{i-}^h} + \sum_{k=2}^{\#(i,1)} \left(\prod_{j=1}^{k-1} \frac{a_{3,i-j}^h}{\rho b_{i-j}^h} \right) \frac{a_{1,i-k}^h}{\rho b_{i-k}^h} \right) \pi^h {\sigma_x}^2 + \left(\frac{a_{1,i-}^h}{\rho b_{i-}^h} \pi_{i-}^h + \sum_{k=2}^{\#(i,1)} \left(\prod_{j=1}^{k-1} \frac{a_{3,i-j}^h}{\rho b_{i-j}^h} \right) \frac{a_{1,i-k}^h}{\rho b_{i-k}^h} \pi_{i-k}^h \right) {\sigma_\epsilon}^2, \\
\mathrm{var}(\tilde{\delta}_i) &= \left(\frac{a_{1,i-}^h}{\rho b_{i-}^h} + \sum_{k=2}^{\#(i,1)} \left(\prod_{j=1}^{k-1} \frac{a_{3,i-j}^h}{\rho b_{i-j}^h} \right) \frac{a_{1,i-k}^h}{\rho b_{i-k}^h} \right)^2 {\sigma_x}^2 + \left(\left(\frac{a_{1,i-}^h}{\rho b_{i-}^h} \right)^2 + \sum_{k=2}^{\#(i,1)} \left(\prod_{j=1}^{k-1} \frac{a_{3,i-j}^h}{\rho b_{i-j}^h} \right)^2 \left(\frac{a_{1,i-k}^h}{\rho b_{i-k}^h} \right)^2 \right) {\sigma_\epsilon}^2,
\end{aligned}$$

and the covariance matrix of $\tilde{X}$ and $(\tilde{s}_i, \tilde{p}, \tilde{\delta}_i)$ is

$$W_i^h = {\sigma_x}^2 \begin{bmatrix} 1 \\ \pi^h \\ \frac{a_{1,i-}^h}{\rho b_{i-}^h} + \sum_{k=2}^{\#(i,1)} \left(\prod_{j=1}^{k-1} \frac{a_{3,i-j}^h}{\rho b_{i-j}^h} \right) \frac{a_{1,i-k}^h}{\rho b_{i-k}^h} \end{bmatrix}.$$

Normal distribution theory dictates

$$\begin{aligned}
\begin{bmatrix} a_{11}^h & a_{21}^h \end{bmatrix} &= W_1^{h\prime} (V_1^h)^{-1}, && \text{(9.16a)} \\
b_1^h &= {\sigma_x}^2 - W_1^{h\prime} (V_1^h)^{-1} W_1^h, && \text{(9.16b)} \\
a_{01}^h &= \mu_x - W_1^{h\prime} (V_1^h)^{-1} \begin{bmatrix} \mu_x \\ \pi_0^h + \pi^h \mu_x \end{bmatrix}, && \text{(9.16c)}
\end{aligned}$$

and for all $i > 1$

$$\begin{aligned}
\begin{bmatrix} a_{1i}^h & a_{2i}^h & a_{3i}^h \end{bmatrix} &= W_i^{h\prime}(V_i^h)^{-1}, && (9.16d)\\
b_i^h &= {\sigma_x}^2 - W_i^{h\prime}(V_i^h)^{-1}W_i^h, && (9.16e)\\
a_{0i}^h &= \mu_x - W_i^{h\prime}(V_i^h)^{-1}\begin{bmatrix} \mu_x \\ \pi_0^h + \pi^h\mu_x \\ \left(\frac{a_{1,i-}^h}{\rho b_{i-}^h} + \sum_{k=2}^{\#(i,1)}\left(\prod_{j=1}^{k-1}\frac{a_{3,i-j}^h}{\rho b_{i-j}^h}\right)\frac{a_{1,i-k}^h}{\rho b_{i-k}^h}\right)\mu_x \end{bmatrix}. && (9.16f)
\end{aligned}$$

After elementary but somewhat tedious manipulations, one acquires the following result:

$$\begin{aligned}
\frac{a_{11}^h}{\rho b_1^h} &= \frac{1}{\rho{\sigma_\epsilon}^2}\frac{\left(\sum_{k=1}^n(\pi_k^h)^2 - \pi^h\pi_1^h\right){\sigma_\epsilon}^2 + (\gamma^h)^2{\sigma_L}^2}{\sum_{k\neq 1}(\pi_k^h)^2{\sigma_\epsilon}^2 + (\gamma^h)^2{\sigma_L}^2},\\
\frac{a_{21}^h}{\rho b_1^h} &= \frac{\pi^h - \pi_1^h}{\rho\left(\sum_{k\neq 1}(\pi_k^h)^2{\sigma_\epsilon}^2 + (\gamma^h)^2{\sigma_L}^2\right)},
\end{aligned}$$

and for $i > 1$

$$\begin{aligned}
\frac{a_{1i}^h}{\rho b_i^h} &= \frac{1}{\rho\beta_i}\Bigg[\left(\left(\sum_{k=1}^n(\pi_k^h)^2 - \pi^h\pi_i^h\right){\sigma_\epsilon}^2 + (\gamma^h)^2{\sigma_L}^2\right)\left(\left(\frac{a_{1,i-}^h}{\rho b_{i-}^h}\right)^2 + \sum_{k=2}^{\#(i,1)}\left(\prod_{j=1}^{k-1}\frac{a_{3,i-j}^h}{\rho b_{i-j}^h}\right)^2\left(\frac{a_{1,i-k}^h}{\rho b_{i-k}^h}\right)^2\right) +\\
&\quad \left(\frac{a_{1,i-}^h}{\rho b_{i-}^h}(\pi_i^h - \pi_{i-}^h) + \sum_{k=2}^{\#(i,1)}\left(\prod_{j=1}^{k-1}\frac{a_{3,i-j}^h}{\rho b_{i-j}^h}\right)\frac{a_{1,i-k}^h}{\rho b_{i-k}^h}(\pi_i^h - \pi_{i-k}^h)\right)\times\\
&\quad \left(\frac{a_{1,i-}^h}{\rho b_{i-}^h}\pi_{i-}^h + \sum_{k=2}^{\#(i,1)}\left(\prod_{j=1}^{k-1}\frac{a_{3,i-j}^h}{\rho b_{i-j}^h}\right)\frac{a_{1,i-k}^h}{\rho b_{i-k}^h}\pi_{i-k}^h\right){\sigma_\epsilon}^2\Bigg],\\
\frac{a_{2i}^h}{\rho b_i^h} &= \frac{1}{\rho\beta_i}\Bigg[(\pi^h - \pi_i^h)\left(\left(\frac{a_{1,i-}^h}{\rho b_{i-}^h}\right)^2 + \sum_{k=2}^{\#(i,1)}\left(\prod_{j=1}^{k-1}\frac{a_{3,i-j}^h}{\rho b_{i-j}^h}\right)^2\left(\frac{a_{1,i-k}^h}{\rho b_{i-k}^h}\right)^2\right){\sigma_\epsilon}^2 -\\
&\quad \left(\frac{a_{1,i-}^h}{\rho b_{i-}^h} + \sum_{k=2}^{\#(i,1)}\left(\prod_{j=1}^{k-1}\frac{a_{3,i-j}^h}{\rho b_{i-j}^h}\right)\frac{a_{1,i-k}^h}{\rho b_{i-k}^h}\right)\left(\frac{a_{1,i-}^h}{\rho b_{i-}^h}\pi_{i-}^h + \sum_{k=2}^{\#(i,1)}\left(\prod_{j=1}^{k-1}\frac{a_{3,i-j}^h}{\rho b_{i-j}^h}\right)\frac{a_{1,i-k}^h}{\rho b_{i-k}^h}\pi_{i-k}^h\right){\sigma_\epsilon}^2\Bigg],\\
\frac{a_{3i}^h}{\rho b_i^h} &= \frac{1}{\rho\beta_i}\Bigg[\left(\sum_{k\neq i}(\pi_k^h)^2{\sigma_\epsilon}^2 + (\gamma^h)^2{\sigma_L}^2\right)\left(\frac{a_{1,i-}^h}{\rho b_{i-}^h} + \sum_{k=2}^{\#(i,1)}\left(\prod_{j=1}^{k-1}\frac{a_{3,i-j}^h}{\rho b_{i-j}^h}\right)\frac{a_{1,i-k}^h}{\rho b_{i-k}^h}\right) -\\
&\quad (\pi^h - \pi_i^h)\left(\frac{a_{1,i-}^h}{\rho b_{i-}^h}\pi_{i-}^h + \sum_{k=2}^{\#(i,1)}\left(\prod_{j=1}^{k-1}\frac{a_{3,i-j}^h}{\rho b_{i-j}^h}\right)\frac{a_{1,i-k}^h}{\rho b_{i-k}^h}\pi_{i-k}^h\right){\sigma_\epsilon}^2\Bigg];\\
\beta_i &= {\sigma_\epsilon}^2\Bigg[\left(\sum_{k\neq i}(\pi_k^h)^2{\sigma_\epsilon}^2 + (\gamma^h)^2{\sigma_L}^2\right)\left(\left(\frac{a_{1,i-}^h}{\rho b_{i-}^h}\right)^2 + \sum_{k=2}^{\#(i,1)}\left(\prod_{j=1}^{k-1}\frac{a_{3,i-j}^h}{\rho b_{i-j}^h}\right)^2\left(\frac{a_{1,i-k}^h}{\rho b_{i-k}^h}\right)^2\right) -\\
&\quad \left(\frac{a_{1,i-}^h}{\rho b_{i-}^h}\pi_{i-}^h + \sum_{k=2}^{\#(i,1)}\left(\prod_{j=1}^{k-1}\frac{a_{3,i-j}^h}{\rho b_{i-j}^h}\right)\frac{a_{1,i-k}^h}{\rho b_{i-k}^h}\pi_{i-k}^h\right)^2{\sigma_\epsilon}^2\Bigg].
\end{aligned}$$

Now define

$$\theta_{01}^h \equiv \frac{a_{01}^h}{\rho b_1^h} \qquad \theta_{11}^h \equiv \frac{a_{11}^h}{\rho b_1^h}, \qquad \theta_{21}^h \equiv \frac{a_{21}^h}{\rho b_1^h},$$

and for $i > 1$

$$\theta_{0i}^h \equiv \frac{a_{0i}^h}{\rho b_i^h} \qquad \theta_{1i}^h \equiv \frac{a_{1i}^h}{\rho b_i^h}, \qquad \theta_{2i}^h \equiv \frac{a_{2i}^h}{\rho b_i^h}, \qquad \theta_{3i}^h \equiv \frac{a_{3i}^h}{\rho b_i^h}.$$

We can rewrite the equations above as follows

$$\theta_{11}^h = \frac{1}{\rho {\sigma_\epsilon}^2} \frac{\left(\sum_{k=1}^n (\pi_k^h)^2 - \pi^h \pi_1^h\right) {\sigma_\epsilon}^2 + (\gamma^h)^2 {\sigma_L}^2}{\sum_{k\neq 1} (\pi_k^h)^2 {\sigma_\epsilon}^2 + (\gamma^h)^2 {\sigma_L}^2}, \tag{9.18a}$$

$$\theta_{21}^h = \frac{\pi^h - \pi_1^h}{\rho\left(\sum_{k\neq 1} (\pi_k^h)^2 {\sigma_\epsilon}^2 + (\gamma^h)^2 {\sigma_L}^2\right)}, \tag{9.18b}$$

$$\begin{aligned}\theta_{1i}^h = \frac{1}{\rho\beta_i} \Bigg[& \left(\left(\sum_{k=1}^n (\pi_k^h)^2 - \pi^h \pi_i^h\right) {\sigma_\epsilon}^2 + (\gamma^h)^2 {\sigma_L}^2\right)\left(\left(\theta_{1,i^-}^h\right)^2 + \sum_{k=2}^{\#(i,1)} \left(\prod_{j=1}^{k-1} \theta_{3,i^-j}^h\right)^2 \left(\theta_{1,i^-k}^h\right)^2\right) + \\ & \left(\theta_{1,i^-}^h (\pi_i^h - \pi_{i^-}^h) + \sum_{k=2}^{\#(i,1)} \left(\prod_{j=1}^{k-1} \theta_{3,i^-j}^h\right) \theta_{1,i^-k}^h (\pi_i^h - \pi_{i^-k}^h)\right) \times \\ & \left(\theta_{1,i^-}^h \pi_{i^-}^h + \sum_{k=2}^{\#(i,1)} \left(\prod_{j=1}^{k-1} \theta_{3,i^-j}^h\right) \theta_{1,i^-k}^h \pi_{i^-k}^h\right) {\sigma_\epsilon}^2 \Bigg],\end{aligned} \tag{9.18c}$$

$$\begin{aligned}\theta_{2i}^h = \frac{1}{\rho\beta_i} \Bigg[& (\pi^h - \pi_i^h)\left(\left(\theta_{1,i^-}^h\right)^2 + \sum_{k=2}^{\#(i,1)} \left(\prod_{j=1}^{k-1} \theta_{3,i^-j}^h\right)^2 \left(\theta_{1,i^-k}^h\right)^2\right) {\sigma_\epsilon}^2 - \\ & \left(\theta_{1,i^-}^h + \sum_{k=2}^{\#(i,1)} \left(\prod_{j=1}^{k-1} \theta_{3,i^-j}^h\right) \theta_{1,i^-k}^h\right)\left(\theta_{1,i^-}^h \pi_{i^-}^h + \sum_{k=2}^{\#(i,1)} \left(\prod_{j=1}^{k-1} \theta_{3,i^-j}^h\right) \theta_{1,i^-k}^h \pi_{i^-k}^h\right) {\sigma_\epsilon}^2 \Bigg],\end{aligned} \tag{9.18d}$$

$$\begin{aligned}\theta_{3,i}^h = \frac{1}{\rho\beta_i} \Bigg[& \left(\sum_{k\neq i} (\pi_k^h)^2 {\sigma_\epsilon}^2 + (\gamma^h)^2 {\sigma_L}^2\right)\left(\theta_{1,i^-}^h + \sum_{k=2}^{\#(i,1)} \left(\prod_{j=1}^{k-1} \theta_{3,i^-j}^h\right) \theta_{1,i^-k}^h\right) - \\ & (\pi^h - \pi_i^h)\left(\theta_{1,i^-}^h \pi_{i^-}^h + \sum_{k=2}^{\#(i,1)} \left(\prod_{j=1}^{k-1} \theta_{3,i^-j}^h\right) \theta_{1,i^-k}^h \pi_{i^-k}^h\right) {\sigma_\epsilon}^2 \Bigg];\end{aligned} \tag{9.18e}$$

$$\begin{aligned}\beta_i = {\sigma_\epsilon}^2 \Bigg[& \left(\sum_{k\neq i} (\pi_k^h)^2 {\sigma_\epsilon}^2 + (\gamma^h)^2 {\sigma_L}^2\right)\left(\left(\theta_{1,i^-}^h\right)^2 + \sum_{k=2}^{\#(i,1)} \left(\prod_{j=1}^{k-1} \theta_{3,i^-j}^h\right)^2 \left(\theta_{1,i^-k}^h\right)^2\right) - \\ & \left(\theta_{1,i^-}^h \pi_{i^-}^h + \sum_{k=2}^{\#(i,1)} \left(\prod_{j=1}^{k-1} \theta_{3,i^-j}^h\right) \theta_{1,i^-k}^h \pi_{i^-k}^h\right)^2 {\sigma_\epsilon}^2 \Bigg].\end{aligned}$$

Moreover, from (9.16b) and (9.16e) we have

$$\begin{aligned} b_1^h &= {\sigma_x}^2 \left(1 - a_{11}^h - a_{21}^h \pi^h\right), \\ b_i^h &= {\sigma_x}^2 \left(1 - a_{1i}^h - a_{2i}^h \pi^h - a_{3i}^h \left[\frac{a_{1,i^-}^h}{\rho b_{i^-}^h} + \sum_{k=2}^{\#(i,1)} \left(\prod_{j=1}^{k-1} \frac{a_{3,i^-j}^h}{\rho b_{i^-j}^h}\right) \frac{a_{1,i^-k}^h}{\rho b_{i^-k}^h}\right]\right), \qquad i > 1. \end{aligned}$$

Following these equations, one can easily acquire

$$b_1^h = \frac{1}{\rho}\left(\frac{1}{\rho {\sigma_x}^2} + \theta_{11}^h + \theta_{21}^h \pi^h\right)^{-1}, \tag{9.19a}$$

$$b_i^h = \frac{1}{\rho}\left(\frac{1}{\rho {\sigma_x}^2} + \theta_{1i}^h + \theta_{2i}^h \pi^h + \theta_{3i}^h \left[\theta_{1,i^-}^h + \sum_{k=2}^{\#(i,1)} \left(\prod_{j=1}^{k-1} \theta_{3,i^-j}^h\right) \theta_{1,i^-k}^h\right]\right)^{-1}, \quad i > 1. \tag{9.19b}$$

We also have from (9.16c) and (9.16f)

$$\begin{aligned} a_{01}^h &= \mu_x - a_{11}^h \mu_x - a_{21}^h \left(\pi_0^h + \pi^h \mu_x\right), \\ a_{0i}^h &= \mu_x - a_{1i}^h \mu_x - a_{2i}^h \left(\pi_0^h + \pi^h \mu_x\right) - a_{3i}^h \left(\frac{a_{1,i^-}^h}{\rho b_{i^-}^h} + \sum_{k=2}^{\#(i,1)} \left(\prod_{j=1}^{k-1} \frac{a_{3,i^-j}^h}{\rho b_{i^-j}^h}\right) \frac{a_{1,i^-k}^h}{\rho b_{i^-k}^h}\right) \mu_x, \qquad i > 1. \end{aligned}$$

Hence the following holds

$$\theta_{01}^h = \mu_x \left(\frac{1}{\rho b_1^h} - \theta_{11}^h - \theta_{21}^h \pi^h \right) - \theta_{21}^h \pi_0^h, \tag{9.20a}$$

$$\theta_{0i}^h = \mu_x \left(\frac{1}{\rho b_i^h} - \theta_{1i}^h - \theta_{2i}^h \pi^h - \theta_{3i}^h \left[\theta_{1,i^-}^h + \sum_{k=2}^{\#(i,1)} \left(\prod_{j=1}^{k-1} \theta_{3,i^{-j}}^h \right) \theta_{1,i^{-k}}^h \right] \right) - \theta_{2i}^h \pi_0^h. \tag{9.20b}$$

On the other hand, we have the market clearing condition given by

$$z_1(s_1, p) + \sum_{i>1} z_i(s_i, p, \delta_{i^-}^i) = L.$$

Substituting from (9.13a)-(9.13b) and solving for p, we get

$$p = \left(\sum_{i=1}^n \frac{1 - a_{2i}^h}{\rho b_i^h} \right)^{-1} \left(\sum_{i=1}^n \frac{a_{0i}^h}{\rho b_i^h} + \sum_{i=1}^n \frac{a_{1i}^h}{\rho b_i^h} s_i + \sum_{i>1} \frac{a_{3i}^h}{\rho b_i^h} \delta_{i^-}^i - L \right).$$

A further substitution from (9.14) results with

$$p = \left(\sum_{i=1}^n \frac{1 - a_{2i}^h}{\rho b_i^h} \right)^{-1} \left(\sum_{i=1}^n \frac{a_{0i}^h}{\rho b_i^h} + \sum_{i=1}^n \frac{a_{1i}^h}{\rho b_i^h} s_i + \sum_{i>1} \sum_{k=1}^{\#(i,1)} \left(\prod_{j=0}^{k-1} \frac{a_{3,i^{-j}}^h}{\rho b_{i^{-j}}^h} \right) \frac{a_{1,i^{-k}}^h}{\rho b_{i^{-k}}^h} s_{i^{-k}} - L \right).$$

Rearranging terms and substituting $\{\theta_{j,i}^h\}_{j=1,2,3}$ for $\{\frac{a_{j,i}^h}{\rho b_i^h}\}_{j=1,2,3}$ yields

$$p = \left(\sum_{i=1}^n \frac{1}{\rho b_i^h} - \theta_{2i}^h \right)^{-1} \left(\sum_{i=1}^n \theta_{0i}^h + \sum_{i=1}^n \left(1 + \sum_{m \in H^{-1}(i)} \prod_{j=0}^{\#(m,i)-1} \theta_{3,m^{-j}}^h \right) \theta_{1i}^h s_i - L \right). \tag{9.21}$$

We observe that the market clearing price induced by hypothesis (9.12) is again linear in private signals and liquidity supply. Expectations formed relying on (9.12) would be rational if and only if the coefficients π_0^h, $\{\pi_i^h\}_{i=1,...,n}$, γ^h in (9.12) are the same as the corresponding coefficients in (9.21). Therefore the following conditions hold:

$$\pi_i^h = \gamma^h \left(1 + \sum_{m \in H^{-1}(i)} \prod_{j=0}^{\#(m,i)-1} \theta_{3,m^{-j}}^h \right) \theta_{1i}^h, \qquad i = 1, ..., n \tag{9.22a}$$

$$\gamma^h = \left(\sum_{i=1}^n \frac{1}{\rho b_i^h} - \theta_{2i}^h \right)^{-1}, \tag{9.22b}$$

$$\pi_0^h = \gamma^h \sum_{i=1}^n \theta_{0i}^h. \tag{9.22c}$$

Recall that θ_{0i}^h, θ_{1i}^h, θ_{2i}^h, θ_{3i}^h, and b_i^h depend on the price coefficients $\pi_0^h, \pi_1^h, ..., \pi_n^h, \gamma^h$ (see Eqns. (9.18a)-(9.20b)). The existence of a linear REESI price as given by (9.12) is equivalent to the existence of a solution to the system of equations (9.22a)-(9.22c) and (9.18a)-(9.20b) in arguments $(\pi_0^h, \pi_1^h, ..., \pi_n^h, \gamma^h)$ with $\gamma^h \neq 0$, which essentially presents a *fixed point problem* as given in the statement of this lemma. □

Proof of Theorem 1: Suppose price $\tilde{p}$ is given by (9.12). Let $\pi_0^h, \pi_1^h, ..., \pi_n^h, \gamma^h$ be the coefficients of our hypothetical price $\tilde{p}$. To establish that there exists a linear REESI price in a hierarchic interaction scheme, it suffices to show that there exists a solution to the fixed point problem given in Lemma 3.

Define

$$\begin{aligned} Q_i^h &\equiv \frac{\pi_i^h}{\gamma^h}, \qquad i = 1, 2, ..., n, \\ Q^h &\equiv \frac{\pi^h}{\gamma^h}, \end{aligned}$$

where $\pi^h = \sum_{i=1}^n \pi_i^h$. Also let

$$\Theta_{11}^h = \frac{1}{\rho {\sigma_\epsilon}^2} \frac{\left(\sum_{k=1}^n (Q_k^h)^2 - Q^h Q_1^h\right) {\sigma_\epsilon}^2 + {\sigma_L}^2}{\sum_{k\neq 1} (Q_k^h)^2 {\sigma_\epsilon}^2 + {\sigma_L}^2}, \tag{9.23a}$$

$$\Theta_{21}^h = \frac{Q^h - Q_1^h}{\rho \left(\sum_{k\neq 1} (Q_k^h)^2 {\sigma_\epsilon}^2 + {\sigma_L}^2\right)}, \tag{9.23b}$$

and for $i = 2, ..., n$

$$\begin{aligned} \Theta_{1i}^h = \tfrac{1}{\rho B_i} \Bigg[& \left(\left(\sum_{k=1}^n (Q_k^h)^2 - Q^h Q_i^h\right) {\sigma_\epsilon}^2 + {\sigma_L}^2\right) \left(\left(\Theta_{1,i-}^h\right)^2 + \sum_{k=2}^{\#(i,1)} \left(\prod_{j=1}^{k-1} \Theta_{3,i-j}^h\right)^2 \left(\Theta_{1,i-k}\right)^2 \right) + \\ & \left(\Theta_{1,i-}^h (Q_i^h - Q_{i-}^h) + \sum_{k=2}^{\#(i,1)} \left(\prod_{j=1}^{k-1} \Theta_{3,i-j}^h\right) \Theta_{1,i-k}^h (Q_i^h - Q_{i-k}^h)\right) \times \\ & \left(\Theta_{1,i-}^h Q_{i-}^h + \sum_{k=2}^{\#(i,1)} \left(\prod_{j=1}^{k-1} \Theta_{3,i-j}^h\right) \Theta_{1,i-k}^h Q_{i-k}^h\right) {\sigma_\epsilon}^2 \Bigg], \end{aligned} \tag{9.23c}$$

$$\begin{aligned} \Theta_{2i}^h = \tfrac{1}{\rho B_i} \Bigg[& (Q^h - Q_i^h) \left(\left(\Theta_{1,i-}^h\right)^2 + \sum_{k=2}^{\#(i,1)} \left(\prod_{j=1}^{k-1} \Theta_{3,i-j}^h\right)^2 \left(\Theta_{1,i-k}^h\right)^2 \right) {\sigma_\epsilon}^2 - \\ & \left(\Theta_{1,i-}^h + \sum_{k=2}^{\#(i,1)} \left(\prod_{j=1}^{k-1} \Theta_{3,i-j}^h\right) \Theta_{1,i-k}^h\right) \left(\Theta_{1,i-}^h Q_{i-}^h + \sum_{k=2}^{\#(i,1)} \left(\prod_{j=1}^{k-1} \Theta_{3,i-j}^h\right) \Theta_{1,i-k}^h Q_{i-k}^h\right) {\sigma_\epsilon}^2 \Bigg], \end{aligned} \tag{9.23d}$$

$$\begin{aligned} \Theta_{3i}^h = \tfrac{1}{\rho B_i} \Bigg[& \left(\sum_{k\neq i} (Q_k^h)^2 {\sigma_\epsilon}^2 + {\sigma_L}^2\right) \left(\Theta_{1,i-}^h + \sum_{k=2}^{\#(i,1)} \left(\prod_{j=1}^{k-1} \Theta_{3,i-j}^h\right) \Theta_{1,i-k}^h\right) - \\ & (Q^h - Q_i^h) \left(\Theta_{1,i-}^h Q_{i-}^h + \sum_{k=2}^{\#(i,1)} \left(\prod_{j=1}^{k-1} \Theta_{3,i-j}^h\right) \Theta_{1,i-k}^h Q_{i-k}^h\right) {\sigma_\epsilon}^2 \Bigg]; \end{aligned} \tag{9.23e}$$

$$\begin{aligned} B_i = {\sigma_\epsilon}^2 \Bigg[& \left(\sum_{k\neq i} (Q_k^h)^2 {\sigma_\epsilon}^2 + {\sigma_L}^2\right) \left(\left(\Theta_{1,i-}^h\right)^2 + \sum_{k=2}^{\#(i,1)} \left(\prod_{j=1}^{k-1} \Theta_{3,i-j}^h\right)^2 \left(\Theta_{1,i-k}^h\right)^2 \right) - \\ & \left(\Theta_{1,i-}^h Q_{i-}^h + \sum_{k=2}^{\#(i,1)} \left(\prod_{j=1}^{k-1} \Theta_{3,i-j}^h\right) \Theta_{1,i-k}^h Q_{i-k}^h\right)^2 {\sigma_\epsilon}^2 \Bigg]. \end{aligned} \tag{9.23f}$$

Notice that

$$\begin{aligned} \Theta_{1i}^h &= \theta_{1i}^h, \\ \Theta_{3i}^h &= \theta_{3i}^h; \\ \Theta_{2i}^h &= \theta_{2i}^h \gamma^h, \qquad i = 1, ..., n, \end{aligned}$$

where θ_{ji}, $j = 1, 2, 3$, $i = 1, 2, ..., n$, are given as in Lemma 3.

From Lemma 3, one further has

$$\frac{1}{\rho b_1^h} = \frac{1}{\rho {\sigma_x}^2} + \Theta_{11}^h + \Theta_{21}^h Q^h, \tag{9.24a}$$

$$\frac{1}{\rho b_i^h} = \frac{1}{\rho {\sigma_x}^2} + \Theta_{1i}^h + \Theta_{2i}^h Q^h + \Theta_{3i}^h \left(\Theta_{1,i^-}^h + \sum_{k=2}^{\#(i,1)} (\prod_{j=1}^{k-1} \Theta_{3,i^{-j}}^h) \Theta_{1,i^{-k}}^h \right), \qquad i = 2, ..., n; \tag{9.24b}$$

$$\theta_{01}^h = \frac{\mu_x}{\rho {\sigma_x}^2} - \Theta_{21}^h \frac{\pi_0^h}{\gamma^h}, \tag{9.24c}$$

$$\theta_{0i}^h = \frac{\mu_x}{\rho {\sigma_x}^2} - \Theta_{2i}^h \frac{\pi_0^h}{\gamma^h}, \qquad i = 2, ..., n. \tag{9.24d}$$

And also:

$$Q_i^h = \left(1 + \sum_{m \in \mathcal{H}^{-1}(i)} \prod_{j=0}^{\#(m,i)-1} \Theta_{3,m^{-j}}^h \right) \Theta_{1i}^h, \qquad i = 1, ..., n \tag{9.25a}$$

$$\gamma^h = \left(\sum_{i=1}^n \frac{1}{\rho b_i^h} - \frac{\Theta_{2i}^h}{\gamma^h} \right)^{-1}, \tag{9.25b}$$

$$\pi_0^h = \gamma^h \sum_{i=1}^n \theta_{0i}^h, \tag{9.25c}$$

Since Θ_{ji}^h, $j = 1, 3$, $i = 1, ..., n$, only depend on $\{Q_i^h\}_{i=1,...,n}$, equation (9.25a) can be analyzed independently from equations (9.25b)-(9.25c). Essentially (9.25a) is a fixed point problem in the arguments $(Q_1^h, ..., Q_n^h)$. To show the existence of a solution (i.e., a fixed point) for this problem, we proceed as follows: let

$$\Omega^h \equiv \left[0, \frac{2}{\rho {\sigma_\epsilon}^2} (1 + n2^n) \right]^n \in \mathbb{R}_+^n$$

and define $\mathcal{F} : \Omega^h \to \mathbb{R}^n$, $\mathcal{G} : \Omega^h \to \Omega^h$ by the conditions

$$\left(\mathcal{F}(Q_1^h, ..., Q_n^h) \right)_i = \left(1 + \sum_{m \in \mathcal{H}^{-1}(i)} \prod_{j=0}^{\#(m,i)-1} \Theta_{3,m^{-j}}^h \right) \Theta_{1i}^h, \qquad i = 1, ..., n;$$

and

$$\begin{aligned}
&\left(\mathcal{G}(Q_1^h, ..., Q_n^h) \right)_i = 0 && \text{if} \quad \left(\mathcal{F}(Q_1^h, ..., Q_n^h) \right)_i < 0, \\
&\left(\mathcal{G}(Q_1^h, ..., Q_n^h) \right)_i = \left(\mathcal{F}(Q_1^h, ..., Q_n^h) \right)_i && \text{if} \quad 0 \le \left(\mathcal{F}(Q_1^h, ..., Q_n^h) \right)_i \le \tfrac{2}{\rho {\sigma_\epsilon}^2} (1 + n2^n), \\
&\left(\mathcal{G}(Q_1^h, ..., Q_n^h) \right)_i = \tfrac{2}{\rho {\sigma_\epsilon}^2} (1 + n2^n) && \text{if} \quad \left(\mathcal{F}(Q_1^h, ..., Q_n^h) \right)_i > \tfrac{2}{\rho {\sigma_\epsilon}^2} (1 + n2^n), \qquad i = 1, ..., n.
\end{aligned}$$

Due to assumptions $S1$ and $S2$, Ω^h is compact and $\mathcal{G}$ is continuous. Thus we can employ *Brouwer's Theorem*, which implies $\mathcal{G}$ has a fixed point

$$(\hat{Q}_1^h, ..., \hat{Q}_n^h) \in \Omega^h.$$

Now we would like to show

$$0 < \hat{Q}_i^h < \frac{2}{\rho {\sigma_\epsilon}^2} (1 + n2^n), \qquad i = 1, ..., n$$

so that we end up with

$$(\hat{Q}_1^h, ..., \hat{Q}_n^h) = \mathcal{F}(\hat{Q}_1^h, ..., \hat{Q}_n^h).$$

We will be able to show this *given sufficiently high level of liquidity variance* ${\sigma_L}^2$ *in price*. However, before proceeding with our arguments, let us introduce the following notation:

$$\hat{\Theta}_{ji} = \Theta_{ji}\Big|_{(Q_1^h,...,Q_n^h)=(\hat{Q}_1^h,...,\hat{Q}_n^h)}, \qquad j = 1, 2, 3, \quad i = 1, ..., n.$$

Claim 3 *There exists* $\underline{\sigma_L}^2 < \infty$ *such that* $\forall\, {\sigma_L}^2 > \underline{\sigma_L}^2$ *and* $\forall\, (\hat{Q}_1^h, ..., \hat{Q}_n^h) \in \Omega^h$

$$\begin{aligned} 0 &< \hat{\Theta}_{1i} < \frac{2}{\rho {\sigma_\epsilon}^2}, \qquad i = 1, 2, ..., n, \\ 0 &< \hat{\Theta}_{3i} < 2, \qquad i = 2, ..., n. \end{aligned}$$

Proof. Since $0 \leq \hat{Q}_i^h \leq \frac{2}{\rho {\sigma_\epsilon}^2}(1 + n2^n)$, $i = 1, ..., n$,, following (9.23a)

$$\lim_{{\sigma_L}^2 \to \infty} \Theta_{11}^h = \frac{1}{\rho {\sigma_\epsilon}^2}.$$

Furthermore, from (9.23e) and the equality above one derives

$$\lim_{{\sigma_L}^2 \to \infty} \Theta_{32}^h = 1.$$

Now proceed by induction: suppose for all $i \leq k$,

$$\begin{aligned} \lim_{{\sigma_L}^2 \to \infty} \Theta_{1,i-1}^h &= \frac{1}{\rho {\sigma_\epsilon}^2}, \\ \lim_{{\sigma_L}^2 \to \infty} \Theta_{3,i}^h &= 1. \end{aligned}$$

Then from (9.23c)

$$\lim_{{\sigma_L}^2 \to \infty} \Theta_{1,k}^h = \frac{1}{\rho {\sigma_\epsilon}^2},$$

and from (9.23e)

$$\begin{aligned} \lim_{{\sigma_L}^2 \to \infty} \Theta_{3,k+1}^h &= \frac{1}{\rho {\sigma_\epsilon}^2} \frac{\frac{1}{\rho {\sigma_\epsilon}^2} + \frac{1}{\rho {\sigma_\epsilon}^2} \sum_{k=2}^{\#(i,1)} 1}{\left(\frac{1}{\rho {\sigma_\epsilon}^2}\right)^2 + \left(\frac{1}{\rho {\sigma_\epsilon}^2}\right)^2 \sum_{k=2}^{\#(i,1)} 1} \\ &= 1. \end{aligned}$$

Thus, by the (strong) induction argument above, we have

$$\begin{aligned} \lim_{{\sigma_L}^2 \to \infty} \Theta_{1i}^h &= \frac{1}{\rho {\sigma_\epsilon}^2}, \qquad i = 1, 2, ..., n, \\ \lim_{{\sigma_L}^2 \to \infty} \Theta_{3i}^h &= 1, \qquad i = 2, ..., n. \end{aligned}$$

By continuity of Θ_{1i}^h and Θ_{3i}^h as functions of ${\sigma_L}^2$, there exists $\underline{\sigma_L}^2 < \infty$ such that $\forall {\sigma_L}^2 \geq \underline{\sigma_L}^2$

$$\begin{aligned} 0 &< \hat{\Theta}_{1i} < \frac{2}{\rho {\sigma_\epsilon}^2}, \qquad i = 1, 2, ..., n, \\ 0 &< \hat{\Theta}_{3i} < 2, \qquad i = 2, ..., n. \end{aligned}$$

□ [*end of claim*]

Claim 4 *For all* ${\sigma_L}^2 > \underline{\sigma_L}^2$

$$\hat{Q}_i^h > 0, \qquad i = 1, ..., n.$$

Proof. Let us fix liquidity variance ${\sigma_L}^2$ so that ${\sigma_L}^2 > \underline{\sigma_L}^2$. Suppose there exists $i_{1 \leq i \leq n}$ such that $\hat{Q}_i^h = 0$. Then due to the way we defined $\mathcal{F}$ and $\mathcal{G}$, the following must hold:

$$\left(\left(1 + \sum_{m \in \mathcal{H}^{-1}(i)} \prod_{j=0}^{\#(m,i)-1} \hat{\Theta}_{3,m^{-j}}^h \right) \hat{\Theta}_{1i}^h \right) \Bigg|_{\hat{Q}_i^h = 0} \leq 0.$$

However Claim 3 shows that $\forall \left(\hat{Q}_1^h, ..., \hat{Q}_n^h\right) \in \Omega^h = \left[0, \frac{2}{\rho {\sigma_\epsilon}^2} \left(1 + n2^n\right)\right]^n$

$$\begin{aligned} \hat{\Theta}_{1i}^h &> 0, \qquad i = 1, 2, ..., n, \\ \hat{\Theta}_{3i}^h &> 0, \qquad i = 2, ..., n. \end{aligned}$$

Then we have a clear violation of the inequality above since

$$\left(\left(1 + \sum_{m \in \mathcal{H}^{-1}(i)} \prod_{j=0}^{\#(m,i)-1} \hat{\Theta}_{3,m^{-j}}^h \right) \hat{\Theta}_{1i}^h \right) \Bigg|_{\hat{Q}_i^h = 0} > 0.$$

Thus it must be true that $\hat{Q}_i^h > 0$, $i = 1, ..., n$. □ [*end of claim*]

Claim 5 *For all* ${\sigma_L}^2 > \underline{\sigma_L}^2$

$$\hat{Q}_i^h < \frac{2}{\rho {\sigma_\epsilon}^2} \left(1 + n2^n\right), \qquad i = 1, ..., n.$$

Proof. Fix liquidity variance ${\sigma_L}^2$ so that ${\sigma_L}^2 > \underline{\sigma_L}^2$, and suppose there exists $i_{1 \leq i \leq n}$ such that $\hat{Q}_i^h = \frac{2}{\rho {\sigma_\epsilon}^2} \left(1 + n2^n\right)$. Once again, due to the way we defined $\mathcal{F}$ and $\mathcal{G}$, this implies

$$\left(\left(1 + \sum_{m \in \mathcal{H}^{-1}(i)} \prod_{j=0}^{\#(m,i)-1} \hat{\Theta}_{3,m^{-j}}^h \right) \hat{\Theta}_{1i}^h \right) \Bigg|_{\hat{Q}_i^h = \frac{2}{\rho {\sigma_\epsilon}^2} (1 + n2^n)} \geq \frac{2}{\rho {\sigma_\epsilon}^2} \left(1 + n2^n\right).$$

But following Claim 3, $\forall \left(\hat{Q}_1^h, ..., \hat{Q}_n^h\right) \in \Omega^h = \left[0, \frac{2}{\rho {\sigma_\epsilon}^2} \left(1 + n2^n\right)\right]^n$

$$\begin{aligned} \hat{\Theta}_{1i}^h &< \frac{2}{\rho {\sigma_\epsilon}^2}, \qquad i = 1, 2, ..., n, \\ \hat{\Theta}_{3i}^h &< 2, \qquad i = 2, ..., n. \end{aligned}$$

Then one necessarily has

$$\begin{aligned} \left(\left(1 + \sum_{m \in \mathcal{H}^{-1}(i)} \prod_{j=0}^{\#(m,i)-1} \hat{\Theta}_{3,m^{-j}}^h \right) \hat{\Theta}_{1i}^h \right) \Bigg|_{\hat{Q}_i^h = \frac{2}{\rho {\sigma_\epsilon}^2} (1 + n2^n)} &< \frac{2}{\rho {\sigma_\epsilon}^2} \left(1 + \sum_{m \in \mathcal{H}^{-1}(i)} 2^{\#(m,i)} \right) \\ &< \frac{2}{\rho {\sigma_\epsilon}^2} \left(1 + n2^n\right), \end{aligned}$$

which violates the inequality stated in the beginning of our proof. Thus $\hat{Q}_i^h < \frac{2}{\rho\sigma_\epsilon{}^2}(1+n2^n), \;\; i=1,...,n.$ $\square$ [*end of claim*]

Claims 3, 4 and 5 prove that $(\hat{Q}_1^h,...,\hat{Q}_n^h)$ is in fact a fixed point of $\mathcal{F}$ for $\sigma_L{}^2 > \underline{\sigma_L}{}^2$, which directly implies that $(\hat{Q}_1^h,...,\hat{Q}_n^h)$ is a solution to equation (9.25a) for $\sigma_L{}^2 > \underline{\sigma_L}{}^2$.

Now if we can show that there exists a solution to the original fixed point problem, namely the one given in Lemma 3, then we will have established the existence of a linear equilibrium price. Rewriting (9.25b) and (9.25c) to solve for γ^h, π_0^h, and substituting $(\hat{Q}_1^h,...,\hat{Q}_n^h)$ for $(Q_1^h,...,Q_n^h)$, one gets

$$\begin{aligned}
\hat{\gamma}^h &= \frac{1+\sum_{i=1}^n \hat{\Theta}_{2i}^h}{\frac{n}{\rho\sigma_x{}^2}+\sum_{i=1}^n\left(\hat{\Theta}_{1i}^h+\hat{\Theta}_{2i}^h\sum_{k=1}^n\hat{Q}_k^h\right)+\sum_{i=2}^n\hat{\Theta}_{3i}^h\left(\hat{\Theta}_{1,i^-}^h+\sum_{k=2}^{\#}(i,1)\left(\prod_{j=1}^{k-1}\hat{\Theta}_{3,i^{-j}}^h\right)\hat{\Theta}_{1,i^{-k}}^h\right)}, \\
\hat{\pi}_0^h &= \hat{\gamma}^h\frac{\frac{n\mu_x}{\rho\sigma_x{}^2}}{1+\sum_{i=1}^n\hat{\Theta}_{2i}^h},
\end{aligned}$$

where $\{\hat{\Theta}_{ji}^h\}_{j=1,2,3;i=1,...,n}$ are functions of $(\hat{Q}_1^h,...,\hat{Q}_n^h)$. Also substituting for $\hat{\pi}_i^h = \hat{\gamma}^h\hat{Q}_i^h, \quad i=1,...,n$, one derives $(\hat{\pi}_1^h,...,\hat{\pi}_n^h,\hat{\gamma}^h,\hat{\pi}_0^h)$ as a solution to the fixed point problem given in Lemma 3 for all $\sigma_L{}^2 > \underline{\sigma_L}{}^2$. Hence we have the desired existence result for the linear equilibrium price.

However, before ending this proof, we need to verify one last thing: whether γ^h is nonzero (since it is required by 9.12). First, note that $\hat{\Theta}_{2i}^h \to 0 \;\forall\, i$ as $\sigma_L{}^2 \to \infty$, and consecutively γ^h converges to

$$\frac{1}{\frac{n}{\rho\sigma_x{}^2}+\frac{n}{\rho\sigma_\epsilon{}^2}+\frac{1}{\rho\sigma_\epsilon{}^2}\sum_{i=1}^n\#(i,1)}.$$

So for sufficiently large $\sigma_L{}^2$, γ^h will be nonzero, and (if need be) by redefining $\underline{\sigma_L}{}^2$, we can always guarantee this. $\square$

Proof of Proposition 3: The proof of Theorem 1 guarantees that for sufficiently large $\sigma_L{}^2$ there exists a linear REESI price in the hierarchic scheme with

$$0 \le \frac{\pi_i^h}{\gamma^h} < \frac{2}{\rho\sigma_\epsilon{}^2}(1+n2^n), \qquad \forall\, i=1,...,n.$$

Then for this linear equilibrium, one can show that

$$\lim_{\sigma_L{}^2\to\infty}\theta_{1i}^h = \frac{1}{\rho\sigma_\epsilon{}^2}, \qquad i=1,...,n,$$

and

$$\lim_{\sigma_L{}^2\to\infty}\theta_{3i}^h = 1, \qquad i=2,...,n$$

with arguments similar to those used in Claim 3 within the proof of Theorem 1. One also has

$$\gamma^h \to \frac{1}{\frac{n}{\rho\sigma_x{}^2}+\frac{n}{\rho\sigma_\epsilon{}^2}+\frac{1}{\rho\sigma_\epsilon{}^2}\sum_{i=1}^n\#(i,1)}$$

as ${\sigma_L}^2 \to \infty$. Using Lemma 3, it can be further shown

$$\begin{aligned}
\lim_{{\sigma_L}^2 \to \infty} \theta_{2i}^h &= 0, \qquad i = 1, ..., n, \\
\lim_{{\sigma_L}^2 \to \infty} b_i^h &= \frac{1}{\rho \left(\frac{1}{\rho {\sigma_x}^2} + (\#(i,1)+1) \frac{1}{\rho {\sigma_\epsilon}^2} \right)}, \qquad i = 1, ..., n.
\end{aligned}$$

for the same linear equilibrium. Remainder of the proof will directly follow from the equations that yield price coefficients in Lemma 3. □

THE STAR (CENTER SPONSORED) INTERACTION SCHEME

Proof of Proposition 4: We assume risky security price $\tilde{p}$ is given by a linear function of the form

$$\tilde{p} = P(\tilde{s}_1, ..., \tilde{s}_n; \tilde{L}) = \pi_0^* + \sum_{i=1}^n \pi_i^* \, \tilde{s}_i - \gamma^* \, \tilde{L} \quad \text{with non-zero } \gamma^*.$$

Since star scheme is a special case of hierarchic interaction, we will be able to employ Lemma 3 in our proof. In particular, this scheme imposes the characteristic that

$$\#(i,1) = 1, \qquad \forall i \in \{2, ..., n\}.$$

Throughout the proof, we will specify the variables pertaining to the star scheme with the super-index $*$. So in the case of star scheme, $(\{\theta_{0i}^*, \theta_{1i}^*, \theta_{2i}^*, b_i^*\}_{i=1,...,n}, \{\theta_{3i}^*\}_{i=2,...,n})$ will represent the demand coefficients such that

$$\begin{aligned}
z_1(s_1, p) &= \theta_{01}^* + \theta_{11}^* s_1 + \left(\theta_{21}^* - \frac{1}{\rho b_1^*} \right) p, \\
z_i(s_i, p, \delta_1^i) &= \theta_{0i}^* + \theta_{1i}^* s_i + \left(\theta_{2i}^* - \frac{1}{\rho b_i^*} \right) p + \theta_{3i}^* \delta_1^i, \qquad i > 1.
\end{aligned}$$

Once again we will use the convention (as in Lemma 3) that

$$\pi^* = \sum_{i=1}^n \pi_i^*.$$

From Lemma 3, one observes that following hold for the star scheme:

$$\pi_1^* = \gamma^* \left(1 + \sum_{i=2}^n \theta_{3i}^* \right) \theta_{11}^*, \tag{9.26a}$$

$$\pi_i^* = \gamma^* \theta_{1i}^*, \tag{9.26b}$$

$$\gamma^* = \left(\sum_{i=1}^n \frac{1}{\rho b_i^*} - \theta_{2i}^* \right)^{-1}, \tag{9.26c}$$

$$\pi_0^* = \gamma^* \sum_{i=1}^n \theta_{0i}^*, \tag{9.26d}$$

where

$$\theta_{11}^* = \frac{1}{\rho\sigma_\epsilon{}^2}\frac{\left(\sum_{k=1}^n(\pi_k^*)^2-\pi^*\pi_1^*\right)\sigma_\epsilon{}^2+(\gamma^*)^2\sigma_L{}^2}{\sum_{k\neq 1}(\pi_k^*)^2\sigma_\epsilon{}^2+(\gamma^*)^2\sigma_L{}^2}, \tag{9.27a}$$

$$\theta_{21}^* = \frac{1}{\rho}\frac{\pi^*-\pi_1^*}{\sum_{k\neq 1}(\pi_k^*)^2\sigma_\epsilon{}^2+(\gamma^*)^2\sigma_L{}^2}, \tag{9.27b}$$

$$b_1^* = \frac{1}{\rho}\left(\frac{1}{\rho\sigma_x{}^2}+\theta_{11}^*+\theta_{21}^*\pi^*\right)^{-1}, \tag{9.27c}$$

$$\theta_{01}^* = \mu_x\left(\frac{1}{\rho b_1^*}-\theta_{11}^*-\theta_{21}^*\pi^*\right)-\theta_{21}^*\pi_0^*; \tag{9.27d}$$

and for $i = 2, ..., n$

$$\theta_{1i}^* = \frac{1}{\rho\sigma_\epsilon{}^2}\frac{\left(\sum_{k=1}^n(\pi_k^*)^2-\pi^*\pi_i^*+\pi_1^*(\pi_i^*-\pi_1^*)\right)\sigma_\epsilon{}^2+(\gamma^*)^2\sigma_L{}^2}{\sum_{k\neq i}(\pi_k^*)^2\sigma_\epsilon{}^2-(\pi_1^*)^2\sigma_\epsilon{}^2+(\gamma^*)^2\sigma_L{}^2}, \tag{9.27e}$$

$$\theta_{2i}^* = \frac{1}{\rho}\frac{\pi^*-\pi_i^*-\pi_1^*}{\sum_{k\neq i}(\pi_k^*)^2\sigma_\epsilon{}^2-(\pi_1^*)^2\sigma_\epsilon{}^2+(\gamma^*)^2\sigma_L{}^2}, \tag{9.27f}$$

$$\theta_{3i}^* = \frac{1}{\rho\sigma_\epsilon{}^2}\frac{\left(\sum_{k\neq i}(\pi_k^*)^2-\pi_1^*(\pi^*-\pi_i^*)\right)\sigma_\epsilon{}^2+(\gamma^*)^2\sigma_L{}^2}{\sum_{k\neq i}(\pi_k^*)^2\sigma_\epsilon{}^2-(\pi_1^*)^2\sigma_\epsilon{}^2+(\gamma^*)^2\sigma_L{}^2}\frac{1}{\theta_{11}^*}, \tag{9.27g}$$

$$b_i^h = \frac{1}{\rho}\left(\frac{1}{\rho\sigma_x{}^2}+\theta_{1i}^*+\theta_{2i}^*\pi^*+\theta_{3i}^*\theta_{11}^*\right)^{-1}, \tag{9.27h}$$

$$\theta_{0i}^* = \mu_x\left(\frac{1}{\rho b_i^*}-\theta_{1i}^*-\theta_{2i}^*\pi^*-\theta_{3i}^*\theta_{11}^*\right)-\theta_{2i}^*\pi_0^*. \tag{9.27i}$$

Equations (9.26a)-(9.26d) and (9.27a)-(9.27i) together determine the linear equilibrium price through the following equations:

$$\pi_1^* = \gamma^*\left(\frac{1}{\rho\sigma_\epsilon{}^2}\frac{\left(\sum_{k=1}^n(\pi_k^*)^2-\pi^*\pi_1^*\right)\sigma_\epsilon{}^2+(\gamma^*)^2\sigma_L{}^2}{\sum_{k\neq 1}(\pi_k^*)^2\sigma_\epsilon{}^2+(\gamma^*)^2\sigma_L{}^2}+\frac{1}{\rho\sigma_\epsilon{}^2}\sum_{i=2}^n\frac{\left(\sum_{k\neq i}(\pi_k^*)^2-\pi_1^*(\pi^*-\pi_i^*)\right)\sigma_\epsilon{}^2+(\gamma^*)^2\sigma_L{}^2}{\sum_{k\neq i}(\pi_k^*)^2\sigma_\epsilon{}^2-(\pi_1^*)^2\sigma_\epsilon{}^2+(\gamma^*)^2\sigma_L{}^2}\right), \tag{9.28a}$$

$$\pi_i^* = \gamma^*\frac{1}{\rho\sigma_\epsilon{}^2}\frac{\left(\sum_{k=1}^n(\pi_k^*)^2-\pi^*\pi_i^*+\pi_1^*(\pi_i^*-\pi_1^*)\right)\sigma_\epsilon{}^2+(\gamma^*)^2\sigma_L{}^2}{\sum_{k\neq i}(\pi_k^*)^2\sigma_\epsilon{}^2-(\pi_1^*)^2\sigma_\epsilon{}^2+(\gamma^*)^2\sigma_L{}^2}, \tag{9.28b}$$

$$\begin{aligned}\frac{1}{\gamma^*} &= \frac{n}{\rho\sigma_x{}^2}+\frac{1}{\rho\sigma_\epsilon{}^2}\frac{\left(\sum_{k=1}^n(\pi_k^*)^2+\pi^*(\pi^*-2\pi_1^*)-\pi^*+\pi_1^*\right)\sigma_\epsilon{}^2+(\gamma^*)^2\sigma_L{}^2}{\sum_{k\neq 1}(\pi_k^*)^2\sigma_\epsilon{}^2+(\gamma^*)^2\sigma_L{}^2}+\\ &\quad \sum_{k=2}^n\frac{1}{\rho\sigma_\epsilon{}^2}\frac{\left(2\sum_{k\neq 1,i}(\pi_k^*)^2+(\pi_1^*+\pi_i^*)^2+\pi^*(\pi^*-2\pi_1^*-2\pi_i^*)-\pi^*+\pi_1^*+\pi_i^*\right)\sigma_\epsilon{}^2+2(\gamma^*)^2\sigma_L{}^2}{\sum_{k\neq i}(\pi_k^*)^2\sigma_\epsilon{}^2-(\pi_1^*)^2\sigma_\epsilon{}^2+(\gamma^*)^2\sigma_L{}^2},\end{aligned} \tag{9.28c}$$

$$\pi_0^* = \gamma^*\left(n\frac{\mu_x}{\rho\sigma_x{}^2}-\frac{\pi_0^*}{\rho}\left(\frac{\pi^*-\pi_1^*}{\sum_{k\neq 1}(\pi_k^*)^2\sigma_\epsilon{}^2+(\gamma^*)^2\sigma_L{}^2}+\sum_{k=2}^n\frac{\pi^*-\pi_i^*-\pi_1^*}{\sum_{k\neq i}(\pi_k^*)^2\sigma_\epsilon{}^2-(\pi_1^*)^2\sigma_\epsilon{}^2+(\gamma^*)^2\sigma_L{}^2}\right)\right). \tag{9.28d}$$

It turns out that Eqs. (9.28a) and (9.28b) can be analyzed independently of (9.28c) and (9.28d). Define

$$Q_i^* = \frac{\pi_i^*}{\gamma^*}, \qquad i = 1, ..., n,$$
$$Q^* = \frac{\pi^*}{\gamma^*}.$$

Then Eqs (9.28a)-(9.28b) yield the following:

$$Q_1^* = \frac{1}{\rho\sigma_\epsilon{}^2}\frac{\left(\sum_{k=1}^n(Q_k^*)^2-Q^*Q_1^*\right)\sigma_\epsilon{}^2+\sigma_L{}^2}{\sum_{k\neq 1}(Q_k^*)^2\sigma_\epsilon{}^2+\sigma_L{}^2}+\frac{1}{\rho\sigma_\epsilon{}^2}\sum_{i=2}^n\frac{\left(\sum_{k\neq i}(Q_k^*)^2-Q_1^*(Q^*-Q_i^*)\right)\sigma_\epsilon{}^2+\sigma_L{}^2}{\sum_{k\neq i}(Q_k^*)^2\sigma_\epsilon{}^2-(Q_1^*)^2\sigma_\epsilon{}^2+\sigma_L{}^2}, \tag{9.29a}$$

$$Q_i^* = \frac{1}{\rho\sigma_\epsilon{}^2}\frac{\left(\sum_{k=1}^n(Q_k^*)^2-Q^*Q_i^*+Q_1^*(Q_i^*-Q_1^*)\right)\sigma_\epsilon{}^2+\sigma_L{}^2}{\sum_{k\neq i}(Q_k^*)^2\sigma_\epsilon{}^2-(Q_1^*)^2\sigma_\epsilon{}^2+\sigma_L{}^2}, \qquad i=2,...,n. \tag{9.29b}$$

The following result states that agents $i = 2, ..., n$ do not differ with respect to their proportional weights in the price, namely Q_i^*. This is actually straightforward given the complete homogeneity between them in terms of signal precision, risk aversion, and location in the interaction pattern. The rigorous proof goes as follows:

Claim 6 *For all* $i, j \in \{2, ..., n\}$,

$$Q_i^* = Q_j^*.$$

Proof. Suppose not, i.e, suppose $\exists\, i, j \in \{2, ..., n\}$ such that $Q_i^* \neq Q_j^*$. Without loss of generality, we can assume $Q_i^* > Q_j^*$. Following (9.29b), we have

$$\frac{\left(\sum_{k=1}^{n}(Q_k^*)^2 - Q^* Q_i^* + Q_1^*(Q_i^* - Q_1^*)\right)\sigma_\epsilon{}^2 + \sigma_L{}^2}{\sum_{k\neq i}(Q_k^*)^2\sigma_\epsilon{}^2 - (Q_1^*)^2\sigma_\epsilon{}^2 + \sigma_L{}^2} \quad > \quad \frac{\left(\sum_{k=1}^{n}(Q_k^*)^2 - Q^* Q_j^* + Q_1^*(Q_j^* - Q_1^*)\right)\sigma_\epsilon{}^2 + \sigma_L{}^2}{\sum_{k\neq j}(Q_k^*)^2\sigma_\epsilon{}^2 - (Q_1^*)^2\sigma_\epsilon{}^2 + \sigma_L{}^2}.$$

Since denominators on LHS and RHS of the inequality above are equal and positive, dropping them does not change the direction of inequality. Further simplifications yield

$$Q_i^*(Q_1^* - Q^*) > Q_j^*(Q_1^* - Q^*).$$

Then it must be true that $Q_i^* < Q_j^*$ since $Q^* \geq Q_1^*$. This violates our very first assumption that $Q_i^* > Q_j^*$. Thus we are done. □ [*end of claim*]

Following this claim, we know that $\exists$ some q^* such that

$$Q_i^* = q^*, \qquad \forall i \in \{2, ..., n\}.$$

Now one can rewrite (9.29a)-(9.29b) as

$$Q_1^* = \frac{1}{\rho\sigma_\epsilon{}^2}\frac{(n-1)q^*(q^* - Q_1^*)\sigma_\epsilon{}^2 + \sigma_L{}^2}{(n-1)(q^*)^2\sigma_\epsilon{}^2 + \sigma_L{}^2} + \frac{n-1}{\rho\sigma_\epsilon{}^2}\frac{(n-2)q^*(q^* - Q_1^*)\sigma_\epsilon{}^2 + \sigma_L{}^2}{(n-2)(q^*)^2\sigma_\epsilon{}^2 + \sigma_L{}^2}, \tag{9.30a}$$

$$q^* = \frac{\sigma_L{}^2}{\rho\sigma_\epsilon{}^2}\frac{1}{(n-2)(q^*)^2\sigma_\epsilon{}^2 + \sigma_L{}^2}, \qquad i{=}2,...,n. \tag{9.30b}$$

Notice that (9.30b) is a cubic polynomial in q^* without a quadratic component. From *Cardano's formula*, the unique (real) solution to (9.30b) is given by

$$q^* = \sqrt[3]{\frac{\sigma_L{}^2}{2(n-2)\rho\sigma_\epsilon{}^4}}\left(\sqrt[3]{1 + \sqrt{1 + \frac{4}{27}\frac{\rho^2\sigma_L{}^2\sigma_\epsilon{}^2}{n-2}}} - \sqrt[3]{-1 + \sqrt{1 + \frac{4}{27}\frac{\rho^2\sigma_L{}^2\sigma_\epsilon{}^2}{n-2}}}\right). \tag{9.31}$$

Then Q_1^* is derived uniquely as follows:

$$Q_1^* = n\left(\rho\sigma_\epsilon{}^2 + \frac{(n-1)q^*\sigma_\epsilon{}^2}{(n-1)(q^*)^2\sigma_\epsilon{}^2 + \sigma_L{}^2} + (n-1)\frac{(n-2)q^*\sigma_\epsilon{}^2}{(n-2)(q^*)^2\sigma_\epsilon{}^2 + \sigma_L{}^2}\right)^{-1} \tag{9.32}$$

Solving for γ^* from (9.28c), we get

$$\gamma^* = \frac{1 + \frac{1}{\rho}\frac{(n-1)q}{(n-1)(q^*)^2\sigma_\epsilon{}^2 + \sigma_L{}^2} + \frac{1}{\rho}\frac{(n-1)(n-2)q^*}{(n-2)(q^*)^2\sigma_\epsilon{}^2 + \sigma_L{}^2}}{\frac{n}{\rho\sigma_x{}^2} + \frac{2n-1}{\rho\sigma_\epsilon{}^2} + \frac{1}{\rho}\frac{(n-1)^2(q^*)^2}{(n-1)(q^*)^2\sigma_\epsilon{}^2 + \sigma_L{}^2} + \frac{1}{\rho}\frac{(n-1)(n-2)^2(q^*)^2}{(n-2)(q^*)^2\sigma_\epsilon{}^2 + \sigma_L{}^2}}. \tag{9.33}$$

Also solving for π_0^* from (9.28d) gives us

$$\pi_0^* \quad = \quad \frac{\gamma^* n \frac{\mu_x}{\rho {\sigma_x}^2}}{1+\frac{1}{\rho}\frac{(n-1)q^*}{(n-1)(q^*)^2{\sigma_\epsilon}^2+{\sigma_L}^2}+\frac{1}{\rho}\frac{(n-1)(n-2)q^*}{(n-2)(q^*)^2{\sigma_\epsilon}^2+{\sigma_L}^2}}. \tag{9.34}$$

Now substituting for $\pi_i^* = \gamma^* q^*$, $i = 2, ..., n$, and $\pi_1^* = \gamma^* Q_1^*$, we have the desired result through the equations (9.31)-(9.34). □

Proof of Corollary 1: Keeping in mind that $1^- = \emptyset$ and $i^- = 1, \ \forall\, i = 2, ..., n$, in the star scheme, we have a limit linear REESI price given by (5.4) immediately from Corollary 3. Since there exists a unique linear REESI price in the star scheme, the derived limit equilibrium is also the unique limit equilibrium.[23] □

SOCIAL INTERACTION AND INFORMATION AGGREGATION

Proof of Proposition 6: Following Proposition 4, the following inequality holds for generic exogenous parameters:

$$q^* \neq \frac{n}{\rho{\sigma_\epsilon}^2 + \frac{(n-1)q^*{\sigma_\epsilon}^2}{(n-1)(q^*)^2{\sigma_\epsilon}^2+{\sigma_L}^2} + (n-1)\frac{(n-2)q^*{\sigma_\epsilon}^2}{(n-2)(q^*)^2{\sigma_\epsilon}^2+{\sigma_L}^2}},$$

where

$$q^* = \sqrt[3]{\frac{{\sigma_L}^2}{2(n-2)\rho{\sigma_\epsilon}^4}}\left(\sqrt[3]{1+\sqrt{1+\frac{4}{27}\frac{\rho^2{\sigma_L}^2{\sigma_\epsilon}^2}{n-2}}}-\sqrt[3]{-1+\sqrt{1+\frac{4}{27}\frac{\rho^2{\sigma_L}^2{\sigma_\epsilon}^2}{n-2}}}\right).$$

Then the price coefficients π_i^* generically satisfy

$$\pi_1^* \neq \pi_i, \qquad \pi_i = \pi_j \qquad \text{for } i, j \in \{2, ..., n\}.$$

Suppose to the contrary, the informational content of price $\tilde{p}$, namely $\sum_{i=1}^n \pi_i^* \tilde{s}_i$, is (generically) a sufficient statistic for the joint distribution $(\tilde{s}_1, ..., \tilde{s}_n, \sum_{i=1}^n \pi_i^* \tilde{s}_i)$ conditional on $\tilde{X}$. Following (6.1) and the fact that $\sum_{i=1}^n \pi_i^* \tilde{s}_i$ adds no information on top of $(\tilde{s}_1, ..., \tilde{s}_n)$,

$$\begin{aligned} \mathrm{E}[\tilde{X}|\tilde{s}_1, ..., \tilde{s}_n] &= \mathrm{E}\left[\tilde{X}\middle|\tilde{s}_1, ..., \tilde{s}_n, \sum_{i=1}^n \pi_i^* \tilde{s}_i\right] \\ &= \mathrm{E}\left[\tilde{X}\middle|\sum_{i=1}^n \pi_i^* \tilde{s}_i\right]. \end{aligned} \tag{9.35}$$

On the other hand, since $\tilde{\mathcal{S}} = \frac{1}{{\sigma_x}^2+{\sigma_\epsilon}^2}\sum_{i=1}^n \tilde{s}_i$ is also a sufficient statistic for the joint distribution $(\tilde{s}_1, ..., \tilde{s}_n, \tilde{\mathcal{S}})$ conditional on $\tilde{X}$, by a similar argument

$$\begin{aligned} \mathrm{E}[\tilde{X}|\tilde{s}_1, ..., \tilde{s}_n] &= \mathrm{E}[\tilde{X}|\tilde{s}_1, ..., \tilde{s}_n, \tilde{\mathcal{S}}] \\ &= \mathrm{E}[\tilde{X}|\tilde{\mathcal{S}}]. \end{aligned} \tag{9.36}$$

[23] One can get the same result using Proposition 4, however, it will require a bit more work.

(9.35) and (9.36) imply

$$\begin{aligned}
\mathrm{E}\left[\tilde{X}\Big|\sum_{i=1}^{n}\pi_i^*\tilde{s}_i\right] &= \mathrm{E}[\tilde{X}|\tilde{\mathcal{S}}] \\
\Longrightarrow \quad \mu_x + \frac{\mathrm{cov}\left(\tilde{X},\sum_{i=1}^{n}\pi_i^*\tilde{s}_i\right)}{\mathrm{var}\left(\sum_{i=1}^{n}\pi_i^*\tilde{s}_i\right)}\left(\sum_{i=1}^{n}\pi_i^*\tilde{s}_i - \mathrm{E}\left[\sum_{i=1}^{n}\pi_i^*\tilde{s}_i\right]\right) &= \mu_x + \frac{\mathrm{cov}(\tilde{X},\tilde{\mathcal{S}})}{\mathrm{var}(\tilde{\mathcal{S}})}(\tilde{\mathcal{S}} - \mathrm{E}[\tilde{\mathcal{S}}]) \\
\Longrightarrow \quad \frac{\sum_{i=1}^{n}\pi_i^*{\sigma_x}^2}{\sum_{i=1}^{n}(\pi_i^*)^2({\sigma_x}^2+{\sigma_\epsilon}^2)}\left(\sum_{i=1}^{n}\pi_i^*\tilde{s}_i - \sum_{i=1}^{n}\pi_i^*\mu_x\right) &= {\sigma_x}^2\left(\frac{1}{{\sigma_x}^2+{\sigma_\epsilon}^2}\sum_{i=1}^{n}\tilde{s}_i - \frac{n}{{\sigma_x}^2+{\sigma_\epsilon}^2}\mu_x\right) \\
\Longrightarrow \quad \frac{\sum_{i=1}^{n}\pi_i^*}{\sum_{i=1}^{n}(\pi_i^*)^2}\sum_{i=1}^{n}\pi_i^*(\tilde{s}_i-\mu_x) &= \sum_{i=1}^{n}(\tilde{s}_i-\mu_x)
\end{aligned}$$

The last equality should hold for all realizations $(s_1, ..., s_n)$ of $(\tilde{s}_1, ..., \tilde{s}_n)$. Consider the case where $s_1 > 0$ and $s_i = 0$, $i = 2, ..., n$. Then we must have

$$\pi_1^*\sum_{i=1}^{n}\pi_i^* = \sum_{i=1}^{n}(\pi_i^*)^2.$$

Given that $\pi_1^* \neq \pi_i$ and $\pi_i = \pi_j$ for $i, j \in \{2, ..., n\}$ for generic exogenous parameters, the last equality will (generically) fail to hold. □

NOISY SOCIAL INFERENCE IN THE CYCLIC INTERACTION SCHEME

Proof of Remark 1: Suppose there exists a linear noisy REESI price of the following form:

$$\tilde{p} = \Pi_0 + \sum_{i=1}^{3}\Pi_i\,\tilde{s}_i + \sum_{i=1}^{3}\Lambda_i\,\tilde{\eta}_i - \Gamma\,\tilde{L} \qquad \text{with nonzero } \Gamma. \tag{9.37}$$

Recall that Δ_i is the (realized) gaussian information that agent i derives from the observation of

$$z_{i-1} + \eta_{i-1},$$

for $i = 1, 2, 3$ (where $i - 1$ is considered in modulo 3). Each agent i's information set is of the form $I_i = (s_i, p, \Delta_i)$. Now let $\mathcal{V}_i$ denote the variance-covariance matrix of $(\tilde{s}_i, \tilde{p}, \tilde{\Delta}_i)$, and $\mathcal{W}_i$ be the covariance matrix of $\tilde{X}$ and $(\tilde{s}_i, \tilde{p}, \tilde{\Delta}_i)$. The expectations of the agents are given by

$$\begin{aligned}
\mathrm{E}[\tilde{X}|I_i] &= \alpha_{0i} + \alpha_{1i}s_i + \alpha_{2i}p + \alpha_{3i}\Delta_i, \\
\mathrm{var}(\tilde{X}|I_i) &= v_i,
\end{aligned}$$

where the values of the coefficients $\alpha_{0i}, \alpha_{1i}, \alpha_{2i}, \alpha_{3i}, v_i$ depend on $\mathcal{V}_i$ and $\mathcal{W}_i$. Due to the homogeneity of agents (in terms of risk aversion, signal precision, distribution of noise terms in inferences from social interaction, and symmetry in the interaction pattern), we necessarily have $\alpha_0, \alpha_1, \alpha_2, \alpha_3, v$ such that

$$\alpha_{0i} = \alpha_0, \quad \alpha_{1i} = \alpha_1, \quad \alpha_{2i} = \alpha_2, \quad \alpha_{3i} = \alpha_3, \quad v_i = v; \quad \forall i = 1, ..., n.$$

Given the CARA-Gaussian setup, demands of agents will be of the following form:

$$\begin{aligned} z_i(s_i, p, \Delta_i) &= \frac{\mathrm{E}[\tilde{X}|s_i, p, \Delta_i] - p}{\rho \mathrm{var}(\tilde{X}|s_i, p, \Delta_i)} \\ &= \frac{\alpha_0 + \alpha_1 s_i + (\alpha_2 - 1)p + \alpha_3 \Delta_i}{\rho v}, \qquad i = 1, ..., n. \end{aligned} \tag{9.38}$$

Gaussian informations inferred from demand observations are verified as follows:

Claim 7 (Inference from Noisy Observation) *For each $i = 1, 2, 3$, the gaussian information, which agent i infers from demand of agent $i - 1$, is*[24]

$$\tilde{\Delta}_i = \left(1 - \left(\frac{\alpha_3}{\rho v}\right)^3\right)^{-1} \sum_{k=1}^{3} \left(\frac{\alpha_3}{\rho v}\right)^{k-1} \left(\frac{\alpha_1}{\rho v}\, \tilde{s}_{i-k} + \tilde{\eta}_{i-k}\right). \tag{9.39}$$

Proof. Each agent i knows the demand coefficients of her uphill neighbor $i - 1$, namely $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta$ (due to N0). Therefore the additional information i derives (on top of p) from the observation of agent $i - 1$'s demand is explicitly of the form[25]

$$\Delta_i = \frac{\alpha_1 s_{i-1} + \alpha_3 \Delta_{i-1}}{\rho v} + \eta_{i-1}. \tag{9.40}$$

Using the very same relation to substitute for $\{\Delta_{i-k}\}_{k=1,...,n}$ in (9.40) above, delivers us

$$\Delta_i = \sum_{k=1}^{3} \left(\frac{\alpha_3}{\rho v}\right)^{k-1} \left(\frac{\alpha_1}{\rho v}\, s_{i-k} + \eta_{i-k}\right) + \left(\frac{\alpha_3}{\rho v}\right)^3 \Delta_{i-3}.$$

Since $i - 3 \equiv i \pmod 3$, this equation yields

$$\Delta_i = \left(1 - \left(\frac{\alpha_3}{\rho v}\right)^3\right)^{-1} \sum_{k=1}^{3} \left(\frac{\alpha_3}{\rho v}\right)^{k-1} \left(\frac{\alpha_1}{\rho v}\, s_{i-k} + \eta_{i-k}\right).$$

The equation above holds for all realizations $\{s_k, \eta_k\}_{k=1,...,n}$ of $\{\tilde{s}_k, \tilde{\eta}_k\}_{k=1,...,n}$, hence (9.39) is acquired. The normality of $\tilde{\Delta}_i$ follows from the linear formulation. □ [*end of claim*]

On the other hand, one can easily verify that all agents have exactly equal weights in price, i.e., there exist Π, Λ dependent on expectation coefficients $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ and variance v such that for all $i = 1, 2, 3$

$$\Pi_i = \Pi, \qquad \Lambda_i = \Lambda.$$

This follows once again from the strict homogeneity in the economy (in terms of risk aversion, signal precision, distribution of noise term in the inference from social interaction, symmetric interaction pattern). In particular, using market clearing condition

$$\sum_{i=1}^{3} z_i(s_i, p, \Delta_i) = L,$$

[24] Index $i - k$ is considered modulo 3.

[25] Because with the knowledge of demand coefficients of agent $i - 1$ and realization of p, agent i can drop $\frac{\alpha_0}{\rho v}$ and $\frac{\alpha_2 - 1}{\rho v}\, p$ from the noisy observation $z_{i-1} + \eta_{i-1}$.

and solving for p, one gets

$$\Pi = \Gamma \frac{\frac{\alpha_1}{\rho v}}{1-\frac{\alpha_3}{\rho v}}, \tag{9.41a}$$

$$\Lambda = \Gamma \frac{\frac{\alpha_3}{\rho v}}{1-\frac{\alpha_3}{\rho v}}, \tag{9.41b}$$

$$\Gamma = \frac{1}{3\left(\frac{1}{\rho v}-\frac{\alpha_2}{\rho v}\right)}, \tag{9.41c}$$

$$\Pi_0 = \Gamma \frac{n\alpha_0}{\rho v}. \tag{9.41d}$$

Now we can write down the variance-covariance matrix $\mathcal{V}_i$ of $(\tilde{s}_i, \tilde{p}, \tilde{\Delta}_i)$:

$$\mathcal{V}_i = \mathcal{V} = \begin{bmatrix} {\sigma_x}^2+{\sigma_\epsilon}^2 & n\Pi\,{\sigma_x}^2+\Pi\,{\sigma_\epsilon}^2 & \frac{\frac{\alpha_1}{\rho v}}{1-\frac{\alpha_3}{\rho v}}\,{\sigma_x}^2+\frac{\left(\frac{\alpha_3}{\rho v}\right)^{n-1}}{1-\left(\frac{\alpha_3}{\rho v}\right)^n}\,\frac{\alpha_1}{\rho v}\,{\sigma_\epsilon}^2 \\ n\Pi\,{\sigma_x}^2+\Pi\,{\sigma_\epsilon}^2 & n^2\Pi^2\,{\sigma_x}^2+n\Pi^2\,{\sigma_\epsilon}^2+n\Lambda^2\,{\sigma_\eta}^2+\Gamma^2\,{\sigma_L}^2 & n\Pi\,\frac{\frac{\alpha_1}{\rho v}}{1-\frac{\alpha_3}{\rho v}}\,{\sigma_x}^2+\Pi\,\frac{\frac{\alpha_1}{\rho v}}{1-\frac{\alpha_3}{\rho v}}\,{\sigma_\epsilon}^2+\Lambda\,\frac{1}{1-\frac{\alpha_3}{\rho v}}\,{\sigma_\eta}^2 \\ \frac{\frac{\alpha_1}{\rho v}}{1-\frac{\alpha_3}{\rho v}}\,{\sigma_x}^2+\frac{\left(\frac{\alpha_3}{\rho v}\right)^{n-1}}{1-\left(\frac{\alpha_3}{\rho v}\right)^n}\,\frac{\alpha_1}{\rho v}\,{\sigma_\epsilon}^2 & n\Pi\,\frac{\frac{\alpha_1}{\rho v}}{1-\frac{\alpha_3}{\rho v}}\,{\sigma_x}^2+\Pi\,\frac{\frac{\alpha_1}{\rho v}}{1-\frac{\alpha_3}{\rho v}}\,{\sigma_\epsilon}^2+\Lambda\,\frac{1}{1-\frac{\alpha_3}{\rho v}}\,{\sigma_\eta}^2 & \mathrm{var}(\tilde{\Delta}_i) \end{bmatrix},$$

where

$$\mathrm{var}(\tilde{\Delta}_i) = \frac{\left(\frac{\alpha_1}{\rho v}\right)^2}{\left(1-\frac{\alpha_3}{\rho v}\right)^2}\,{\sigma_x}^2+\frac{1}{1-\left(\frac{\alpha_3}{\rho v}\right)^2}\,\frac{1+\left(\frac{\alpha_3}{\rho v}\right)^n}{1-\left(\frac{\alpha_3}{\rho v}\right)^n}\left(\left(\frac{\alpha_1}{\rho v}\right)^2{\sigma_\epsilon}^2+{\sigma_\eta}^2\right).$$

The covariance matrix of $\tilde{X}$ and $(\tilde{s}_i, \tilde{p}, \tilde{\Delta}_i)$ is given by

$$\mathcal{W}_i = \mathcal{W} = {\sigma_x}^2 \begin{bmatrix} 1 \\ n\Pi \\ \frac{\frac{\alpha_1}{\rho v}}{1-\frac{\alpha_3}{\rho v}} \end{bmatrix}.$$

Normal distribution theory dictates

$$[\alpha_1 \quad \alpha_2 \quad \alpha_3] = (\mathcal{W})'(\mathcal{V})^{-1}, \tag{9.42a}$$

$$v = {\sigma_x}^2 - (\mathcal{W})'(\mathcal{V})^{-1}\mathcal{W}, \tag{9.42b}$$

$$\alpha_0 = \mu_x - (\mathcal{W})'(\mathcal{V})^{-1}\begin{bmatrix}\mu_x & \Pi_0+n\Pi\mu_x & \frac{\frac{\alpha_1}{\rho v}}{1-\frac{\alpha_3}{\rho v}}\mu_x\end{bmatrix}'. \tag{9.42c}$$

Now define

$$\theta_0 \equiv \frac{\alpha_0}{\rho v}, \quad \theta_1 \equiv \frac{\alpha_1}{\rho v}, \qquad \theta_2 \equiv \frac{\alpha_2}{\rho v}, \qquad \theta_3 \equiv \frac{\alpha_3}{\rho v}.$$

Plugging in the values

$$\rho = {\sigma_x}^2 = {\sigma_\epsilon}^2 = {\sigma_L}^2 = {\sigma_\eta}^2 = 1,$$

Eqns. (9.42a)-(9.42c) give us the following:

$$\theta_1 = \frac{-\left(-\Gamma^2-2\Lambda^2+(\Gamma^2+4\Lambda^2)\theta_3-(\Gamma^2+2\Lambda^2)\theta_3^2+\Lambda\Pi\theta_1(1+\theta_3-2\theta_3^2-3\theta_3^3-3\theta_3^4)+(\Gamma^2+3\Lambda^2)\theta_1^2(-1+\theta_3+\theta_3^3+\theta_3^4)\right)}{D}, \tag{9.43a}$$

$$\theta_2 = \frac{-\left(-2\Pi(1-\theta_3+\theta_3^2)-\Lambda\theta_1(-1-\theta_3+\theta_3^3+\theta_3^4)+\Pi\theta_1^2(-1+3\theta_3-3\theta_3^2-2\theta_3^3+\theta_3^4+3\theta_3^5+3\theta_3^6)\right)}{D}, \tag{9.43b}$$

$$\theta_3 = \frac{(-1+\theta_3^3)\left(2\Lambda\Pi+(\Gamma^2+3\Lambda^2)\theta_1(-1+\theta_3^2)\right)}{D}, \tag{9.43c}$$

$$\frac{1}{\mathring{v}} = 1+\theta_1+n\Pi\theta_2+\frac{\theta_1}{1-\theta_3}\theta_3, \tag{9.43d}$$

$$\theta_0 = \mu_x\left(1-\theta_1-n\Pi\theta_2-\frac{\theta_1}{1-\theta_3}\theta_3\right)-\Pi_0\theta_2, \tag{9.43e}$$

where

$$\begin{aligned} D = {} & \Gamma^2+2\Lambda^2+2\Pi^2-(\Gamma^2+4\Lambda^2+2\Pi^2)\theta_3+(\Gamma^2+2\Lambda^2+2\Pi^2)\theta_3^2+2\Lambda\Pi\theta_1(-1-\theta_3+\theta_3^3+\theta_3^4)- \\ & \theta_1^2\left(-\Gamma^2-3\Lambda^2-\Pi^2+(\Gamma^2+3(\Lambda^2+\Pi^2))\theta_3-(\Gamma^2+3(\Lambda^2+\Pi^2))\theta_3^2-2\Pi^2\theta_3^3+(\Gamma^2+3\Lambda^2+\Pi^2)\theta_3^4+(\Gamma^2+3(\Lambda^2+\Pi^2))\theta_3^5+(\Gamma^2+3(\Lambda^2+\Pi^2))\theta_3^6\right). \end{aligned}$$

So the existence of a linear noisy REESI price given by (9.37) is equivalent to the existence of a solution to the system of equations (9.41a)-(9.41d), (9.43a)-(9.43e) and Γ being nonzero. Substituting for Π, Λ, Λ, Π_0 from (9.41a)-(9.41d), the equations (9.43a)-(9.43e) present a fixed point problem in the arguments $(\theta_1, \theta_2, \theta_3, v, \theta_0)$. Now taking Γ nonzero and plugging in for the price coefficients, equations (9.43a) and (9.43c) will reduce to

$$\begin{aligned} \theta_1 &= \frac{-\left(-1-2\left(\frac{\theta_3}{1-\theta_3}\right)^2+(1+4\left(\frac{\theta_3}{1-\theta_3}\right)^2)\theta_3-(1+2\left(\frac{\theta_3}{1-\theta_3}\right)^2)\theta_3^2+\left(\frac{\theta_3}{1-\theta_3}\right)\left(\frac{\theta_1}{1-\theta_3}\right)\theta_1(1+\theta_3-2\theta_3^2-3\theta_3^3-3\theta_3^4)+(1+3\left(\frac{\theta_3}{1-\theta_3}\right)^2)\theta_1^2(-1+\theta_3+\theta_3^3+\theta_3^4)\right)}{d}, \\ \theta_3 &= \frac{(-1+\theta_3^3)\left(2\left(\frac{\theta_3}{1-\theta_3}\right)\left(\frac{\theta_1}{1-\theta_3}\right)+(1+3\left(\frac{\theta_3}{1-\theta_3}\right)^2)\theta_1(-1+\theta_3^2)\right)}{d}. \end{aligned}$$

with

$$\begin{aligned} d = {} & 1+2\left(\frac{\theta_3}{1-\theta_3}\right)^2+2\left(\frac{\theta_1}{1-\theta_3}\right)^2-(1+4\left(\frac{\theta_3}{1-\theta_3}\right)^2+2\left(\frac{\theta_1}{1-\theta_3}\right)^2)\theta_3+(1+2\left(\frac{\theta_3}{1-\theta_3}\right)^2+2\left(\frac{\theta_1}{1-\theta_3}\right)^2)\theta_3^2+2\left(\frac{\theta_3}{1-\theta_3}\right)\left(\frac{\theta_1}{1-\theta_3}\right)\theta_1(-1-\theta_3+\theta_3^3+\theta_3^4) \\ & -\theta_1^2\Bigg(-1-3\left(\frac{\theta_3}{1-\theta_3}\right)^2-\left(\frac{\theta_1}{1-\theta_3}\right)^2+(1+3(\left(\frac{\theta_3}{1-\theta_3}\right)^2+\left(\frac{\theta_1}{1-\theta_3}\right)^2))\theta_3(1+3(\left(\frac{\theta_3}{1-\theta_3}\right)^2+\left(\frac{\theta_1}{1-\theta_3}\right)^2))\theta_3^2- \\ & 2\left(\frac{\theta_1}{1-\theta_3}\right)^2\theta_3^3+(1+3\left(\frac{\theta_3}{1-\theta_3}\right)^2+\left(\frac{\theta_1}{1-\theta_3}\right)^2)\theta_3^4+(1+3(\left(\frac{\theta_3}{1-\theta_3}\right)^2+\left(\frac{\theta_1}{1-\theta_3}\right)^2))\theta_3^5+(1+3(\left(\frac{\theta_3}{1-\theta_3}\right)^2+\left(\frac{\theta_1}{1-\theta_3}\right)^2))\theta_3^6\Bigg). \end{aligned}$$

The last two equations above can be analyzed independently from other equations. Solving for $(\theta_1, \theta_3) \in \mathbb{R}^2$, we find the following solution set

$$\begin{aligned} & \Big\{(0.194071,1.38522),(-0.194071,1.38522),(0.189114,1.31422),(-0.189114,1.31422),(0.57609,0.172628), \\ & (-0.57609,0.172628),(2.59494,-1.3336),(-2.59494,-1.3336),(0.560385,-1.99196),(-0.560385,-1.99196)\Big\}. \end{aligned} \tag{9.44}$$

Since θ_1 and θ_3 are nonzero (as verified above), the following can be acquired through (9.41a)-(9.41d), (9.43b), and (9.43d):

$$\begin{aligned} \theta_2 = {} & \left(\frac{1+\theta_1-\theta_3}{\theta_1}-\frac{3\theta_1(\theta_3-1)\left(-2-\theta_3(-3+\theta_3+\theta_3^3+\theta_3^4)+\theta_1^2(-1+\theta_3(3+\theta_3(1+\theta_3)(-3+\theta_3+3\theta_3^3)))\right)}{-(-1+\theta_3)^2(1+\theta_3(-1+3\theta_3))+\theta_1^4(-1+\theta_3(3+\theta_3(1+\theta_3)(-3+\theta_3+3\theta_3^3)))+\theta_1^2\left(-3+\theta_3\left(7+(\theta_3-1)\theta_3(7+\theta_3(1+\theta_3(6+\theta_3(9+6\theta_3+4\theta_3^2)))))\right)}\right) \\ & \left(1+\frac{(\theta_3-1)^2(1+\theta_3(-1+3\theta_3))-\theta_1^4(-1+\theta_3(3+\theta_3(1+\theta_3)(-3+\theta_3+3\theta_3^3)))+\theta_1^2\left(3-\theta_3(7+(\theta_3-1)\theta_3(7+\theta_3(1+\theta_3(6+\theta_3(9+6\theta_3+4\theta_3^2)))))\right)}{3\theta_1(\theta_3-1)(-2-\theta_3(-3+\theta_3+\theta_3^3+\theta_3^4)+\theta_1^2(-1+\theta_3(3+\theta_3(1+\theta_3(-3+\theta_3+3\theta_3^3)))))}\right)^{-1}, \end{aligned} \tag{9.45a}$$

$$\frac{1}{v} = \frac{1+\theta_1-\theta_3}{\theta_1} - \frac{3\theta_1(\theta_3-1)\Big(-2-\theta_3(-3+\theta_3+\theta_3^3+\theta_3^4)+\theta_1^2(-1+\theta_3(3+\theta_3(1+\theta_3)(-3+\theta_3+3\theta_3^3)))\Big)}{-(\theta_3-1)^2(1+\theta_3(-1+3\theta_3))+\theta_1^4(-1+\theta_3(3+\theta_3(1+\theta_3(-3+\theta_3+3\theta_3^3)))+\theta_1^2(-3+\theta_3(7+(\theta_3-1)\theta_3(7+\theta_3(1+\theta_3(6+\theta_3(9+6\theta_3+4\theta_3^2))))))}. \quad (9.45b)$$

Finally, substituting from (9.44) for (θ_1, θ_3) and using (9.41a)-(9.41d), (9.43e) with (9.45a)-(9.45b), we get the linear noisy REESI coefficients $(\Pi_0, \Pi, \Lambda, \Gamma; \theta_0, \theta_1, \theta_2, \theta_3, v)$ as given in Remark 1. $\square$

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