Common Failings: How Corporate Defaults are Correlated[1]

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Abstract

We develop, and apply to data on U.S. corporations from 1987-2000, tests of the standard doubly-stochastic assumption under which firms' default times are correlated only as implied by correlation of their default intensity processes, for example through dependence on common or correlated observable risk factors. The data do not support the joint hypothesis of well specified default intensities and the doubly-stochastic assumption, although we provide evidence that this may be due to mis-specification of the default intensities, which do not include macroeconomic default-prediction covariates. There is at most weak evidence of default clustering in excess of that implied by the doubly-stochastic model and correlation of the firm-specific default covariates.

1 Introduction

Why do corporate defaults cluster? Several explanations have been explored in the literature. First, firms may be exposed to common or correlated risk factors whose co-movements cause correlated changes in the conditional probabilities of defaults across firms. Second, the event of default by one firm may be "contagious," in that this event itself can push other firms toward default. For example, there could be a "domino" or cascade effect, under which corporate failures directly induce other corporate failures, as with the collapse of Penn Central Railway in 1970. A third channel for default correlation is learning from defaults. For example, the defaults of Enron and WorldCom may have revealed accounting irregularities that could be present in other firms, and thus may have had a direct impact on the conditional default probabilities of other firms.

Our primary objective is to examine whether correlation in default intensities, that is, the first channel on its own, is sufficient to account for the degree of default clustering that we find in the data.

Specifically, we test whether our data are consistent with the standard doubly-stochastic model of default, under which, conditional on the path of risk factors determining all firms' default intensities, defaults are independent Poisson arrivals at these (conditionally deterministic) intensities. This model is particularly convenient for computational and statistical purposes, although its empirical relevance for default correlation has been unresolved. We develop, and apply to default data for U.S. corporations during the period 1987-2000, a test of the doubly-stochastic assumption. We reject this hypothesis, taking as correct our source of conditional default probabilities. We also provide, however, evidence that this rejection may be due to mis-specification of our default probability data, which do not incorporate any direct dependence of default probabilities on macroeconomic covariates that may be responsible for some clustering of defaults. In any case, we do not find substantial evidence of default clustering beyond that predicted by the doubly-stochastic model and our data.

Understanding how corporate defaults are correlated is particularly important for the risk management of portfolios of corporate debt. For example, as backing for the performance of their loan portfolios, banks retain capital at levels designed to withstand default clustering at extremely high confidence levels, such as 99.9%. Some banks do so on the basis of models in which default correlation is captured by common risk factors determining conditional default probabilities, as in those of Gordy [2003] and Vasicek [1987]. (Banks do, however, attempt to capture the effects of contagion that arise from parent-subsidiary and other direct contractual links.) If defaults are more heavily clustered in time than currently envisioned in their default-risk models, then significantly greater capital might be required in order to survive default losses at high confidence levels. An understanding of the sources and degree of default clustering is also crucial for the rating and risk analysis of structured credit products that are exposed to correlated default, such as collateralized debt obligations (CDOs) and options on portfolios of default swaps. The Bank of America has reported that synthetic CDO volumes reached over $500 billion in 2003, an annual growth rate of over 130%.

While there is some empirical evidence regarding the correlation of conditional corporate default probabilities (see, for example, Das, Freed, Geng and Kapadia, [2001]), there is relatively little evidence regarding the presence of highly clustered defaults. Renault and Servigny [2002] have estimated historical average one-year default correlations, but do not address the issue of clustering. Collin-Defresne, Goldstein, and Helwege [2003] find that default events are associated with significant increases in the credit spreads of other firms, consistent with a clustering effect in excess of that suggested by the doubly-stochastic model, or at least a failure of the doubly-stochastic model under risk-neutral probabilities. That is, their findings may be due to default-induced increases in the conditional default probabilities of other firms, or could be due to default-induced increases in default risk premia[1] of other firms, as envisioned by Kusuoka [1999]. Both effects could be at play. Collin-Dufresne, Goldstein, and Helwege do not disentangle these two channels for default-induced widenings of spreads. Explicitly considering a failure of the doubly-stochastic hypothesis. Collin-Defresne, Goldstein, and Helwege [2003], Giesecke [2002], Jarrow and Yu [?], and Schönbucher [2004] explore learning-from-default interpretations, based on the statistical modeling of frailty, under which default intensities include unobservable covariates. In a frailty setting, the arrival of a default causes, via Bayes' Rule, a jump in

[1]Collin-Dufresne, Goldstein, and Huggonier [2002] provide a simple method for incorporating the pricing impact of failure, under risk-neutral probabilities, of the doubly-stochastic hypothesis. Other theoretical work on the impact of contagion on default pricing includes that of Cathcart and El Jahel [2002], Davis and Lo [2000], Giesecke [2002], Kusuoka [1999], Schönbucher and Schubert [2001], Teremtyev [2003], Yu [2003], and Zhou [2001].

the conditional distribution of hidden covariates, and therefore a jump in the conditional default probabilities of any other firms whose default intensities depend on the same unobservable covariates. For example, the collapses of Enron and WorldCom could have caused a sudden reduction in the perceived precision of accounting leverage measures of other firms. Indeed, Yu [2004] finds that, other things equal, a reduction in the measured precision of accounting variables is associated with a widening of credit spreads. Lang and Stulz [1992] explore evidence of default contagion in equity prices.

Before describing our data, methods, and results in detail, we offer a brief synopsis. Our data on actual default times and on monthly estimates of conditional probabilities of default within one year (PDs) were provided to us by Moodys, and cover the period January, 1987 to October, 2000. These data are described in Section 3, with further details in Appendix A. After dropping firms for which we had missing data, we were left with 241 individual issuer defaults among a total of 1,990 firms over 216,859 firm-months of data.

From the time-series of PD data for each firm, we estimate default intensities for each firm, using a simple time-series model of intensities. For this, we assume that the default intensity process for each firm is a Feller diffusion (also known as a Cox-Ingersoll-Ross process, or a square-root diffusion). The fitting procedure is explained in Section 3.2. The current intensity level measured from the one-year default probability is relatively robust to mis-specification, since intensities and one-year conditional default probabilities are relatively close for a wide range of alternative intensity models and reasonable paramaters.

We then exploit the following result, demonstrated in Section 2. Consider a change of time scale under which the passage of one unit of "new time" coincides with a period of calendar time over which the cumulative total of all firms' default intensities increases by one unit. Under the doubly-stochastic assumption, and under this new time scale, the cumulative number of defaults to date defines a standard (constant mean arrival rate) Poisson process. For example, fixing any scalar $c > 0$, if we define successive non-overlapping time intervals each lasting for c units of new time (corresponding to periods that include an accumulated total default intensity, across all firms, of c), the doubly-stochastic assumption implies that the number of defaults in the successive time intervals (X_1 defaults in the first interval lasting for c units, X_2 defaults in the second interval, and so on), are independent Poisson distributed random variables with mean c. This time-changed Poisson-process property is the basis for most of our tests, outlined as follows.

1. We apply a Fisher dispersion test for consistency of the emprical distribution of the numbers $X_1, X_k, \ldots$ of defaults in succssive time bins of a given accumulated intensity size c, with the theoretical Poisson distribution associated with the doubly-stochastic model.

2. We test whether the mean of the upper quartile of our sample $X_1, X_2, \ldots, X_K$ of numbers of defaults in successive time bins of a given size c is significantly larger than the mean of the upper quartile of a sample of like size drawn from the Poisson distribution with parameter c. An analogous test is based on the median of the upper quartile. These tests are designed to detect default clustering in excess of that implied by the default intensities and the doubly-stochastic assumption. We also extend this test to be applicable across all bin sizes.

3. Fixing the size c of time bins, we test for serial correlation of $X_1, X_2, \ldots$ by fitting an autoregressive model. The presence of serial correlation would imply a failure of the independent-increments property of Poisson processes, and, if the serial correlation is positive, could lead to default clustering in excess of that associated with the doubly-stochastic assumption.

4. In order to avoid reliance on specific bin sizes, we provide the results of a test due to Prahl [1999] for clustering of default arrival times (in our new time scale) in excess of that associated with a Poisson process.

We find the data broadly consistent with a rejection of the joint hypothesis of correctly specified intensities and the doubly-stochastic hypothesis, at standard confidence levels. We also test for the presence of missing covariates in the PD model, which was estimated from only firm-specific covariates such as leverage, asset volatility, and credit rating. We are especially concerned about missing covariates that might be associated with default clustering, such as business-cycle variables. Indeed, we find evidence, in some tests, that certain macroeconomic business-cycle variables should probably have been included as default-prediction covariates. For example, the number of defaults in a given bin, in excess of its conditional mean, is in theory uncorrelated with any variables in the information set of the observer before the time bin begins. Among other related results, however, we find some evidence of correlation between X_k, the number of defaults in bin k, and macroeconomic variables such as U.S. GDP growth that were observed before bin k begins. It is possible that missing covariates, rather than a failure of the doubly-stochastic property, is responsible for the relatively poor fit of the data to the joint hypothesis that we test.

The rest of the paper comprises the following. In Section 2, we derive the property that the total default arrival process is a Poisson process with constant intensity under a time rescaling based on default intensity accumulation. This property is the basis for our test statistics. Section 3 describes our data, comprising default probabilities and default times over a period of fourteen years. This section also describes the conversion of default probabilities into intensities. Section 4 provides various tests of the doubly-stochastic hypothesis, and Section 5 looks at the independence of increments in the process governing default arrival. In Section 6 we undertake tests for missing covariates with a view to assessing whether the data on default probabilities is mis-specified. Section 7 concludes. The appendices contain further details on the data and estimation procedures.

2 Time Rescaling for Poisson Defaults

In this section, we define the doubly-stochastic default property that rules out default correlation beyond that implied by correlated default intensities, and provide some testable implications of this property.

We fix a probability space $(\Omega, \mathcal{F}, P)$ and an observer's information filtration $\{\mathcal{F}_t : t \geq 0\}$, satisfying the usual conditions. This and other standard technical definitions that we rely on may be found in Protter [2003]. We suppose that, for each firm i of n firms, default occurs at the first jump time τ_i of a non-explosive counting process N_i with stochastic intensity process λ_i. (Here, N_i is $(\mathcal{F}_t)$-adapted and λ_i is $(\mathcal{F}_t)$-predictable.)

The key question at hand is whether the joint distribution of, in particular any correlation among, the default times $\tau_1, \ldots, \tau_n$ is determined by the joint distribution of the intensities. Violation of this assumption means, in essence, that even after conditioning on the default intensities of all firms, the times of default can be correlated.

A standard version of the assumption that default correlation is captured by co-movement in default intensities is the assumption that the multi-dimensional counting process $N = (N_1, \ldots, N_n)$ is doubly stochastic. That is, conditional on the path $\{\lambda_t = (\lambda_{1t}, \ldots, \lambda_{nt}) : t \geq 0\}$ of all intensity processes, as well as the information $\mathcal{F}_T$ available at any given stopping time T, the counting processes $\hat{N}_1, \ldots, \hat{N}_n$, defined by $\hat{N}_i(u) = N_i(u+T)$, are independent Poisson processes with respective (conditionally deterministic) intensities $\hat{\lambda}_1, \ldots, \hat{\lambda}_n$ defined by $\hat{\lambda}_i(u) = \lambda_i(u+T)$. In this case, we also say that $(\tau_1, \ldots, \tau_n)$ is doubly-stochastic with intensity $(\lambda_1, \ldots, \lambda_n)$. In particular, the doubly-stochastic assumption implies that the default times $\tau_1, \ldots, \tau_n$ are independent given the intensities.

We will test the following key implication of the doubly stochastic assumption.

Proposition. *Suppose that* $(\tau_1, \ldots, \tau_n)$ *is doubly stochastic with intensity* $(\lambda_1, \ldots, \lambda_n)$*. Let* $K(t) = \#\{i : \tau_i \leq t\}$ *be the cumulative number of defaults by* t*, and let* $U(t) = \int_0^t \sum_{i=1}^n \lambda_i(u) 1_{\tau_i > u}\, du$ *be the cumulative aggregate intensity of surviving firms, to time* t*. Then* $J = \{J(s) = K(U^{-1}(s)) : s \geq 0\}$ *is a Poisson process with rate parameter 1.*

Proof: Let $S_0 = 0$ and $S_j = \inf\{s : J(s) > J(S_{j-1})\}$ be the jump times, in the new time scale, of J. By Billingsley [1986], Theorem 23.1, it suffices to show that the inter-jump times $\{Z_j = S_j - S_{j-1} : j \geq 1\}$ are *iid* exponential with parameter 1. Let $T(j) = \inf\{t : K(t) \geq j\}$. By construction,

$$Z_j = \int_{T_{j-1}}^{T_j} \sum_{1=1}^{n} \lambda_i(u) 1_{\tau_i > u}\, du.$$

By the doubly-stochastic assumption, given $\{\lambda_t = (\lambda_{1t}, \ldots, \lambda_{nt}) : t \geq 0\}$ and $\mathcal{F}_{T_j}$, we know that $\tilde{N}_{j+1} = \{\tilde{N}(u) = \sum_{1=1}^n N_i(u+T_j) 1_{\tau_i > T_j}\, du,\ u \geq T_j\}$ is a sum of independent Poisson processes, and therefore itself a Poisson process, with intensity $\tilde{\lambda}_{j+1}(u) = \sum_{1=1}^n \lambda_i(u+T_j) 1_{\tau_i > T_j}\, du$. Thus Z_{j+1} is exponential with parameter 1.

In order to check the independence of $Z_1, Z_2 \ldots$, consider any integer $k > 1$ and any bounded Borel functions $f_1, \ldots, f_k$. By the doubly-stochastic property and the law of iterated expectations, applied recursively,

$$E[f_1(Z_1) f(Z_2) \cdots f_{k-1}(Z_{k-1}) f_k(Z_k)]$$

$$
\begin{aligned}
&= E[f_1(Z_1)f(Z_2)\cdots f_{k-1}(Z_{k-1})E[f_k(Z_k)\,|\lambda, \mathcal{F}_{T_{k-1}}]] \\
&= E[f_1(Z_1)f(Z_2)\cdots f_{k-1}(Z_{k-1})]\int_0^\infty f_k(z)e^{-z}\,dz \\
&\vdots \\
&= \prod_{i=1}^{k}\int_0^\infty f_i(z)e^{-z}\,dz.
\end{aligned}
$$

Thus, $Z_1, Z_2 \ldots$ are indeed independent, and J is a Poisson process with parameter 1, completing the proof.

Using this result, some of the properties of the doubly-stochastic assumption that we shall test are based on the following characterization.

Poisson property: *For any $c > 0$, the random variables*

$$J(c), J(2c) - J(c), J(3c) - J(2c), \ldots$$

are iid Poisson with parameter c.

That is, if we divide our sample period into "bins" that each have an equal cumulative aggregate intensity of c, then we can test the doubly stochastic assumption by testing whether the number of defaults in each bin is distributed Poisson with parameter c.

3 Data

Our empirical tests are based on a dataset of default probabilities and default events, both of which are obtained from Moody's Investor Services.

3.1 Description of the Data

The data on default probabilities consists of a monthly time-series of estimated conditional one-year default probabilities for public non-financial North American firms over the period January, 1987 to October, 2000. These default probabilities are the output of a logit model estimated from the history of firm-specific financial covariates and default times. A key covariate is the 'distance-to-default' measure suggested by the Merton [1974] model, which is an estimate of the number of standard deviations of annual asset growth by which assets exceed a measure of book liabilities. Other covariates include financial statement information and Moody's rating, when available. Details of the model and its econometric fit and performance are described in Sobehart, Stein, Mikityanskaya and Li [2000] and Sobehart Keenan and Stein [2000]. This database of estimated default probabilities was part of Moody's *RiskCalc* system. (Moody's subsequently distributed a related default probability estimate, the Moody's KMV EDF, also based on distance to default.)

Key advantages of this PD dataset include: *(i)* it is relatively comprehensive, and *(ii)* it is consistent with Moody's database of historical defaults over the sample period. In particular, the database, covering 1,990 firms, includes almost all firms that have been rated by Moody's over this period.

Using a separate database of defaults also obtained from Moody's, we identify a total of 241 defaults of the rated firms in our database. As the default probabilities and defaults are from separate databases, much of the matching is done manually by matching company names. Given that the default probabilities have been computed by fitting to observed defaults, we can verify the completeness of the matching by comparing the mean default rate implied by the default probabilities to the actual number of defaults. We discuss this in more detail in our analysis below. Appendix A provides further details on the construction of the database.

Figure 1 shows a plot of the monthly cross-sectional sample mean of estimated one-year conditional default probabilities. The plot shows evidence of positive correlation of default intensities, in that the cross-sectional mean one-year conditional probability of default ranges from 0.69% to 3.11%, and increases markedly with the U.S. recesssion that occurred around 2000-2001. The number of firms in our sample at a given time increases from a low of 1,081 firms at the beginning of the sample period in 1987 to a high of 1,554 firms in the second half of 1998. Figure 2 shows a plot of the number of defaults over this period, month by month, ranging from 0 to a maximum of 8 per month, as well as a plot of the total of the estimated default intensities of all sampled firms. We turn next to the estimation of these intensities from one-year default probabilities.

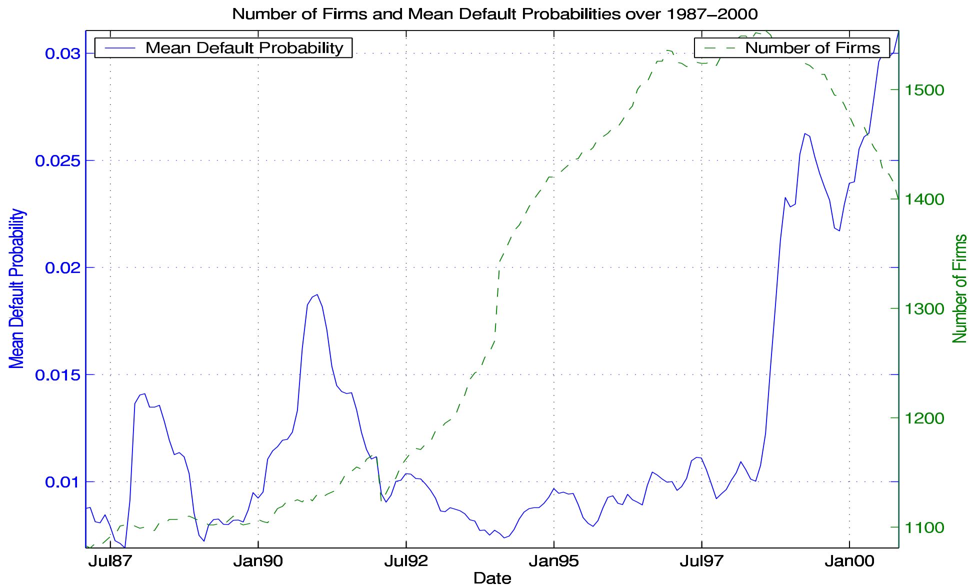

Figure 1: Cross-sectional sample mean of one-year default probabilities, and the number of firms covered, January 1987 to October 2000.

3.2 From PDs to Intensities

In order to test the doubly-stochastic assumption using the new-time-scale Poisson process described in Proposition above, we estimate default intensities, firm by firm, from the PD data on one-year default probabilities, as follows.

For a given firm, the default intensity process λ is assumed to satisfy a stochastic differential equation of the form

$$d\lambda_t = k(\theta - \lambda_t)\, dt + \sigma\sqrt{\lambda_t}\, dz_t, \tag{1}$$

where z is a standard Brownian motion, and where k, θ, and σ are positive numbers. The doubly-stochastic assumption implies that the T-maturity survival probability at time t, for a currently surviving firm, is

$$s_t(T) = E\left[\exp\left(-\int_t^{t+T} \lambda_u\, du\right) \,\middle|\, \lambda_t\right]. \tag{2}$$

Cox, Ingersoll, and Ross [1985] have provided the well-known solution:

$$s_t(T) = A(T)\exp\left[-\lambda_t B(T)\right], \tag{3}$$

where

$$A(T) = \left(\frac{2\gamma e^{(k+\gamma)T/2}}{(k+\gamma)(e^{\gamma T}-1)+2\gamma}\right)^{\frac{2k\theta}{\sigma^2}} \quad (4)$$

$$B(T) = \frac{2e^{\gamma T}-1}{(k+\gamma)(e^{\gamma T}-1)+2\gamma} \quad (5)$$

$$\gamma = \sqrt{k^2+2\sigma^2}. \quad (6)$$

Inverting equation (3), we get, for any time horizon T,

$$\lambda_t = -\frac{1}{B(T)} \ln\left[\frac{s_t(T)}{A(T)}\right]. \quad (7)$$

Our PD data are data are monthly observations of the one-year default probability, $1 - s_t(1)$. We estimate the parameters $\{k, \theta, \sigma\}$, and the default intensities of each firm, by a method-of-moments estimator provided in Appendix B. The estimator matches the time-series behavior of λ_t implied by the Feller diffusion, using the relationship between default intensity and PD given by (7). Maximum likelihood estimation has also been used in similar settings, and is efficient in large samples, but is notoriously biased in small samples. Our method-of-moments estimator is robust and computationally efficient, usually able to fit a given firm's default intensity model in a couple of seconds. In any case, the fit is relatively robust to mis-specification of the time-series model and to fitting error, as intensities are relatively close to one-year default probabilities. Figure 2 shows the total of the estimated intensities of all firms, as well as the monthly arrivals of defaults.

4 Goodness-of-Fit Tests

Having estimated default intensities λ_{it} for each firm i and each date t (with λ_t taken to be constant within months), and letting $\tau(i)$ denote the default time of name i, we let $U(t) = \int_0^t \sum_{i=1}^n \lambda_{is} 1_{\tau(i)>s}\, ds$ be the total accumulative default intensities of all surviving firms. Fixing time bins containing c units of accumulative default intensity, we then construct calendar times $t_0, t_1, t_2, \ldots$ such $t_0 = 0$ and $U(t_i) - U(t_{i-1}) = c$, and let $X_k = \sum_{i=1}^n 1_{t_k \le \tau(i) < t_{k+1}}$ denote the number of defaults in the k-th time bin. Figure 3 illustrates the the time bins of size $c = 8$ over the last five calendar years of our data set.

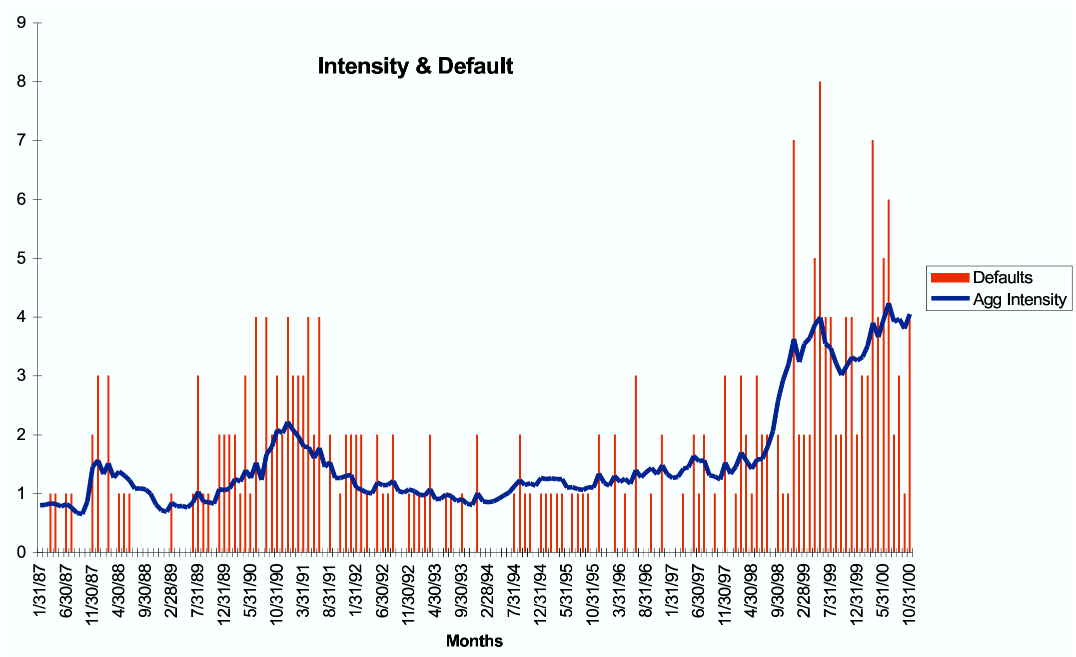

Figure 2: Aggregate (across firms) default intensities and firm defaults from 1987-2000.

Table 1 presents a comparison of the empirical and theoretical moments of the distribution of defaults per bin, for each of several bin sizes.[2] The actual bin sizes differ very slightly from the integer bin sizes shown because of daily granularity in the construction of the binning times $t_1, t_2, \ldots$. The approximate match between a bin size and the associated sample mean $(X_1 + \cdots + X_K)/K$ of the number of defaults per bin offers some confirmation that the underlying PD data are reasonably well estimated, however this is to be expected given the within-sample nature of the estimates. For larger bin sizes, the empirical variances are bigger than their theoretical counterparts under the null of correctly specified doubly-stochastic intensity model of defaults.

Figure 4 presents the observed default frequency distribution, and the associated theoretical Poisson distribution, for bin sizes 2 and 8. For bin sizes 4 and 8, there is a tendency for bi-modality (two peaks), as opposed to

[2]Under the Poisson distribution, $P(X_i = k) = \frac{e^{-c}c^k}{k!}$. The associated moments of X_k are a mean and variance of c, a skewness of $c^{-0.5}$, and a kurtosis of $3 + c^{-1}$.

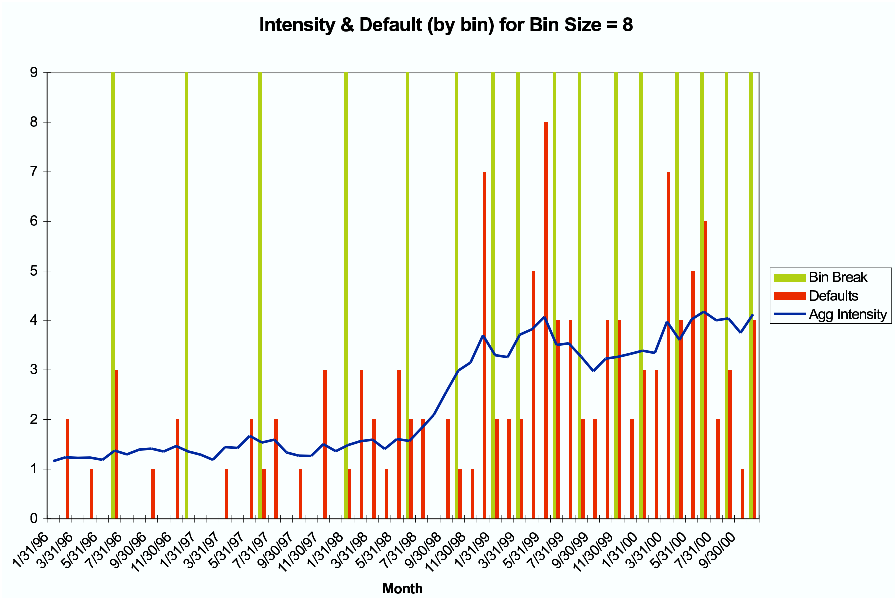

Figure 3: Aggregate intensities and defaults by month, 1996-2000, with time bin delimiters marked for intervals that include a total accumulated default intensity of $c = 8$ per bin.

the uni-model theoretical Poisson distribution associated with the hypothesis of doubly-stochastic defaults.

4.1 Fisher's Dispersion Test

Our first goodness-of-fit test of the hypothesis of correctly measured default intensities and the doubly-stochastic property is Fisher's dispersion test of the agreement of the empirical distribution of defaults per bin, for a given bin size c, to the theoretical Poisson distribution with parameter c.

Fixing the bin size c, a simple test of the null hypothesis that $X_1, \ldots, X_K$ are independent Poisson distributed variables with mean parameter c is Fisher's dispersion test (Cochran [1954]). Under this null,

$$W = \sum_{i=1}^{K} \frac{(X_i - c)^2}{c}, \tag{8}$$

is distributed as a χ^2 random variable with $K - 1$ degrees of freedom. An

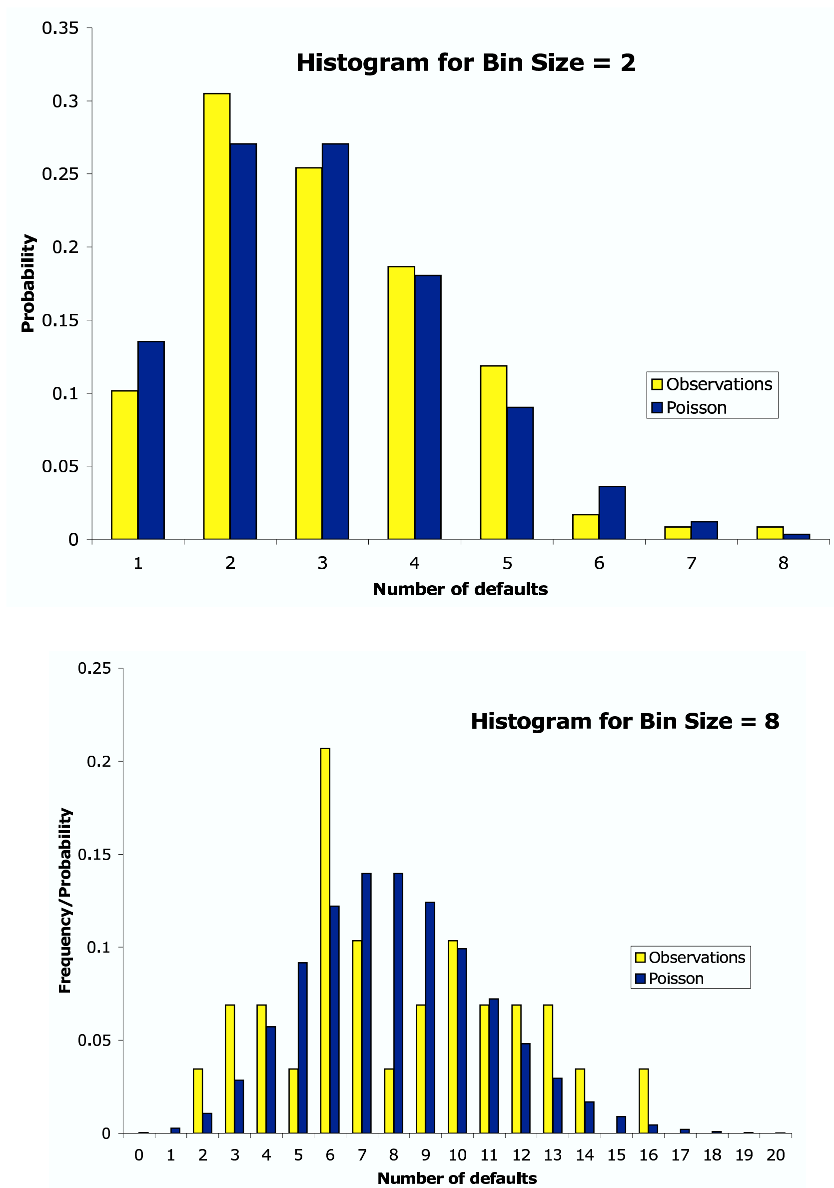

Figure 4: Comparison of the empirical and Poisson distributions of defaults for bin sizes 2 and 8.

Table 1: Comparison of empirical and theoretical moments for the distribution of defaults per bin. The number of bin observations is shown in parentheses under the bin size. The upper-row moments are those of the theoretical Poisson distribution under the doubly-stochastic hypothesis; the lower-row moments are the empirical counterparts.

Bin Size	Mean	Variance	Skewness	Kurtosis
2	2.00	2.00	0.71	3.50
(118)	2.04	1.89	0.71	3.52
4	4.00	4.00	0.50	3.25
(59)	4.07	4.00	0.41	2.06
6	6.00	6.00	0.41	3.17
(39)	6.08	8.07	0.41	2.19
8	8.00	8.00	0.35	3.12
(29)	8.14	13.12	0.26	2.07
10	10.00	10.00	0.32	3.10
(24)	10.04	15.43	0.82	2.25

outcome for W that is large relative to a χ^2 random variable of the associated number of degrees of freedom would cause a small p-value, meaning a surprisingly large amount of clustering if the null hypothesis of doubly stochastic default (and correctly specified conditional default probabilities) applies. The p-values shown in Table 2 indicate that, at standard confidence levels such as 95%, there is a borderline rejection of this null hypothesis for bin sizes 6 and 10.

4.2 Upper tail tests

If defaults are more positively correlated than would be suggested by the co-movement of intensities, then the upper tail of the empirical distribution of defaults per bin could be fatter than that of the associated Poisson distribution. We use a Monte Carlo test of the "size" (mean or median) of the upper quartile of the empirical distribution against the theoretical size of the upper quartile of the Poisson distribution, as follows.

For a given bin size c, suppose there are K bins. We let M denote the

Table 2: Fisher's dispersion test for goodness of fit of the Poisson distribution with mean equal to bin size. Under the joint hypothesis that default intensities are correctly measured and the doubly-stochastic property, W is χ^2-distributed with $K-1$ degrees of freedom.

Bin Size	K	W	p-value
2	118	110.5	0.65
4	59	58.0	0.47
6	39	51.2	0.07
8	29	46.0	0.02
10	24	35.5	0.05

sample mean of the upper quartile of the empirical distribution of distribution of $X_1, \ldots, X_K$. By Monte Carlo simulation, we generated 10,000 data sets, each consisting of K *iid* Poisson random variables with parameter c. We then compute the fraction p of the simulated data sets whose sample upper-quartile size (mean or median) is above the actual sample mean M. Under the null hypothesis that the distribution of the actual sample is Poisson with parameter c, the p-value would be approximately 0.5.

The sample p-values are presented in Table 3, and suggest, for larger bin sizes, fatter upper-quartile tails than those of the theoretical Poisson distribution. (That is, our one-sided tests implies rejection for larger bins of the null joint hypothesis, at typical confidence levels.)

We corroborated these results with an analysis of the tail distributions using the Pearson χ^2 statistic for the theoretical tail distribution associated with the corresponding theoretical Poisson distribution. The results (not reported) imply a strong rejection of a Poisson-distributed upper-quartile distribution at standard confidence levels.

4.3 Prahl's Test of Clustered Defaults

Fisher's dispersion and our tailored upper-tail test do not exploit well the information available across all bin sizes. In this section, we apply a test for "bursty" default arrivals due to Prahl [1999]. Prahl's test is sensitive to cluster-like deviations from the theoretical Poisson process. This test is

Table 3: Tests of median and mean of the upper upper quartile of defaults per bin, against the associated theoretical Poisson distribution. The last line in the table, denoted "All" is the probability, under the hypothesis that time-changed default arrivals are Poisson with parameter 1, that there exists at least one bin size for which the mean (or median) of number of defaults per bin exceeds the corresponding empirical mean (or median).

Bin	Mean of Tails		p-value	Median of Tails		p-value
Size	Data	Simulation		Data	Simulation	
2	3.62	3.63	0.58	3.00	3.18	0.25
4	6.71	6.25	0.21	6.00	5.90	0.17
6	10.00	8.81	0.05	9.50	8.42	0.07
8	12.75	11.12	0.03	12.50	10.69	0.03
10	16.00	13.71	0.02	16.50	13.26	0.00
All			0.70			0.44

particularly suited for detecting clustering of defaults that may arise from more default correlation than would be suggested by co-movement of default intensities alone.

Prahl's test statistic is based on the fact that, in the new time scale under which default arrivals are those of a Poisson process (with rate parameter 1), the inter-arrival times $Z_1, Z_2, \ldots$ are *iid* exponential of mean 1. (Because of date granularity, our mean is slightly larger than 1.) The sample moments of these time-rescaled inter-arrival times are provided in Table 4.

Letting C^* denote the sample mean of $Z_1, \ldots, Z_n$, Prahl shows that

$$M = \frac{1}{n} \sum_{\{Z_k < C^*\}} \left(1 - \frac{Z_k}{C^*}\right). \tag{9}$$

is asymptotically (in n) normally distributed with mean $e^{-1} - \alpha/n$ and variance β^2/n, where

$$\begin{aligned} \alpha &\simeq 0.1839 \\ \beta &\simeq 0.2431. \end{aligned}$$

Using our data, for $n = 240$ default times,

$$M = 0.3681$$

Table 4: Selected moments of the distribution of cumulative amount of intensities between successive default times. Under the joint hypothesis of doubly-stochastic defaults and correctly measured default intensities, the distribution is exponential.

Moment	Empirical	Exponential
Mean	1.07	1.07
Variance	1.19	1.16
Skewness	2.13	2.00
Kurtosis	7.46	6.00

$$\begin{aligned}\mu(M) &= \frac{1}{e} - \frac{\alpha}{n} = 0.3671 \\ \sigma(M) &= \frac{\beta}{\sqrt{n}} = 0.0156.\end{aligned}$$

Because the test statistic M is less that one tenth of a standard deviation from the associated asymptotic mean, this test provides no notable evidence of default clustering in excess of that associated with the default intensities under the doubly stochastic model.

We also report a direct Kolmogorov-Smirnov goodness-of-fit test of goodness of fit the exponential distribution of inter-default times in the new time scale. The associated K-S statistic is 1.8681, for a p-value of only 0.002, leading to a strong rejection of the joint hypothesis of corrrectly specified conditional default probabilities and the doubly-stochastic nature of correlated default. Figure 5 shows the empirical distribution of inter-default times after scaling time change by total intensity of defaults, compared to the exponential density implied by the doubly-stochastic model.

In summary, Prahl's test does not indicated default clustering in excess of what would be implied by the doublly stochastic property and co-movement of the default intensities.

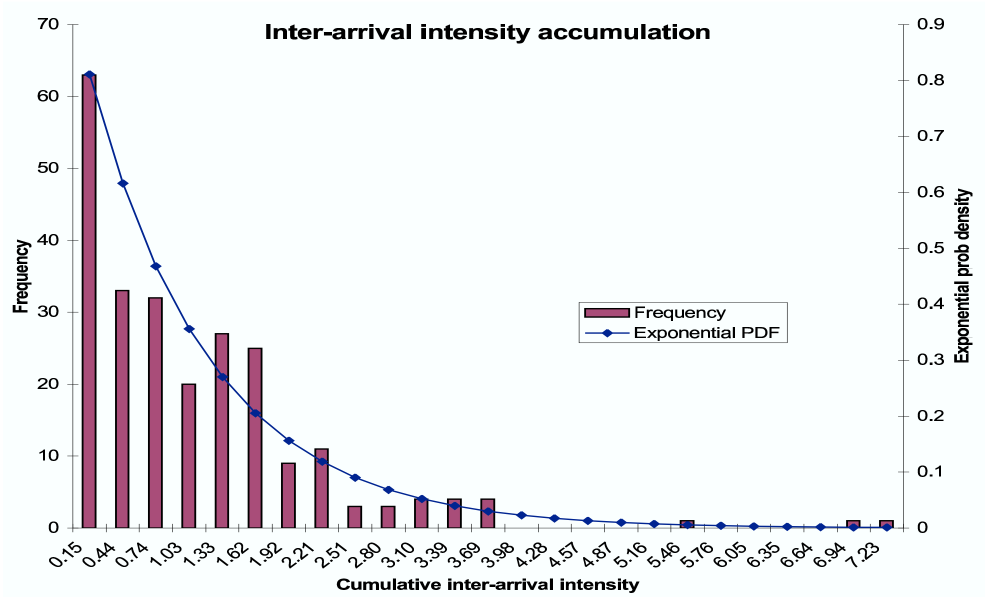

Figure 5: The empirical distribution of inter-default times after scaling time change by total intensity of defaults, compared to the exponential density implied by the doubly-stochastic model.

5 Testing for Independent Increments

Although all of the above tests depend in part on the independent-increments property of Poisson processes, we will test specifically for serial correlation of the number of defaults in successive bins. That is, under the null hypothesis of doubly-stochastic defaults, fixing an accumulative total default intensity of c per time bin, the number of defaults $X_1, X_2, \ldots$ in successive bins are independent and identically distributed. We test for independence by estimating an auto-regressive model for $X_1, X_2, \ldots$, allowing for $X_k = A + BX_{k-1} + \epsilon_k$, for coefficients A and B, and for *iid* innovations $\epsilon_1, \epsilon_2 \ldots$. A large and positive auto-regressive coefficient B would be evidence of a failure of the null hypothesis. Such a failure in small bins could generate the effect of fat tails in larger bins, and could be responsible for the apparent rejections of the Poisson distribution in the larger bins that we reported earlier. Such a failure could perhaps be evidence of mis-specification of the underlying PD model, for example through missing covariates for default prediction. That is, serial correlation of missing covariates could cause an appearanc default clustering

Table 5: Results for AR(1) regressions of defaults of consecutive bins, for each of a range of bin sizes.

Bin Size	No. of Bins	A (t_A)	B (t_B)	R^2 AR(1)
2	118	1.73	0.16	0.03
		(7.66)	(1.72)	0.16
4	59	2.72	0.34	0.12
		(4.83)	(2.73)	0.34
6	39	4.20	0.32	0.10
		(3.97)	(2.01)	0.32
8	29	6.68	0.19	0.03
		(3.83)	(0.96)	0.18
10	24	6.09	0.39	0.15
		(2.75)	(1.93)	0.39

in excess of that implied by the doubly-stochastic property, even if in fact the true default-time model is doubly stochastic.

Table 5 presents the results of this autocorrelation analysis. The AR(1) coefficient is always positive, and sometimes sigificantly larger than zero at traditional confidence levels. Indeed, the auto-regressive coefficient B tends to be "more significant" for small bin sizes, consistent with an interpretation of the earlier failure of the test for Poisson distributed upper tails in large bins as potentially due to a failure of the independence assumption for small bins, and perhaps mis-specification of the underlying intensity model.

6 Tests for Missing Default Covariates

Our lack of support for the joint hypothesis of correctly specified default probabilities and the doubly stochastic property might be related to missing covariates in the PD default-prediction model, a logit-based model that uses only firm-specific covariates. In particular, this default prediction model may be missing covariates that are common to many firms, and would therefore reveal additional default time correlation under the doubly-stochastic model.

Prior work by Shumay [2001], Lennox [1999], Lo [1986], and Duffie and

Ke [2003] indeed suggests that macro-economic performance is an important explanatory variable in default preduction. Among these prior studies, Duffie and Ke included distance to default, the key covariate in Moody's PD model, and found significant additional dependence of default intensities on U.S. personal income growth, for the U.S. machinery and instruments sector for 1971 to 2001.

In this section, we explore the potential role of two macro-economic variables, United States G.D.P. growth rate (GDP) and personal income growth rate (PI). In particular, we examine *(i)* whether the inclusion of these macro-economic variables helps predict defaults in addition to the default intensities, and if so, *(ii)* whether these variables can potentially explain the apparent failure of the doubly-stochastic assumption.

We first examine whether the default intensities based on Moody's default intensities indeed indicate mis-specification from lack of a macro-economic covariate. Under the null hypothesis of no mis-specification, fixing a bin size of c, the number of defaults in a bin in excess of the mean, $Y_k = X_k - c$, is the increment of a martingale, and should therefore be uncorrelated with any variable in the information set available prior to the formation of the k-th bin. Consider the regression,

$$Y_k = \alpha + \beta_1 PI_k + \beta_2 GDP_k + \epsilon_k, \tag{10}$$

where PI_k and GDP_k are the growth rates of U.S. personal income and U.S. growth in gross domestic product observed in the quarter immediately prior to the beginning of the k-th bin. Under the null hypothesis of correct specification, *whether or not the doubly-stochastic assumption is satisfied*, the coefficients β_1 and β_2 are in theory equal to zero. Table 6 reports estimated regression results for a range of bin sizes.

We report the results for the multiple regression as well as for each of the variables separately. For bin sizes of both 2 and 10, the coefficient for GDP growth rate is significant at the 99% level. For each of the bins, the signs of the coefficients in the single equation regressions are negative as one would expect under a mis-specification of missing macro-economic variables. That is, significantly more than the number of defaults predicted by the PD model occur when GDP and personal income growth rates are lower than normal. Overall, there appears to be at least some mild evidence of mis-specification. Given the persistence of macro-economic performance across time, these missing covariates may also be partly responsible for the presence of the apparent auto-correlation in $X_1, X_2, \ldots$ that we reported earlier. One may therefore wish to consider whether any excess clustering of defaults (beyond that implied by the doubly-stochastic property) is related to this potential mis-specification of the default intensity processes. With the presently available data, we are unable to disentangle the role of missing covariates from any potential for contagion for the apparently fat tailed distribution of defaults per bin.

Table 7 specifically tests whether the excess upper quartile defaults (defined as the mean of the upper quartile less the mean of the upper quartile for the Poisson distribution of parameter c) examined previously in Table 3 are correlated with the personal income and GDP growth rates. We report two sets of regressions, the first set based on the prior period's macro-economic variables and the second set based on the growth rates observed within the bin-period.[3]

As for these upper-tail-size regressions, the estimated coefficients for PI and GDP based on the prior period's growth rates are not significant at typical confidence levels. The coefficient for current-period PI for bin-size 4, however, is significant at typical confidence levels, and has a sign consistent with the presence of mis-specification by failure to include macro-economic performance variables in prediction of default.

7 Concluding Comments

Defaults cluster in time both because firms' default intensity processes are correlated and also perhaps because, even after conditioning on these intensities, default occurrence is correlated through additional channels such as contagion and frailty. The latter channels are not admitted in a doubly-stochastic setting. By a time change that reduces the process of cumulative defaults to a standard Poisson process, we provide test statistics of the joint hypothesis that default intensities are correctly measured and the doubly-stochastic property. We are particularly interested in whether defaults are indeed independent given intensities. We believe this to be the first such empirical test. For several types of tests, we reject (at traditional confidence levels) the null of correctly measured intensities and the doubly-stochastic

[3]The within-period growth rates are computed by compounding over the daily growth rates that are consistent with the reported quarterly growth rates.

Table 6: Macroeconomic Variables and Default Intensities. For each bin size c, OLS-estimated coefficients are reported for regression of the number of defaults in excess of the mean, $Y_k = X_k - c$, on the previous quarter's personal income growth rate and the GDP growth rate. The number of observations is the number of bins of size c. Standard errors are corrected for heteroskedasticity; t-statistics are reported in parentheses.

Bin Size	No. Bins	Intercept	Personal Income	GDP	R^2 (%)
2	118	0.10	-11.15		0.37
		(0.58)	(-0.65)		
		0.49		-14.13	5.65
		(2.71)		(-3.03)	
		0.46	27.37	-19.74	6.98
		(2.45)	(1.33)	(-3.35)	
4	59	0.25	-25.39		0.93
		(0.56)	(-0.63)		
		0.76		-21.16	5.74
		(1.53)		(-1.76)	
		0.74	18.55	-24.85	6.07
		(1.43)	(0.41)	(-1.75)	
6	39	0.53	-56.34		2.14
		(0.69)	(-0.73)		
		1.26		-34.38	7.76
		(1.45)		(-1.58)	
		1.24	6.37	-35.53	7.78
		(1.37)	(0.08)	(-1.49)	
8	29	1.06	-28.35		2.93
		(0.74)	(-0.76)		
		0.13		-2.65	0.00
		(0.12)		(-0.03)	
		0.97	65.44	-42.16	4.28
		(0.67)	(0.59)	(-0.91)	
10	24	1.15	-127.60		6.10
		(0.78)	(-0.92)		
		2.62		-71.44	18.99
		(1.76)		(-1.97)	
		2.57	44.67	-81.00	19.39
		(1.61)	(0.37)	(-2.53)	

Table 7: Upper-tail regressions. For each bin size c, OLS-estimated coefficients are shown for regression of the number of defaults observed in the upper quartile less the mean of the upper quartile of the theoretical distribution (with Poisson parameter equal to the bin size) on the previous and current personal income (PI) and GDP growth rates. The number of observations is the number K of bins. Standard errors are corrected for heteroskedasticity; t-statistics are reported in parentheses.

Bin Size	K	Intercept	Previous Qtr PI	Previous Qtr GDP	R^2 (%)
2	40	-0.05	5.23		0.35
		(-0.71)	(0.49)		
		-0.10		3.47	1.26
		(-0.75)		(0.84)	
		-0.09	-7.43	5.54	1.52
		(-0.71)	(-0.48)	(0.89)	
4	17	0.64	-25.07		10.78
		(2.42)	(-1.53)		
		0.67		-8.30	9.64
		(2.52)		(-1.48)	
		0.67	-16.99	-3.52	11.39
		(2.42)	(-0.67)	(-0.39)	

Bin Size	K	Intercept	Current Bin PI	Current Bin GDP	R^2 (%)
2	40	-0.00	-0.94		0.01
		(-0.03)	(-0.09)		
		-0.09		3.13	0.97
		(-0.63)		(0.59)	
		-0.08	-20.53	8.66	2.90
		(-0.58)	(-0.65)	(0.71)	
4	17	0.69	-32.96		15.71
		(2.86)	(-2.85)		
		0.55		-4.26	2.02
		(2.42)		(-0.68)	
		0.55	-72.94	17.45	26.50
		(2.56)	(-2.70)	(1.60)	

property, at traditional confidence levels. We present some evidence, however, of potential mis-specification of these default probability estimates, in that they do not include business-cycle covariates that may offer some predictive power for default above and beyond the role of firm-specific covariates. Moreover, there is at best weak evidence of highly clustered defaults, after controlling for co-movement in intensities by a time change.

The economic impact of a failure of the doubly-stochastic property for the risk management of credit portfolios is of critical interest to investors and bank regulators. For example, the level of economic capital necessary to support levered portfolios at high confidence levels is heavily dependent on the degree to which this often-assumed property actually applies. This is especially the case in light of the upcoming changes in bank capital regulations under the proposed Basel II accord on regulatory capital (see Gordy [2003], Allen and Saunders [2003], and Kayshap and Stein [2004]).

Appendices

A Moody's Data on Defaults

This appendix provides some details of the creation of the data set used in this paper. Our source of data are two separate databases, one containing default probabilities and the other containing information of defaults. For the empirical work in this paper, we need to account for all the defaults that occur over our sample of firms for which we have PDs. Below, we describe how we link the two datasets, and the set of defaults that results from our procedures.

In its default database, Moody's records 628 US and Canadian defaults of non-financial firms in the period 1/87 to 10/2000. A few firms default twice over this time period (Grand Union defaulted three times). Moody's records defaults only for firms that it has rated at some point in the firm's history. The defaults in the database are indexed by Moody's Issuer Number (MIN). Although some of these firms are linked to a Cusip or a Bloomberg ticker, many of the firms do not have a link to any external identifier. However, the name of the defaulted firm is provided, as well as some information regarding the nature of default. Moody's database of default probabilities is created using accounting and equity price data, and is limited to firms that had available data in the sample period. Our sample period is January 1987 to October 2000. This data is indexed by the Gvkey.

The defaulted firms that have a Cusip are matched to the PD database using the Gvkey-Cusip link of the combined Compustat-CRSP database. For the remaining firms, we do a manual match using the company name. After both these matches, many firms remain without a Gvkey. Some of these firms do not have Gvkeys because they are either subsidiaries, or related to the primary public firm that has defaulted. For example, on 7 April 1987, Texaco Capital, Texaco Capital N.V., Texaco Corporation and Texaco Operations Europe are listed as four separate defaults. Of these, only Texaco Corporation is counted in our sample.

The number of defaults that are available for our empirical work is farther reduced as many firms were not rated by Moody's according to our PD database over the period 1987-2000. The final dataset, corresponding to the default of firms that are present in the PD database over our sample period, contains 241 incidents of defaults among a total of 1,990 firms and over 216,859 firm-months of data.

B Estimation of Default Intensities from PDs

This appendix provides the algorithm for our method-of-moments estimator of default intensities.

1. First, we obtain starting coefficient estimates values from the regression, for $h = 1/12$,
$$s_{t+h}(1) - s_t(1) = \alpha + \beta s_t(1) + e_t, \tag{11}$$
where α and β are the ordinary-least-squares (OLS) estimators and e_t denotes the residual. From this regression, we get initial estimates of the three parameters as:
$$k = -\frac{\beta}{h} \tag{12}$$
$$\theta = -\frac{\alpha}{\beta} \tag{13}$$
$$\sigma = \frac{V(e)}{\sqrt{\theta h}}, \tag{14}$$
where $V(e_t)$ denotes the sample standard deviation of the residual e_t.

2. Given starting values of $\{k, \theta, \sigma\}$, we obtain an initial estimate of the default intensity λ_t, for each observation time t, using equation (7).

3. Next, we estimate by OLS,
$$\lambda_{t+h} - \lambda_t = a + b\lambda_t + w_t. \tag{15}$$
New parameter estimates are then given by
$$\hat{k} = -\frac{b}{h}, \quad \hat{\theta} = -\frac{a}{b}, \quad \hat{\sigma} = V\left(\frac{w_t}{\sqrt{h\lambda_t}}\right), \tag{16}$$
where, again, $V(\,\cdot\,)$ denotes sample standard deviation.[4]

4. Given these updated estimates of the parameters $\{k, \theta, \sigma\}$, we return to Steps 2 and 3, and iterate to numerical convergence.

[4]In the current version of our results, we use $V(w_t/\sqrt{\hat{\theta}h})$ in place of the sample standard deviation shown in (16), although our tests indicate that this causes minimal distortion in the estimated intensities.

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