A Two-Factor Model of Value and Growth with Adjustment Costs

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Abstract

This paper examines a two factor model of value and growth, in an economy with many different industries consisting of identical competitive firms. A cyclical (value) factor captures permanent shocks to industry or aggregate demand, and innovations to this shock are assumed to command a generous risk premium. A growth factor captures transitory shocks to the growth rate of aggregate demand or demand for an industry's output, and innovations to this factor are assumed to command a small risk premium. Competitive firms have several distinct mechanisms to adjust capital, each modeled as a quadratic adjustment costs for investment modified by a maximum growth rate cap (fixed exogenously). There is also an option to scrap capital at low valuations. Firms are valued numerically as real options in a model with two state variables: revenues per unit of capital and the expected growth rate of demand. Since demand is more volatile than costs, firms possess "real leverage," which tends to make them more volatile when firms are losing money. It is shown that adjustment cost parameters can be chosen so that stocks with low growth rates ("value stocks") load relatively strongly on the value factor and stocks with high growth rates ("growth stocks") load relatively strongly on the growth factor. In addition, these parameters can be chosen so that growth stocks have relatively high volatility. Returns on value stocks correlate relatively strongly with innovations in demand, while returns on growth stocks correlate relatively strongly with innovations in future demand growth. Distressed stocks, close to scrapping capital, have particularly high returns since they load most strongly on the cyclical factor.

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1 Introduction

This paper examines a two factor model of value and growth, in an economy with many different industries consisting of identical competitive firms. A cyclical (value) factor captures permanent shocks to industry or aggregate demand, and innovations to this shock are assumed to command a generous risk premium. A growth factor captures transitory shocks to the growth rate of aggregate demand or demand for an industry's output, and innovations to this factor are assumed to command a small risk premium. Competitive firms have several distinct mechanisms to adjust capital, each modeled as a quadratic adjustment costs for investment modified by a maximum growth rate cap (fixed exogenously). There is also an option to scrap capital at low valuations. Firms are valued numerically as real options in a model with two state variables: revenues per unit of capital and the expected growth rate of demand. Since demand is more volatile than costs, firms possess "real leverage," which tends to make them more volatile when firms are losing money. It is shown that adjustment cost parameters can be chosen so that stocks with low growth rates ("value stocks") load relatively strongly on the value factor and stocks with high growth rates ("growth stocks") load relatively strongly on the growth factor. In addition, these parameters can be chosen so that growth stocks have relatively high volatility. Returns on value stocks correlate relatively strongly with innovations in demand, while returns on growth stocks correlate relatively strongly with innovations in future demand growth. Distressed stocks, close to scrapping capital, have particularly high returns since they load most strongly on the cyclical factor.

The theoretical model in this paper is similar to the one-factor model of Kogan (forthcoming), and is related to Kogan(2001), which also provides a good review of the literature. It is also related to the theoretical model of Berk, Green, and Naik (1999). The model is consistent with the empirical predictions of Bansal, Dittmar, and Lundblad (forthcoming), and Xing and Zhang (2004) and Zhang (forthcoming).

2 One-Factor Model

Consider a competitive industry of atomistic firms in which the level of demand for the industry's output is subject to permanent shocks and the price elasticity of demand is constant. Let $P(t)$ denote the price for industry output, let $Q(t)$ denote the the equilibrium quantity of industry output, let $A(t)$ denote the stochastic level of the demand curve, and let η denote the constant elasticity of demand. The price can be expressed as

$$P(t) = A(t) \cdot Q(t)^{-1/\eta}. \tag{1}$$

To distinguish the true process from the risk neutral process, we will use superscripts "true" and "RN" where appropriate. Assume that the "true" level of the demand curve $A^{true}(t)$ follows a geometric Brownian motion with growth rate denoted G and instantaneous standard deviation denoted σ_A:

$$dA^{true}(t)/A^{true}(t) = Gdt + \sigma_A dB_A. \tag{2}$$

Assume that there are complete markets in which the risk of betting on innovations in dB_A is rewarded with a constant Sharpe ratio π_A. With continuous trading, this complete markets assumption can be implemented by assuming that investors are able to trade continuously a zero-value security with continuously settled cash flows $\pi_A dt + dB_A$. The risk neutral process for $A(t)$, denoted $A^{RN}(t)$ is then given by

$$dA^{RN}(t)/A^{RN}(t) = (G - \pi_A \sigma_A)dt + \sigma_A dB_A. \tag{3}$$

We assume a production function with fixed proportions between capital and labor, and we require all capacity to be utilized at all times. One unit of capital costs a constant amount c, depreciates at constant rate δ, and employs labor which costs a constant amount w. We assume a quadratic adjustment cost for for allowing capital investment to deviate from the rate of depreciation. Let b denote the quadratic adjustment cost parameter such that investing at rate I per unit of capital incurs a quadratic adjustment cost of $\frac{1}{2}b(I-\delta)^2$ per unit of capital. The only choice variable of firms is the investment rate I. Since all capacity is always utilized, the choice of investment rate I also determines capacity and output. In a steady state equilibrium, the value of one unit of capital, denoted $V(P)$ and the equilibrium investment rate, denoted $\bar{I}(P)$, are both functions of the single state variable P. We will also consider fixed horizon models, in which the value of one unit of capital and the investment rate depend on both price and time, i.e., $V(P,t)$ and $I(P,t)$. Note that the state variable is the price $P(t)$, not the level of demand $A(t)$. If the level of demand shifts outward, prices rise, and this makes the present value of revenues from an extra unit of capital higher. Extra investment occurs, and this drive the price towards its equilibrium level.

Quadratic adjustment costs imply that industry capacity responds gradually to changes in prices. Industry capacity, which equals the size of the industry's capital stock, follows the process

$$dQ/Q = [\bar{I}(P(t)) - \delta]dt \tag{4}$$

The price of a unit of output under the risk neutral measure follows the process

$$dP^{RN}/P^{RN} = \Big(G - \pi_A\sigma_A - [\bar{I}(P(t)) - \delta]/\eta\Big)dt + \sigma_A dB_A, \tag{5}$$

and the price of a unit of output under the true measure follows the process

$$dP^{RN}/P^{RN} = \Big(G - [\bar{I}(P(t)) - \delta]/\eta\Big)dt + \sigma_A dB_A \tag{6}$$

Let the risk-free rate of interest, denoted r, be constant.

2.1 Continuous Problem

Calculation of the industry equilibrium is a dynamic programming problem, which determines the value of a unit of capital as a function of the output price, denoted $V(P)$, and the optimal investment policy as a function of the output price, denoted $\bar{I}(P)$. A discrete-time Euler approximation to the dynamic programming problem is given by

$$\begin{aligned} V(P,t|\bar{I}(.)) = \max_I \Big\{ &\big[P - w - cI - b(I-\delta)^2/2\big] \cdot \Delta t \\ &- e^{(r+\delta-I)\Delta t} \cdot E^{RN}\Big\{V(P+\Delta t, t+\Delta t)\Big|P(t) = P, \bar{I}(.)\Big\}\Big\}. \end{aligned} \tag{7}$$

This equation says that, in equilibrium, a firm chooses its investment policy to maximize the value per unit of capital stock, conditional on other firms following the equilibrium investment policy $\bar{I}(.)$. The value of one unit of capital at time t is the value of the current cash flow it generates over the next interval of time, $[P - w - cI - b(I-\delta)^2/2]\Delta t$, plus the present value of the residual value at time $t + \Delta t$. The residual value is discounted at the risk-free rate r since the risk-neutral measure is used to calculate the expectation. The value is also discounted to adjust for the firm's rate of net investment $I - \delta$. The residual value depends on the current output price $P(t)$ as well as the equilibrium investment policy being followed by the firms in the industry $\bar{I}(.)$.

Going to continuous time and applying Ito's lemma, the dynamic programming equation becomes the partial differential equation (Bellman equation)

$$\begin{aligned} -\frac{\partial V}{\partial t} = \max_I \Big\{ &P - w - cI - b(I-\delta)^2/2 - (r+\delta-I)V(P,t) \\ &+ [G - \pi_A\sigma_A - (\bar{I}(P,t) - \delta)/\eta]P\frac{\partial V}{\partial P} + \sigma_A^2 P^2 \frac{\partial^2 V}{\partial P^2}\Big\}. \end{aligned} \tag{8}$$

Risk aversion shows up in the growth rate of the price, where under the risk neutral measure $G - \pi_A\sigma_A$ appears instead of G alone. The effect of competition shows up in the term $[\bar{I}(P,t) - \delta]/\eta$, which says that higher investment by other firms makes revenues fall at a rate which depends also on the elasticity of demand η. Quadratic adjustment costs make the first order condition linear, and the solution for I is

$$I(P,t) = \frac{V(P,t) - c}{b} + \delta. \tag{9}$$

In equilibrium, all firms must invest at the optimal rate, so we must have $I = \bar{I}(P)$. In a steady state, the partial differential equation becomes the second-order ordinary differential equation

$$\begin{aligned} 0 \quad = \quad & P - w - \delta c + \frac{(V(P) - c)^2}{2b} - rV(P) \\ & + \Big(G - \pi_A\sigma_A - \frac{V(P) - c}{b\eta}\Big) P\frac{dV}{dP} + \frac{1}{2}\sigma_A^2 P^2 \frac{d^2V}{dP^2}. \end{aligned} \tag{10}$$

In this equation, the term $[(V(P)-c)^2]/(2b)$ measures the contribution of the competitive firm's real option to choose an investment strategy other than $I = \delta$.

Unfortunately, this differential equation is unstable and difficult to solve for arbitrary values of the parameters. Therefore, rather than solve the differential equation directly, we will discretize dynamic problem and solve the dynamic programming problem below.

2.2 Accounting Concepts

The notation in this problem maps in a simple way into standard finance and accounting concepts. For simplicity, let us assume that adjustment costs are expensed, but the costs required to replace depreciating capital, δc, are capitalized and depreciated at the true rate of depreciation δ. Then the "book value" of a firm with one unit of capital is the constant c. Note that we assume that the firm has no debt and no working capital; in particular, it has no inventories and no cash. Thus, our concept of "book value" corresponds to the book value of physical capital, ignoring all short-term assets. "Tobin's Q" can be defined as $V(P,t)/c$; it equals the market value of capital divided by the book value of capital, where the book value of capital is the replacement cost of capital excluding adjustment costs.

The "price" $P(t)$ corresponds to revenues per unit of capital. To distinguish the price of the firm's output from price of the firm's stock, we can refer to $P(t)$ as "revenues" and to $V(P)$ as the firm's "value." There are two revenue multiples which are easy to define.

Book value per unit of revenue is given by $c/P(t)$, and market value per unit of revenue, the standard "revenue multiple", has the definition $V(P(t))/P(t)$.

The firm's earnings per unit of capital are given by revenues minus costs $P(t) - w - \delta c - b(I - \delta)^2/2$, where costs include wage costs w, depreciation costs δc, and adjustment costs $(I - \delta)^2/2$. Note that since adjustment costs were not capitalized above, they are expensed here. A firm which is aggressively expanding its capital stock (because revenues and stock values are high) has its price-to-earnings ratio inflated because of the expensing of adjustment costs. It is conceivable that such a firm could have negative profits, because high adjustment costs are expensed. Conversely, a firm which is "restructuring" by rapidly decreasing its capital stock because revenues and valuations are low, has its low earnings depressed even further because the adjustment costs of restructuring are expensed. The firm's rate of return on capital (book value) is earnings divided by book value, $[P(t) - w - \delta c - b(I - \delta)^2/2]/c$. The firm's earnings-to-price ratio is defined by $[P(t) - w - \delta c - b(I - \delta)^2/2]/V(P(t))$. Note that negative earnings may well make the rate of return on capital and the earnings-to-price ratios negative.

Cash flow per unit of capital is given by $P(t) - w - Ic - b(I - \delta)^2$. Starting with earnings $P(t) - w - \delta c - b(I - \delta)^2/2$, this is obtained by adding back depreciation δc and subtracting investment costs Ic. Cash flow per unit of capital when investment equals depreciation is given by $P(t) - w - \delta c$, while cash flow per unit of capital when capital investment equals zero is given by $P(t) - w - b\delta^2/2$. The investment strategy which myopically maximizes short-term cash flow can be calculated by differentiating cash flow with respect to investment, yielding $I = \delta - c/b$, i.e., investment falls short of depreciation by c/b; myopically maximizing short-term cash flow yields cash flow equal to $P(t) - w - \delta c + c^2/(2b)$. Myopically maximizing short-term earnings yields investment equals depreciation, $I = \delta$, in which case the myopically maximized earnings equal $P(t) - w - \delta c$. Presumably, one reason that we have accounting rules which capitalize investment is to discourage myopic managers from from disinvesting in order to improve apparent short-term performance.

It is tempting to think of a "cash cow" as a firm which is engaging in little net investment, with investment approximately equal to depreciation. If the capital stock is shrinking, it is tempting to think of such a firm as "restructuring," i.e. aggressively reducing its capital stock (and wage costs) so that some cash can be generated by liquidating capital today and so that wage costs do not have to be incurred in the future. In fact, it is useful to think of a large component of adjustment costs as extraordinary wage expenses. In booming times, these expenses would include extraordinary sign-up bonuses and overtime pay, to hire the labor which goes with the rapidly expanding capital stock. In lean times, the adjustment costs could include labor contract buy-outs or severance packages associated with shedding unwanted labor.

This model has the property that revenues are much more volatile over short periods of time than wage and depreciation expenses. Wage and depreciation expenses change slowly over time as the capital stock changes. Revenues are much more volatile, since the price is subject to random increments, i.e., the stochastic process for prices has both a "dt" and "dB" term, while the process for capacity has only a "dt" term. Thus, to the extent that costs have much volatility, this volatility must be attributed to the adjustment cost component of costs. The model is missing the idea that costs increase when demand increases because more labor is hired to increase output: this model has fixed proportions between capital and labor. It is as if labor is hired with a long-term contract at the same time that capital is installed, and the labor contract has the same horizon as the capital itself.

It would be easy to add taxes and required working capital to this model, but this would not affect the substance of the economic analysis.

2.2.1 Going Concern Value

Finance textbooks teach us that the value of a firm equals its going concern value plus the value of its growth options. If the going concern value, denoted V^{GCV}, is defined as the value of a firm when it follows the sub-optimal policy of always choosing investment to equal depreciation, then the going concern value of a firm solves the differential equation obtained by dropping the growth option term $[(V(P)-c)^2]/(2b)$ from the ordinary differential equation above:

$$\begin{aligned} 0 &= P - w - \delta c - rV^{GCV}(P) \\ &\quad + \Big(G - \pi_A \sigma_A - \bar{I}(P)\Big) P \frac{dV^{GCV}}{dP} + \frac{1}{2}\sigma_A^2 P^2 \frac{d^2 V^{GCV}}{dP^2}. \end{aligned} \tag{11}$$

In this equation, $\bar{I}(p)$ denotes the optimal investment policy followed by other firms, and it satisfies $\bar{I}(P) = [V(P) - c]/(b\eta)$, where $V(p)$ denotes the optimal value of the firm, not just the value of growth options. The value of growth options is the difference between the value of the firm following the optimal investment policy and the going concern value, $V(P) - V^{GCV}(P)$, and this value is always non-negative.

If adjustment costs are low, the real option can be very valuable when $V(p)$ is quite different from c. The term $[V(P) - c]/(b\eta)$ measures how investment by other firms affects the price of the industry's output. If adjustment costs are low, this term will be large in absolute value when the difference between $V(P)$ and c is large in absolute value, and this will make the price of output $P(t)$ mean revert quickly to the level where $V(P)$ is close to c, thus eroding the value of real investment options to firms. Firms compete in the way in which they exercise real options, but this competition lessens the value of the real options.

2.2.2 Special Case: Infinite Adjustment Costs

The infinite horizon steady-state case where adjustment costs are infinite ($b = \infty$) can easily be solved. The solution can be derived mathematically by conjecturing that a solution linear in P should solve the ODE for infinite transactions costs

$$0 = P - w - \delta c - rV(P) + (G - \pi_A\sigma_A -)P\frac{dV}{dP} + \frac{1}{2}\sigma_A^2 P^2 \frac{d^2V}{dP^2}, \tag{12}$$

then solve for the two unknown coefficients of the linear solution. The following simple finance intuition yields the same result: Because of infinite adjustment costs, the firm always invests at rate δ to keep the capital stock constant at one unit. Thus, costs incurred each period equal wage costs plus capital replacement costs $w + \delta c$. Since these costs are fixed and not risky, they have a risk-free perpetuity value of $(w + \delta c)/r$. Revenues $P(t)$ are stochastic, but grow at risk-neutralized rate $G - \pi_A\sigma_A$, so Gorden's growth formula yields a present value of $P(t)/(r + \pi_a\sigma_A - G)$. The solution of the continuous-time problem with infinite adjustment costs is therefore

$$V(P(t)) = \frac{P(t)}{r + \pi_A\sigma_A - G} - \frac{w + \delta c}{r}, \tag{13}$$

and this is easily verified by plugging the derivatives into the differential equation. Note that the denominator of the risky price in this formula can be expressed as the risk-adjusted interest rate $r + \pi_A\sigma_A$ less the growth rate G. In our risk-neutral valuation, the risk premium was subtracted from the growth rate, but in the solution, it can also be thought of equivalently as added to the interest rate. Simple MBA-level intuition makes the solution obvious once it is recognized that the revenues and costs grow at different rates and need to be discounted at different rates.

2.3 Discrete-Time Set-up

Since the differential equation is unstable, we will solve the problem by discretizing the dynamic programming problem, then solving the dynamic programming problem by brute-force backward induction. Instead of using the Euler discretization of the dynamic programming problem in (7), we will use a more accurate mixed discrete and continuous formulation. Firms choose a rate of investment $\bar{I}(P,t)$ at time t, and this investment rate is assumed to be fixed over the interval from t to $t+\Delta t$. For practical purposes, the time period Δt might correspond to one week, one month, or one quarter. Production, consumption, and hedging occur continuously over the interval $[t, t+\Delta t]$. In the limit as the investment decision interval Δt goes to zero, we obtain a continuous time model in which the investment rate also changes continuously. Under these assumptions, the value for a representative firm of one unit of capital as a function of the price $P(t)$ at time t is given by the dynamic programming equation

$$V\Big(P(t),t\Big) = \max_I E\Big\{\int_{s=t}^{t+\Delta t} e^{-(r+\delta-I)(s-t)} \cdot \Big(P(s) - w - cI - \frac{1}{2}b(I-\delta)^2\Big) \cdot ds \\ + e^{-(r+\delta-I)\Delta t} \cdot V\Big(P(t+\Delta t), t+\Delta t\Big)\Big| P(t), \bar{I}(P(t),t)\Big\}. \tag{14}$$

Under the assumption that the investment rate is constant over the interval $[t, t+\Delta t]$, this dynamic programming problem is exact, not an approximation like the Euler discretization. This dynamic programming equation also values a firm as a real option, in which the option to expand or contract is exercised optimally. The complete markets assumption (continuous hedging) allows the capital stock to be valued using the risk neutral process for innovations in demand.

In a competitive equilibrium, the optimal investment strategy for a representative firm must equal the investment strategy of the other firms in the industry, so $I(P(t),t) = \bar{I}(P(t),t)$. Thus, in addition to solving a non-linear maximization problem, it is also necessary to solve a fixed point problem. In the ordinary differential equation (10) obtained from the Euler approximation, equating I to $\bar{I}$ was a trivial issue, because the optimal investment rate I solves a quadratic problem whose solution does not depend on $\bar{I}$. Here, calculating the optimal investment rate I while simultaneously insuring that it equals the equilibrium rate $\bar{I}$ of other firms at each step becomes a more complicated issue, which we address below.

The dynamic programming equation equation says that the value of one unit of capital is the sum of the present value of the cash flows over the interval $[t, t+\Delta t]$ and the residual value the remaining capital at date $t+\Delta t$. The cash flows over the interval $[t, t+\Delta t]$ are the difference between revenues per unit of capital $P(s)$, which is changing randomly over the interval, and the costs per unit of initial capital, $w + cI + bI^2/2$. From (6), the revenues $P(s)$

received by an atomistic competitive firm at date s during the interval $[t, t+\Delta t]$ depend on the equilibrium rate of investment of other firms in the industry $\bar{I}(P(t), t)$, in the following way:

$$P(s) = P(t) \cdot e^{\left[G - \pi_A \sigma_A - (\bar{I}(P(t),t)-\delta)/\eta - \sigma_A^2\right](s-t) + \sigma_A (B_A(s) - B_A(t))}. \tag{15}$$

The expected price at time s, under the risk neutral measure, is therefore given by

$$E^{RN}\{P(s)|P(t)\} = P(t) \cdot e^{\left[G - \pi_A \sigma_A - (\bar{I}(P(t),t)-\delta)/\eta\right](s-t)}. \tag{16}$$

The present value of revenues and costs over the interval $[t, t+\Delta t$ can be expressed exactly using annuity formulas. Let $f_A(r, t)$ denote the present value of an annuity of one dollar for t periods capitalized at constant interest rate r. We have

$$f_A(r, t) = \frac{1}{r}(1 - e^{-rt}) \tag{17}$$

The dynamic programming equation can now be written

$$\begin{aligned} V(P(t), t) = \max_I \Big\{ & f_A\Big(r + \delta - I - G + \pi_A \sigma_A + (\bar{I}(P(t), t) - \delta)/\eta, \Delta t\Big) \cdot P(t) \\ & - f_A(r + \delta - I, \Delta t) \cdot \frac{1}{2}(w + cI + bI^2) \\ & + e^{-(r+\delta-I)\Delta t} \cdot E^{RN}\Big\{ V(P(t+\Delta t, t+\Delta t)) \Big| P(t), \bar{I}(P(t), t) \Big\} \Big\}. \end{aligned} \tag{18}$$

The risk neutral valuation in this dynamic programming equation is implemented by discounting cash flows at the risk-free rate r, while adjusting the expected growth rate of demand G downward by the expected risk premium on innovations in demand $\pi_A \sigma_A$. To discount costs and the residual value of the initial unit of capital at the risk-free rate r, the net investment rate $I - \delta$ at which the capital is growing over the interval $[t, t+\Delta t]$ is first subtracted from the risk-free rate r inside the annuity formula (like the"D-over-R-minus-G" in Gorden's growth formula). To discount the present value of revenues $P(t)$, it is necessary, in addition, to subtract the rate at which prices are expected to increase under the risk neutral measure, $G - \pi_A \sigma_A - (\bar{I}(P(t), t) - \delta)/\eta$, from the discount rate inside the annuity formula.

2.4 Numerical Solution

There are numerous ways to solve the dynamic programming equation. One approach is to convert the problem to continuous time by letting Δt go to zero. An alternative and more practical approach for obtaining an approximate solution is to discretize the dynamic programming problem and solve the discretized problem by backward induction. We do this here by defining an evenly spaced grid in the log of prices (to make transition probabilities have the same variance) and by using the continuous probability density as an approximation proportional to the discrete transition probabilities.

To create a grid of points evenly spaced in $log(P)$, let p denote $log(P)$, define a minimum log-price p_0 and a log-price increment Δp. The log-price grid of $N+1$ points becomes $p_0, p_1 = p_0 + \Delta p, \ldots, p_N = p_0 + N\Delta p$. The minimum log-price p_0 and the maximum log-price p_N should be chosen to be sufficiently small and large, respectively, that the probability of such prices being reached in a long period of time is very small. From equation (15), we can write

$$p(t+\Delta t) = p(t) + \Big(G - \pi_A\sigma_A - (\bar{I}(P(t),t) - \delta)/\eta - \sigma_A^2/2\Big)\Delta t + \sigma_A\sqrt{\Delta t}\cdot\tilde{Z}, \qquad (19)$$

where $\tilde{Z}$ is a normally distributed random variable with mean zero and variance one. The value of $\tilde{Z}$ which takes p_i at date t to p_j at date $t+\Delta t$ is a function of the rate of investment in the industry $\bar{I}$, denoted $Z(i,j|\bar{I})$, which is given by

$$Z(i,j|\bar{I}) = \frac{(j-i)\Delta p - \Big(G - \pi_A\sigma_A - (\bar{I} - \delta)/\eta - \sigma_A^2/2\Big)\Delta t}{\sigma_A\sqrt{\Delta t}}. \qquad (20)$$

Thus, the transition probability density from p_i at date t to p_j at date $t+\Delta t$, denoted $prob(i,j|\bar{I})$, is given by

$$prob\Big(i,j|\bar{I}\Big) = \frac{1}{2\pi}\exp\Big(-\frac{1}{2}Z(i,j|\bar{I})^2\Big). \qquad (21)$$

To construct transition probabilities on the discrete grid, approximate the transition probabilities by these probability densities $prob(i,j|\bar{I})$, re-scaled so that the probabilities sum to one. The expectation $E^{RN}\Big\{V(P(t+\Delta t,t+\Delta t))\Big|P(t),\bar{I}\Big\}$ can be approximated in a straightforward manner using these probabilities. Given values of $V(P(t+\Delta t),t+\Delta t)$ at the grid points, the value of $E^{RN}\Big\{V(P(t+\Delta t,t+\Delta t))\Big|P(t),\bar{I}\Big\}$ can be calculated at grid point $P(t) = P_i$ using the formula

$$E^{RN}\Big\{V(P(t+\Delta t,t+\Delta t))\Big|P_i,\bar{I}\Big\} = \sum_{j=0}^{N} V(P_j,t+\Delta t)\cdot prob(i,j|\bar{I}) \qquad (22)$$

The value of $V(P_i,t)$ can now be obtained from equation (18).

To speed up the numerical calculations, two adjustments are made to the procedure described above. First, whenever $Z(i,j|\bar{I})$ exceeds three standard deviations in absolute value, the transition probability is set to zero. This makes the calculation more sparse.

Second, the grid is extended above and below the maximum and minimum prices p_0 and p_N to accommodate values of $Z(i,j|\bar{I})$ for negative j and $j > N$. These probabilities are calculated, provided the absolute value of $Z(i,j|\bar{I})$ is small enough, and these transition probabilities are added to p_0 or p_N. This has the effect of putting a reflecting barrier at p_0 and p_N.

To obtain an approximation for the steady state value function and steady state investment strategy, time is discretized using monthly intervals $\Delta t = 1/12$. Assuming a liquidation value of zero at $T = 100 years$, a values of a unit of capital on the discrete grid can be calculated by iterating backwards $T/\Delta t = 1200 years$. Note that $\bar{I}$ is really a function of $P(t)$ and t, so that we have $\bar{I} = \bar{I}(P(t), t)$. This implies that the transition probabilities must be recalculated for every time period on every iteration. These recalculations of probabilities represent a significant part of the calculation burden.

It remains to be described how the optimal investment rate $I(P_i, t_n)$ is calculated and how equality between the optimal investment rate $I(P_i, t_n)$ and the equilibrium investment rate $\bar{I}(P_i, t_n)$ is enforced at each node and for each iteration. The easiest approach is to approximate the solution to the optimization problem with the linear solution from the continuous problem, with a one period lag. We then have $I(P_i, t_n) = \bar{I}(P_i, t_n) = [V(P_i, t_{n+1}) - P_i]/c$. For the purpose of calculating steady state values, this approach works well for two reasons. First, after many iterations, $V(P_i, t_n)$ is close to its steady state value, and this is close to $V(P_i, t_{n+1})$. Thus, implementing the optimum with a one period lag does not affect the result in a steady state. Second, using the continuous solution rather than the actual solution to the non-linear first-order condition for the problem in (18) results in an error in I which is of order Δt. Since I is close to the optimum, the error in the valuation $V(P_i, t_n)$ is only of order Δt^2. In this version of the paper, we use this simple approximation for the optimal solution to solve the dynamic programming problem. A more accurate approach would be to linearize the non-linear first order-condition, then use Newton's method to solve the linearized problem repeatedly, using the simple approach above as an initial guess. This iterative approach converges very rapidly to the correct solution, since the problem can probably be shown to be a contraction mapping of order δt.

After calculating the steady-state value function and steady-state transition probabilities, it is possible to calculate the steady-state distribution of firms across price nodes by starting from an arbitrary initial distribution, the applying the transition probabilities repeatedly (e.g. several hundred years) until convergence to a steady state distribution is obtained.

Since there are two sets of transition probabilities, risk neutral and true, there will be two steady state distributions. Steady-state distributions calculated this way are distributions of "numbers of firms," not "book value of capital" or "market value of capital." Steady state distributions for book values of capital will differ from steady state distributions for numbers of firms, since firms in higher price categories have higher investment rates. To calculate the steady state distribution of book value, after applying the transition probabilities, it is also necessary to increase or decrease the amount of capital by the amount of net investment or disinvestment over the period. The steady state distribution of market value of capital can be obtained from the steady state distribution of book value of capital by multiplying the book value of capital at each node by the market value for capital corresponding to that node. We expect the book value of capital to be shifted to the right relative to the market value of capital. The market value of capital will be shifted far to the right, since valuations at high prices are many times higher than at low prices. Not that there will be two steady-state distributions (true and risk neutral) for all three classifications: number of firms, book value, and market value.

Note also that this model does not generate a meaningful concept of firm size. Over time, some firms will become infinitely large and some firms infinitely small, even relative to other firms at the same price node. This is because firm size is non-stationary due to the level of demand being non-stationary, making firm size dependent on the infinite history of industry demand. For empirical purposes, some method should be added to this model to allow firms and industries to shrink if they become too large and to expand if they become too small.

3 An Example of the One-Factor Model

This section describes an example of the one-factor model with the following assumptions: $r = 0.0200$, $\pi_A = 0.40$, $\sigma_A = 0.15$, $\delta = 0.10$, $c = 100$, $b = 20c$, $w = 70$, $G = 0.03$, $\eta == .50$.

In addition, two other assumptions are made about adjustment costs and abandoning capital. First, it is assumed that capital can scrapped for twenty percent of its book value, without occurring any adjustment costs. This assumption has the effect of preventing the value of a unit of capital from being negative. Otherwise, with quadratic adjustment costs, it would be possible for revenues to fall so far below costs that short-run losses would offset any long-term positive cash flows. Second, it is assumed that the maximum percentage rate at which the capital stock can grow is ten percent per year. With a ten percent depreciation rate, this implies a maximum gross investment rate of twenty percent per year. In effect, we assume a linear marginal adjustment cost up until this maximum investment rate, beyond which the adjustment cost rises to infinity. Note that this assumption has two effects. On the one hand, it lowers a firm's real option to expand at a rate faster than ten percent per year, and this lowers the firm's value. On the other hand, it implies that the firm's competitors will be investing less aggressively when revenues are high. This will tend to keep prices high for a longer period of time. In equilibrium, this second effect is more important that the first; industry profits are higher when there is a cap on the rate at which firms can expand when values are high.

The value of one unit of capital as a function of revenues per unit of capital is shown in Figure 1. When revenues are approximately 60, the capital is abandoned for twenty cents on the dollar. At values above 60, the value of one unit of capital rises steadily, but not linearly. Since the value is expressed as a percent of the book value (assumed to be 100), there will be positive net investment when the value is greater than 100. In equilibrium, intuition suggests that the long-run average price of on unit of capital should be slightly above 100, since net positive investment must occur at a rate sufficient to keep up with growing demand. In this case, since prices rise at a rate of three percent per year if there is no net investment, output must rise at a rate of $G/\eta = 1.5$ percent per year to keep up with demand growth, since the price elasticity of demand is 0.50.

It is important to keep in mind the distinction between the "risk neutral" probabilities used in calculating asset prices and the "true" probabilities which describes empirical outcomes. Since the risk neutral probabilities are conservative in the sense that the expected growth rate in demand is reduced by the risk premium, the risk neutral probabilities imply an steady state distribution of values that has a lower mean that the steady state distribution of values implied by the true distribution. Figure 2 shows the steady state distribution of values of capital for both the risk neutral and the true distributions. Note that the risk neutral distribution does have a distinctly lower mean value than the true distribution. Both distributions are truncated at a revenue level of about 60, when capital is scrapped. Under the true distribution it is rare for capital to be worth more than 150 percent of it book value. Figure 2 actually gives two different versions of the risk neutral and true probability distributions, respectively. One version is "equally weighted," in the sense that it is the distribution of the number of firms. The other version is "book-value weighted" and takes account of the fact that firms in higher valuation levels invest more and therefore have more capital. It turns out that there is so much change in capital values over time that the net investment effect which accounts for the difference between these two versions of the distributions does not make much difference in the overall shape of the distributions.

Figure 3 shows the annualized standard deviations of returns as a function of revenues per unit of capital. As prices fall from about 160 to about 65, annualized return volatility declines from about 74 percent to about 20 percent. Intuitively, this decline reflects a "real leverage" effect. The firms revenues are riskier than the firm's costs. Thus, as revenues decline (holding costs constant), the firm becomes effectively more levered, if we think of claims to revenues as a risky equity-like asset, and obligations to pay relatively fixed costs as a debt-like liability. Below revenues of about 65, volatility declines dramatically, as the option to scrap capital puts a floor on how far prices can fall. Above prices of around 160 (rarely seen in the true distribution of values), there is a sudden change in the shape of the graph of volatility from convex (between 65 and 160) to concave (above 160). This change represents the effect of limiting the maximum net investment rate to being ten percent. It raises both the value of a unit of capital and its volatility. Note that the slope of the curve representing the value of a unit of capital is essentially proportional to price volatility at that point, as in option pricing. While Figure 1 makes it seem like the value is an almost linear function of revenues at revenues above 60, Figure 3 makes it clear that there are interesting non-linearities in this pricing function.

3.1 Growth and Value, Boom and Bust

One of the purposes of this paper is to build a theoretical model which captures a distinction between growth stocks and value stocks and a related distinction between boom stocks and cyclical stocks.

A cyclical stock is one in which current increases in demand lead to current increases in revenues and profits. This increase increases the value of the firm's capital stock and encourages the firm to increase its rate of investment. The increased rate of investment reduces prices towards their long term equilibrium. Similarly, after a decrease in demand, prices for output and profits fall, investment is reduced, and the reduced investment encourages prices and profits to rise back towards their long-term equilibrium level.

The one factor model captures well the features of cyclical industries. Although the level of demand follows a random walk with drift, the level of revenues and profits per unit of capital follow a mean reverting process, similar to but not identical to an AR-1. When prices and profits are high, the stocks are “cyclical boom” stocks. When prices and profits are low, the stocks are “cyclical bust” stocks.

Now let us consider the difference between growth stocks and value stocks.

A growth stock is one in which revenues and profits are expected to increase in the future. Growth firms have high valuations per unit of capital and invest aggressively. As a result of the rapid investment, they may have low or negative cash flows in the short run. Growth stocks typically have high volatility, much of which is associated with uncertainties in the expected future growth rate. If the expected future growth rate is high compared to the rate at which firms can easily increase their stock of capital, profits per unit of capital may increase, even though the firms are investing aggressively. A growth stock will not always be a growth stock. Eventually, the rate of growth in demand slows down, and the level of industry capacity catches up with the level of demand. At this point the growth stock becomes a value stock or a cyclical stock.

Value stocks are the opposite of growth stocks, in the sense that the level of demand is growing at a slow rate or decreasing. Firms have low valuations, in the sense that the ratio of market values to book values is low. Prices are low, profits are low, and the firms respond by disinvesting, i.e., by failing to replace capital as it depreciates. Such firms should have low ratios of value to cash flow, and much of their value lies in cash flows that are expected to be received in the short run.

The one factor model does not capture the distinction between growth stocks and value stocks. In the one factor model, if the firm has a high valuation (high market to book), revenues and profitability are expected to fall in the future. While this is a characteristic of cyclical stocks in a boom, this is the opposite of what should happen with a growth stock, where high valuations should predict future earnings and sales growth. When valuations are low, the one factor model predicts that revenues and and profits will increase in the future, the opposite for value stocks. Similarly, the one factor model does not capture the flavor of value stocks either.

There is extensive empirical evidence that value stocks have higher risk-adjusted expected returns than growth stocks. The one factor model does not allow different stocks to have different risk-adjusted returns, since all returns load on one factor. To make the distinction between growth stocks and value stocks and to allow stocks to have different risk adjusted levels of returns, a more complicated two-factor model is needed. The most obvious way to build a two-factor model from the one-factor model discussed above is to allow the growth rate G to be stochastic. Such a model is described in the following section.

4 A Two Factor Model

The two-factor model model is built from the one factor model by allowing the growth rate G to follow an autoregressive (Ornstein-Uhlenbeck) process. Letting α_G denote the mean reversion parameter, σ_G denote the volatility of the growth rate, and $\bar{G}$ denote the long term average rate of growth, we have the "true process"

$$dG^{TRUE}(t) = -\alpha_G(G^{TRUE}(t) - \bar{G})dt + \sigma_G dB_G(t). \tag{23}$$

Letting π_G denote the Sharpe ratio associated with bearing growth risk, the risk neutral process used for valuation is given by

$$dG^{RN}(t) = -\alpha_G(G^{RN}(t) - \bar{G})dt - \pi_G\sigma_G dt + \sigma_G dB_G(t). \tag{24}$$

In comparison with the true process, the risk neutral process reduces the steady-state level of $G(t)$ from $\bar{G}$ to $\bar{G} - \pi_G\sigma_G/\alpha_G$. The only difference between the true process and the risk neutral process is that the risk neutral process makes a more pessimistic prediction about growth rate of the growth rate. This is similar to the difference between the true process and risk neutral process for the cyclical factor, where the risk neutral process for the cyclical factor makes a more pessimistic assumption about the growth rate itself. Risk-adjustments for the cyclical factor are made to growth rates, while risk-adjustments for the growth factor are made to growth rates of growth rates.

An important objective of this paper is to construct a two-factor model of value and growth in which value stocks generate a higher reward for risk-bearing than growth stocks. For value stocks and growth stocks to generate different rewards for risk-bearing, the returns on value stocks and growth stocks should load differently on the two different factors. This can be accomplished if growth stocks load more heavily on the growth factor than value stocks and if the cyclical factor generates a greater reward for risk-bearing than the growth factor. In order to generate economically significant differences in loadings on the growth factor, it is necessary to assume that marginal adjustment costs increase at an increasing rate as the rate of investment increases. This has the effect of restricting the ability of firms to grow when the incentives to grow are great. Although an inability to grow rapidly reduces the value of an individual firm, holding constant the process for prices and growth, the inability to grow affects the process for prices. When growth rates are high and firms face very high costs for investing at a rapid enough rate to keep up with the high growth rate, prices for output rise faster than they would if firms did not face accelerating marginal costs for expansion. This not only has the effect of increasing the value of the firms but it also increases the loading of the firm's return on the growth factor. The intuition is that small changes in growth rates affect value stocks and growth stocks differently. For value stocks, small changes in growth rates are adjusted to relatively easily by changes in investment rates, and the effect of the change in the growth rate on the value of the value stock is small. For growth stocks, small changes in growth rates do not lead to much change in investment but instead lead to large changes in value.

Implementation of a model with increasing and convex marginal costs of investment requires replacing the quadratic cost function, which has linear marginal costs, with something more complicated. Here we adopt the strategy of assuming that the cost function is the sum of several quadratic adjustment cost functions, each with its own minimum and maximum rates of dis-investment and investment. It is as if the firm has several mechanisms for expansion, each of which has linear marginal costs between fixed maximum and minimum capacities. By combining the marginal cost functions, an increasing, convex, piecewise-liner aggregate marginal cost function is created. It makes the notation easier if each component cost function is a function of net investment ΔI rather than gross investment I. Let the adjustment cost parameter, maximum investment rate, and minimum investment rate for the j^{th} component cost function be denoted by b_j, ΔI_j^{MAX}, and Δ_j^{MIN}, respectively. The j^{th} component cost function is then defined by

$$C_j(\Delta I) = \begin{cases} -\infty & if \quad \Delta I < \Delta I_j^{MIN} \\ c\Delta I + \frac{1}{2}b_j(\Delta I)^2 & if \quad \Delta I_j^{MIN} \leq \Delta I \leq \Delta I_j^{MAX} \\ +\infty & if \quad \Delta I < \Delta I_j^{MXN} \end{cases} . \tag{25}$$

The discrete dynamic programming equation is similar to equation (18), with the following modifications. First, the growth rate $G(t)$ becomes a second state variable. Second, gross investment $I(t)$ is changed to net investment $\Delta I(t)$. Third, the quadratic cost function is replaced by the sum of J modified quadratic cost functions as discussed above. The modified dynamic programming equation becomes

$$V(P(t), G(t), t) = \max_{\Delta I_1, \ldots, \Delta I_J} \Bigg\{ \tag{26}$$

$$f_A\Big(r - \sum_{j=1}^{J} \Delta I_j - G + \pi_A \sigma_A + \Big[\sum_{j=1}^{J} \Delta \bar{I}_j(P(t), G(t), t)\Big]/\eta, \Delta t\Big) \cdot P(t)$$

$$
\begin{aligned}
&-f_A\Big(r-\sum_{j=1}^{J}\Delta I_j,\Delta t\Big)\cdot\frac{1}{2}\Big(w+c\delta+\sum_{j=1}^{J}C_j(\Delta I_j)\Big)\\
+&e^{-(r-\sum_{j=1}^{J}\Delta I_j)\Delta t}\cdot E^{RN}\Big\{V\Big(P(t+\Delta t),G(t+\Delta t),t+\Delta t\Big)\\
&\qquad\Big|\quad P(t),\ G(t),\ \sum_{j=1}^{J}\Delta\bar{I}_j(P(t),G(t),t)\Big\}\qquad\Big\}.
\end{aligned}
$$

This two-state-variable dynamic programming problem can be solved by the same brute-force backward induction method used to solve the one-state-variable model. Backward induction now takes place on a two-dimensional array of pairs $< p_i, G_j >$. A discrete grid for $G(t)$ is constructed using an Euler approximation for equation (23). Note that, unlike $P(t)$, the state variable $G(t)$ is assumed to be exogenous. It does not depend on investment rates or other firm policy variables. We assume that the transition probabilities for the two factors are independently distributed. Probability densities from the continuous problem are used as weights to construct a two-dimensional probability distribution on the discrete grid. The optimal level for each net investment component is approximated by the solution to the corresponding continuous problem, with a one period lag:

$$
I(P_i,G_j,t_n)=\bar{I}(P_i,G_j,t_n)=[V(P_i,G_j,t_{n+1})-P_i]/c. \tag{27}
$$

Steady state distributions (both risk neutral and true) can be constructed for number of firms, book value of firms, and market value of firms, using essentially the same approach as for the continuous model.

4.1 Example of a Two-Factor Model

This section describes an example of the two-factor model with the following assumptions: $r = 0.0300$, $\pi_A = 0.30$, $\sigma_A = 0.15$, $\delta = 0.10$, $c = 100$, $c_{SCRAP} = 0.20c$ $b = 20c$, $w = 70$, $\eta == .80$, $\bar{G} = 0.02$, $\pi_G = 0.00$, $\alpha_G = 0.10$, $\sigma_G = 0.05$. There are $J = 4$ components of the adjustment cost function, with the following parameter settings: $b^1 = 10c$, $b^2 = 50c$, $b^3 = 200c$, $b^4 = 200c$, $\Delta b_1^{MAX} = 0.02$, $\Delta b_2^{MAX} = 0.03$, $\Delta b_3^{MAX} = 0.03$, $\Delta b_4^{MAX} = +\infty$, $\Delta b_1^{MIN} = -0.50$,$\Delta b_2^{MIN} = -0.20$, $b_3^{MIN} = -\infty$, $b_4^{MIN} = -\infty$. The cost function parameters are chosen to make the marginal cost of net investment or disinvestment strongly increasing and convex.

The steady state distribution of growth rates has a standard deviation of $\sigma_G/\sqrt{2\alpha_G} = 0.1118$. With a mean growth rate of two percent, a firm with a growth rate two standard deviations above the mean has demand growing at a rate of $G(t)/\eta = 30.45\%$.

Steady state valuations were calculated on a $60x60$ grid, scaled so that the the steady state distributions of number of firms (both true and risk neutral distributions) and value-weighted distribution (true) fit neatly within the grid.

Equilibrium valuations are shown in figure (4). This figure shows level curves for the value of one unit of capital, scaled so that adjacent level curves represent changes in 20% of the book value of capital. Growth rates are plotted on the horizontal axis and revenues per unit of capital are plotted on the vertical axis. Level curves go monotonically higher towards the northeast of the figure and monotonically lower towards the southwest. This indicates that higher revenues per unit of capital (cyclical booms) and higher growth rates both increase the value of capital, while lower revenues (cyclical busts) and lower growth rates both reduce it. Since the increment between curves, 20%, corresponds to the scrap value of capital, the large area in the southwestern section of the figures represents the combinations of price and growth where firms scrap capital. The next four bands indicate firms trading below book value, in a range from $20\% - 100\%$ of book value. Moving further in a northeastern direction, the market-to-book value ratio continues to increase in increments of 20%. The large region in the northeastern section of the figure indicates firms with a market-to-book value ratio greater than five.

The number of firms in the various value categories depends on the steady state distribution of firms. Figure (5) shows the equally-weighted true distribution of firms, figure (6) shows the equally-weighted risk neutral distribution of firms, and figure (7) shows the value-weighted true distribution of firms. All three distributions are truncated along the southwestern boundary, at the point where firms scrap capital. Compared with the true distribution of firms, the risk neutral distribution of firms is shifted to the southwest. This shift results from risk-aversion with respect to cyclical risk, which makes the risk neutral distribution more pessimistic with respect to cyclical risk than the true distribution. Note that it is assumed that that investors are risk neutral with respect to value risk. Compared with the true distribution of firms, the value-weighted true distribution of firms is shifted far towards the northeast. This indicates merely the high valuations of such firms compared with lower valued firms in the southwest. There is still truncation of the distribution where firms are scrapping capital, but such firms have such tiny valuations that this truncation does not show up on the graph. Although the innovations in the growth factor and the cyclical factor are independent, there is positive correlation between growth and revenues in all three graphs. This correlation occurs because fast growth results in levels of investment that do not keep up with the fast growth, thus forcing up prices for output.

Figure (8) shows levels curves for the volatility of firms. The level curves are such that the at the center of the value-weighted true distribution, annualized percentage volatilities of firms' returns is solidly in the low 20% to 30% range. Moving east towards higher growth results also in higher volatility, with a meaningful percentage of very high growth firms having volatilities greater than 40%. Intuitively, this increase in volatility is due to the convexity of marginal adjustment costs. Moving south and west towards low-growth firms or cyclical bust firms, the volatility also increases. Intuitively, this is due to the increasing real leverage of these firms. For cyclical bust firms, the real leverage shows up in low current profits. For low growth firms, the real leverage shows up as low profits expected in the future, as demand continues to shrink.

Although the volatilities may seem to be low, keep in mind that this model assumes that both the risk-free rate and the risk-premium are constant. Fluctuations in risk-premiums and the risk-free rate, if added to the model, could increase volatility. Moreover, the model assumes that all the firms in an industry are identical and thus have perfectly correlated returns. The low volatilities in the example might be high volatilities for particular industries consisting of portfolios of firms which are not in fact identical.

Figure (4) shows the percentage of the variance of returns due to the cyclical factor. These percentages decrease monotonically as firms moves from low growth to high growth. At the extreme west side of the figure, the percentage of volatility from the value factor exceeds 90%. At the extreme east side of the graph, the percentages are less than 40%. This pattern is consistent with the idea that value stocks offer a more favorable risk-return trade-off than growth stocks, if it is also assumed (as it is here) that the cyclical factor offers a higher reward for risk-bearing than the growth factor. In the model here, changes in the Sharpe ratio for individual stocks across firms well within the value-weighted distribution of firm size are relatively modest. For example (using round numbers), a typical growth stock obtains about 36% of its return variance from the cyclical factor. Since the Sharpe ratio of the cyclical factor is assume to be 0.30 and the Sharpe ratio of the growth factor is assume to be zero, the Sharpe ratio for investing in a growth stock is assumed to be $\pi_P * \sqrt(0.36) = 0.18$. In comparison, a typical growth stock obtains about 64% of its volatility from the cyclical factor, yielding a Sharpe ratio of $\pi_P * \sqrt(0.64) = 0.24$ for investments in value stocks. Although these differences in Sharpe ratios for individual investments in value and growth stocks may appear to be empirically modest, the differences in the model could easily be magnified by assuming a negative reward for bearing growth risk ($\pi_G < 0$) and a much greater positive reward for bearing value risk (e.g. $\pi_P = 0.50$).

Note that the highest Sharpe ratios are experienced for firms which are close to scrapping capital due to very low (highly negative) growth rates. If a financial side were added to the firms, this would probably correspond to firms suffering financial distress. Thus, the model suggests high Sharpe ratios for investing in firms experiencing financial distress. Furthermore, if the model were generalized to examine returns on risky bonds or options on stocks, this suggests that it is likely that junk bonds will command high risk premiums, and out-of the money puts will command higher risk premiums than out-of the money calls. Working out the details of these conjectures is, however, a topic for future research.

5 Towards a Macro Model

The model described so far has been one concerning a single industry. It is straightforward to construct an aggregate model by considering the aggregate economy to be the sum of a continuum of industries, each consisting of identical firms. To connect the industries together, assume that the innovations in the cyclical factor and the growth factor are each sums of a common component and and industry-specific component. Using straightforward notation, we have

$$dB_P^i(t) = \gamma_P dZ_P^{MKT}(t) + \sqrt{1-\gamma_P^2}dZ_P^i(t), \tag{28}$$

$$dB_G^i(t) = \gamma_G dZ_G^{MKT}(t) + \sqrt{1-\gamma_G^2}dZ_G^i(t). \tag{29}$$

The parameters γ_P and γ_G measure the correlation between the cyclical factor and the growth factor between two industries. Any risk premiums are earned based on bearing market brownian risks dZ_P^{MKT} and dZ_G^{MKT}, while industry-specific idiosyncratic brownian risks dZ_P^i and dZ_G^j are priced in a risk neutral manner. If the Sharpe ratio for investing in the market factors dZ_P^{MKT} and dZ_P^{MKT} are denoted π_P^{MKT} and π_P^{MKT}, respectively, then the Sharpe ratios for individual firms (industries) are given by

$$\pi_P^i = \gamma\pi_P^{MKT}, \tag{30}$$

$$\pi_G^i = \gamma\pi_G^{MKT}. \tag{31}$$

Using this method of aggregation, each industry, taken in isolation, looks just the same as in our model of individual industries. In diversified portfolios of stocks, the idiosyncratic risks will disappear and the portfolios will be betting only on the two aggregate factors dZ_P^{MKT} and dZ_G^{MKT}.

Now suppose that aggregate consumption is approximated as the sum of the revenues of all the industries in the economy. Then innovations in aggregate consumption are almost perfectly correlated with innovations in the cyclical factor dZ_P^{MKT}. Stocks which load heavily on the cyclical factor thus also load strongly on innovations in consumption. Because we have assumed that cyclical risk is more highly rewarded than growth risk, the value stocks which load strongly on the cyclical factor will also have cash flows and returns which correlate strongly with consumption. If consumption risk is the only type of risk which is priced, it would be appropriate to assume that growth risk dZ_G^{MKT} is not priced at all, as we assumed in the example above. The corrleation of the cyclical factor with consumption may also be related to the results in Bansal, Dittmar, and Lundblad.

Simple aggregation also leads to some predictions about aggregate volatility and returns. There are two types of bull markets, cyclical bull markets and growth bull markets.

In a cyclical bull market, the cyclical factor dZ_P^{MKT} has positive innovations. This pushes the average towards high revenues but not necessarily towards high growth. From figure (4), market-to-book ratios rise. Since rewards for bearing factor risk are assumed constant, the only mechanism for the Sharpe ratio of the market portfolio to change is for the loading of the market portfolio on the two factors to change. From figure (9), it can be seen that expected returns do not change much. Because the level curves in this figure are approximately vertical and a cyclical boom is a northward movement in the graph, approximately along the same level curve, the factor loadings stay approximately constant. From figure (4), it can be seen that there will not be much effect on volatility, since the level curves in this figure tend to run more north-south than east-west in the relevant range near the center of the value-weighted distribution.

In a growth bull market, the growth factor $dZ_G MKT$ has positive innovations. This pushes the average firm towards high growth and this increases valuations. Although the average firm invests more, demand grows faster if the expectations of rapid growth are fulfilled and there will thus also be a tendency for firms to move north along the cyclical axis. From figure (9), the importance of the cyclical factor in returns diminishes and expected returns fall. Similarly, in a growth bear market, Sharpe ratios rise. From figure (8), market volatility increases, since a move mostly east but somewhat north is approximately perpendicular to the level curves, starting near the center of the distribution. Note, however, that since the mean of the distribution is close to a point of minimum volatility along east-west movements, there is likely to be little change in market volatility. Furthermore, a growth bear market could also lead to increasing volatility.

All of these issue can be addressed much more precisely using numerical simulations of aggregate economies.

6 Conclusion

This paper has described a two-factor model industry dynamics, in which a value factor and a cyclical factor drive both the valuation of firms and the output of industries. By making the marginal cost of adjusting capacity convex, the model can be calibrated so that growth stocks have high volatilities and low returns due to high loadings on the value factor. Value stocks can also have high volatilities (due to real leverage), but have high returns due to a heavy loading on the cyclical factor. The model can be given a macroeconomic flavor by simple aggregation. This aggregate model makes it possible to distinguish growth bull markets from cyclical bull markets. The reward for bearing market risks varies randomly across time because loadings on the two factors vary across time, even though factor rewards are constant.

The model suggests numerous issues for future research, both theoretical and empirical. The model can be used to value options on individual stocks and the market portfolio. Given a model of debt and equity, it could also be used to price risky debt and analyze the expected returns on risky debt. An important issue in valuing risky debt is that real leverage makes firms more volatile as they become more distressed, compounding the effect of financial leverage.

In addition to the theoretical exercise in this paper, the model could also be used to fit real data!

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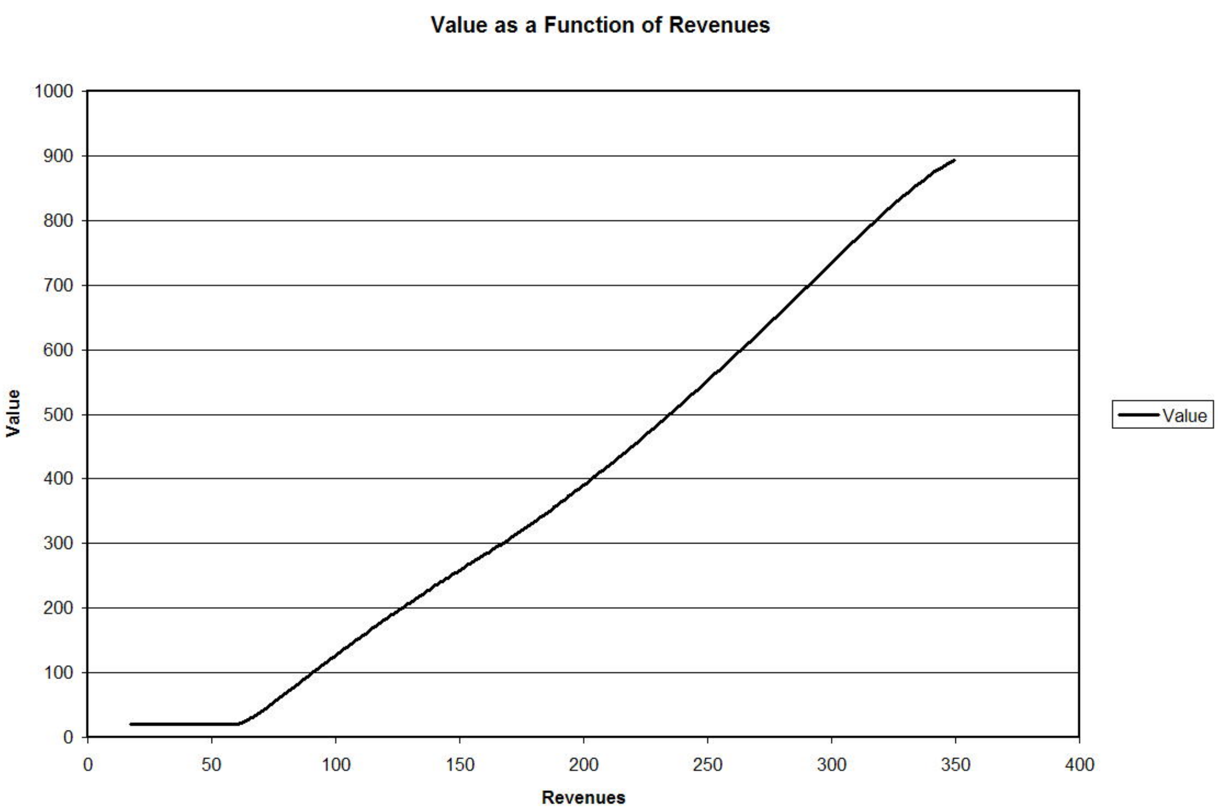
Figure 1:



Value as a Function of Revenues
1000
900
800
700
600
500
400
300
200
100
0
Value
0
50
100
150
200
250
300
350
400
Revenues
Value

Figure 2:

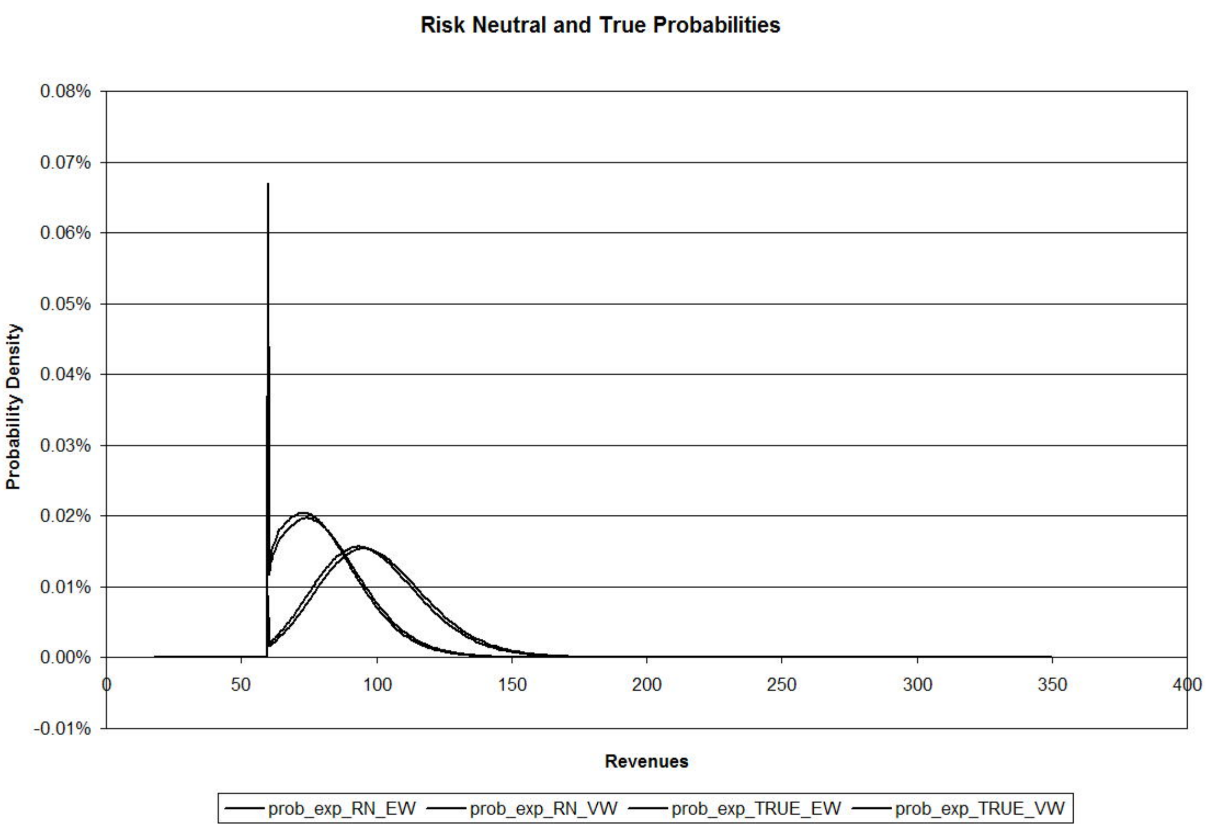
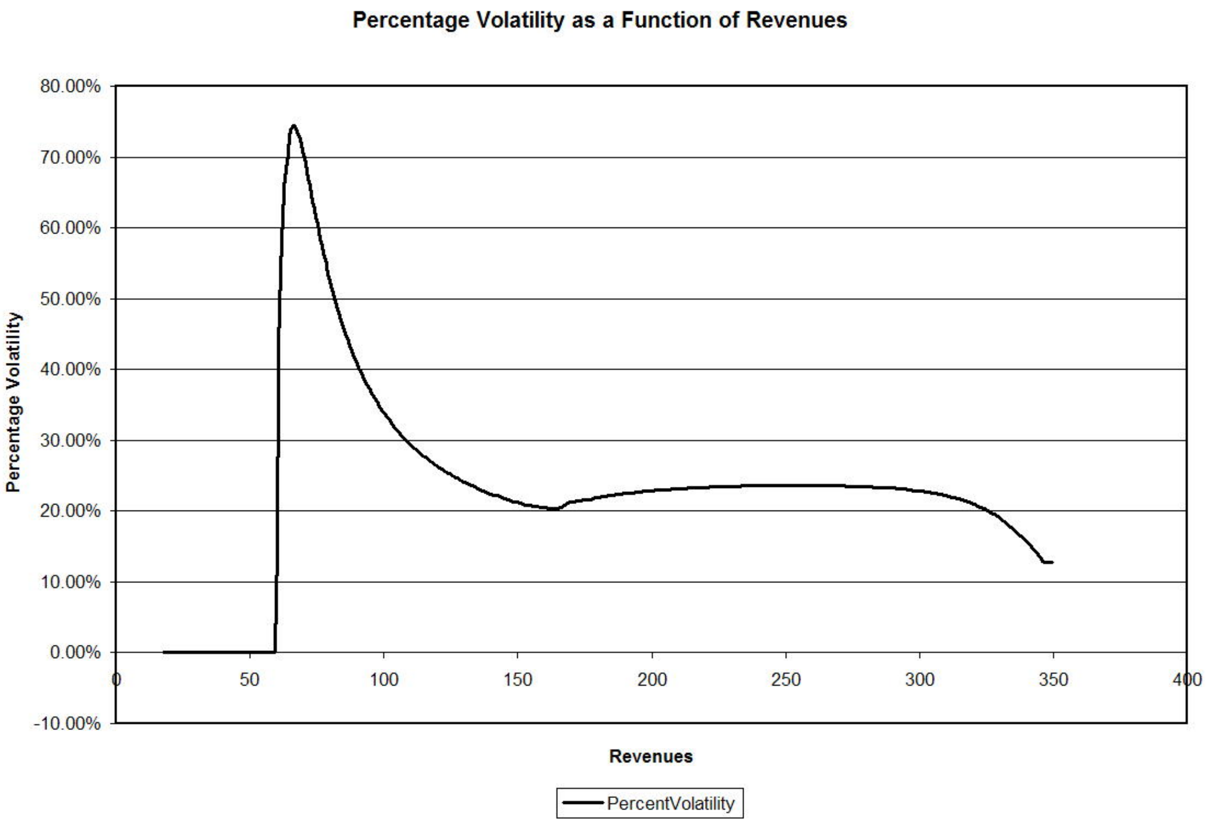
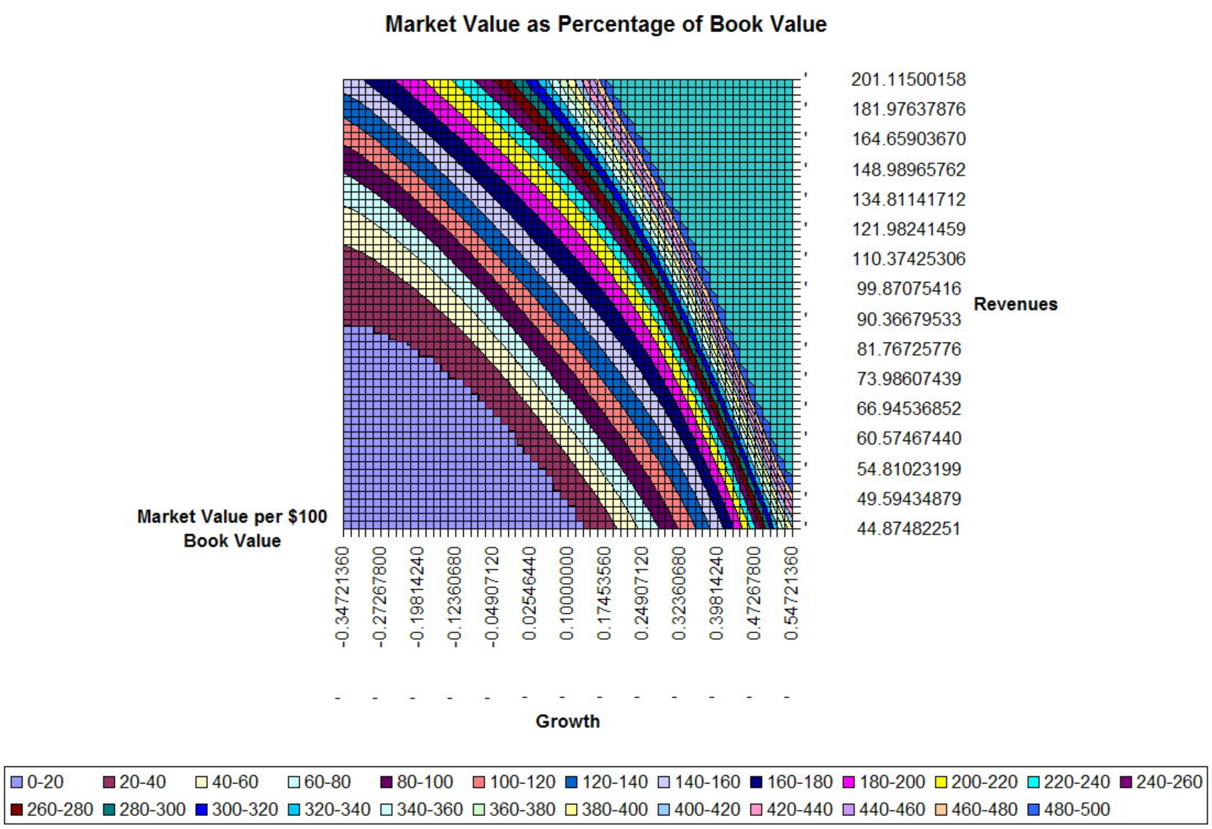

Risk Neutral and True Probabilities
0.08%
0.07%
0.06%
0.05%
0.04%
0.03%
0.02%
0.01%
0.00%
-0.01%
Probability Density
0
50
100
150
200
250
300
350
400
Revenues
prob_exp_RN_EW
prob_exp_RN_VW
prob_exp_TRUE_EW
prob_exp_TRUE_VW

Figure 3:



Percentage Volatility as a Function of Revenues
80.00%
70.00%
60.00%
50.00%
40.00%
30.00%
20.00%
10.00%
0.00%
-10.00%
Percentage Volatility
0
50
100
150
200
250
300
350
400
Revenues
PercentVolatility

Figure 4:



Market Value as Percentage of Book Value
201.11500158
181.97637876
164.65903670
148.98965762
134.81141712
121.98241459
110.37425306
99.87075416
90.36679533
81.76725776
73.98607439
66.94536852
60.57467440
54.81023199
49.59434879
44.87482251
Revenues
Market Value per $100 Book Value
-0.34721360
-0.27267800
-0.19814240
-0.12360680
-0.04907120
0.02546440
0.10000000
0.17453560
0.24907120
0.32360680
0.39814240
0.47267800
0.54721360
Growth
0-20
20-40
40-60
60-80
80-100
100-120
120-140
140-160
160-180
180-200
200-220
220-240
240-260
260-280
280-300
300-320
320-340
340-360
360-380
380-400
400-420
420-440
440-460
460-480
480-500

Figure 5:

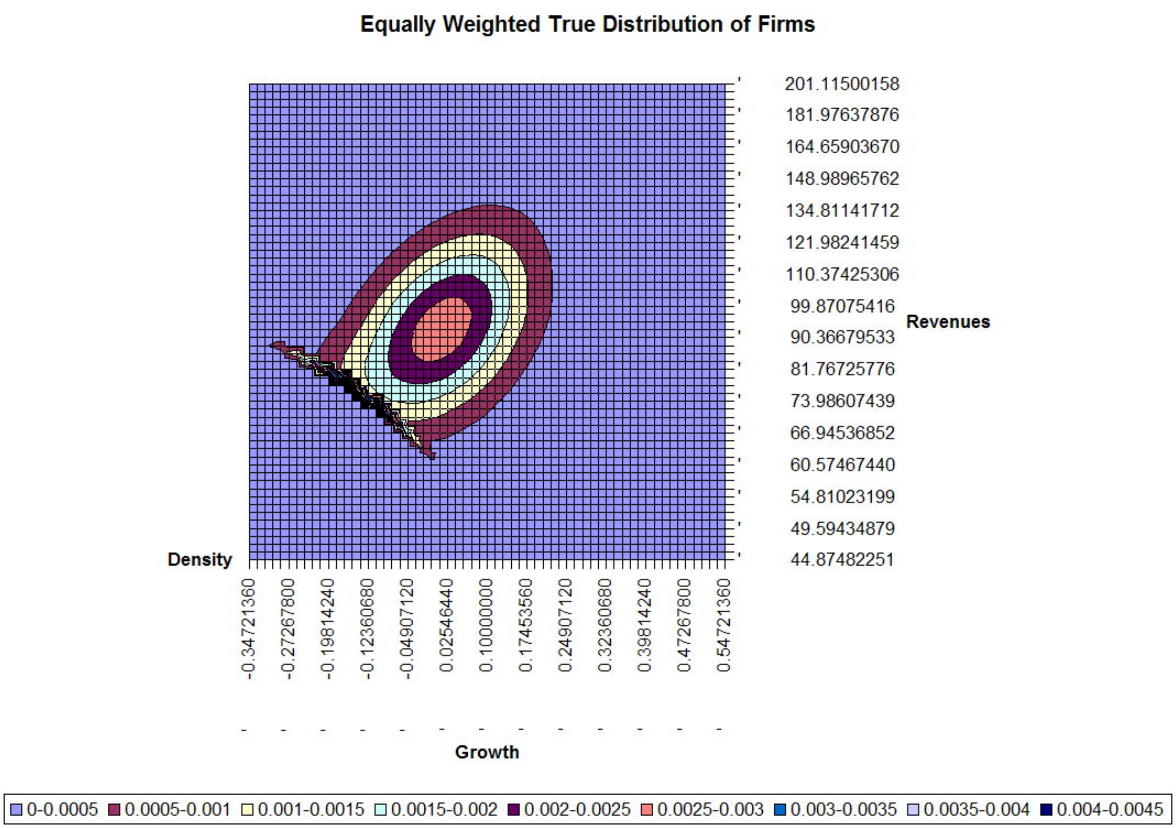

Equally Weighted True Distribution of Firms
201.11500158
181.97637876
164.65903670
148.98965762
134.81141712
121.98241459
110.37425306
99.87075416
90.36679533
81.76725776
73.98607439
66.94536852
60.57467440
54.81023199
49.59434879
44.87482251
Revenues
Density
-0.34721360
-0.27267800
-0.19814240
-0.12360680
-0.04907120
0.02546440
0.10000000
0.17453560
0.24907120
0.32360680
0.39814240
0.47267800
0.54721360
Growth
0-0.0005
0.0005-0.001
0.001-0.0015
0.0015-0.002
0.002-0.0025
0.0025-0.003
0.003-0.0035
0.0035-0.004
0.004-0.0045

Figure 6:

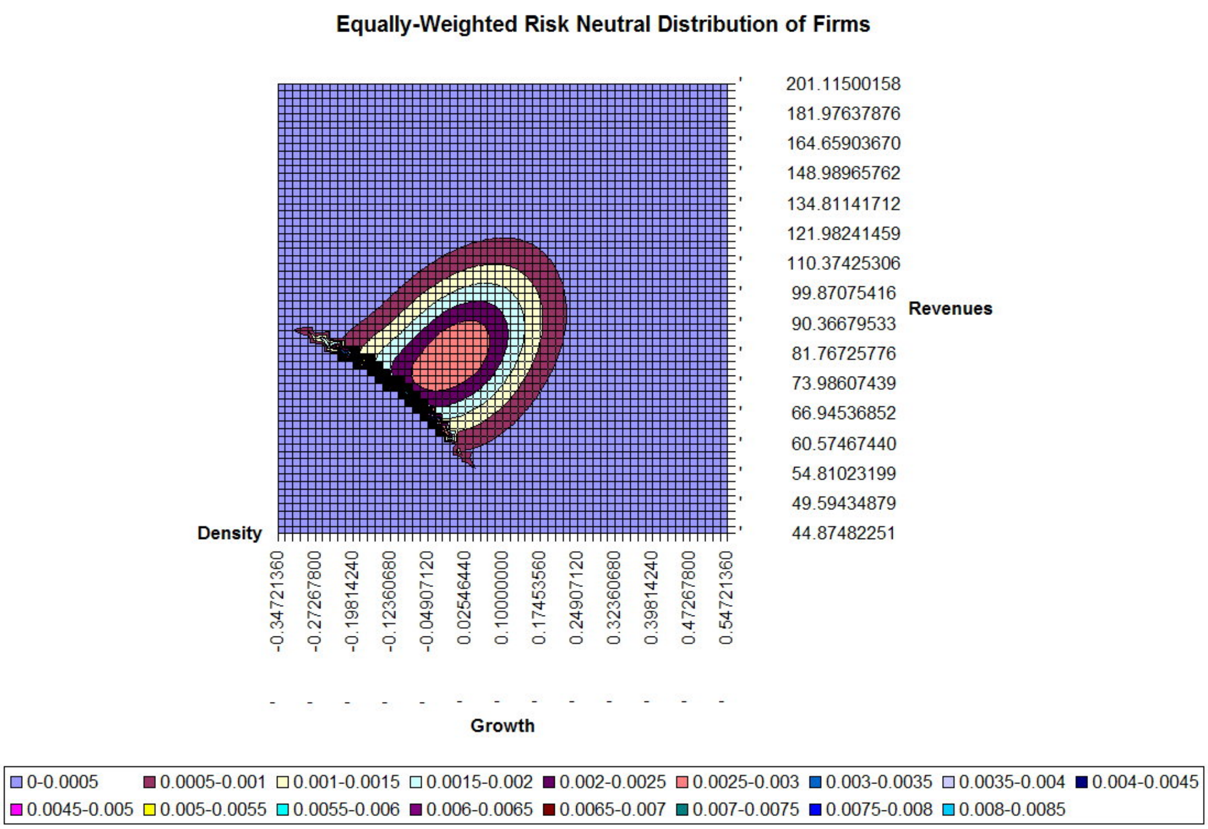

Equally-Weighted Risk Neutral Distribution of Firms
201.11500158
181.97637876
164.65903670
148.98965762
134.81141712
121.98241459
110.37425306
99.87075416
90.36679533
81.76725776
73.98607439
66.94536852
60.57467440
54.81023199
49.59434879
44.87482251
Revenues
Density
-0.34721360
-0.27267800
-0.19814240
-0.12360680
-0.04907120
0.02546440
0.10000000
0.17453560
0.24907120
0.32360680
0.39814240
0.47267800
0.54721360
Growth
0-0.0005
0.0005-0.001
0.001-0.0015
0.0015-0.002
0.002-0.0025
0.0025-0.003
0.003-0.0035
0.0035-0.004
0.004-0.0045
0.0045-0.005
0.005-0.0055
0.0055-0.006
0.006-0.0065
0.0065-0.007
0.007-0.0075
0.0075-0.008
0.008-0.0085

Figure 7:

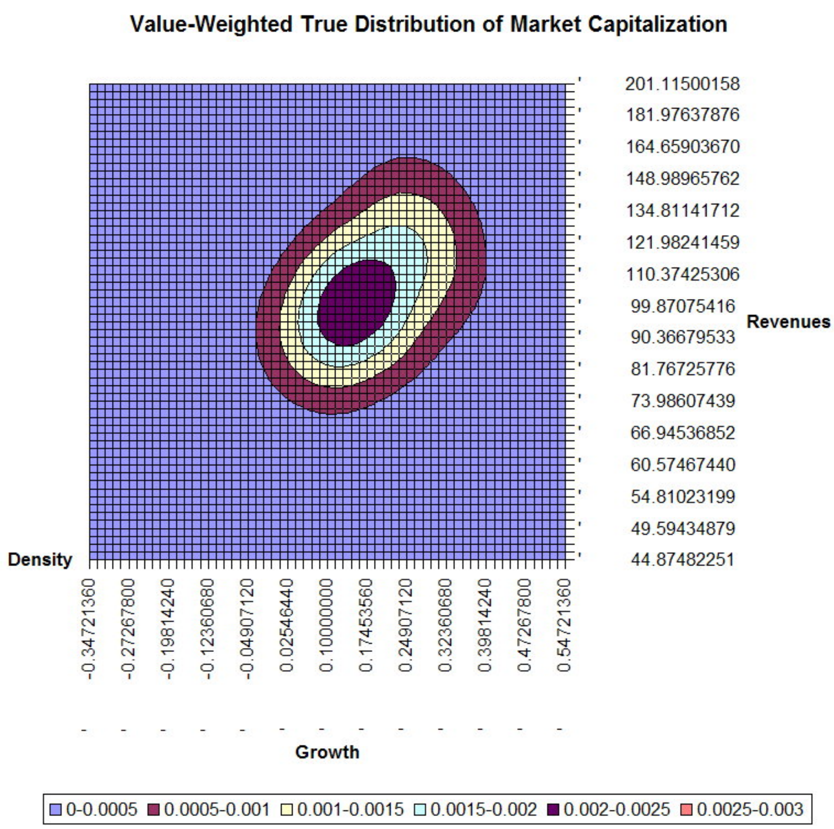
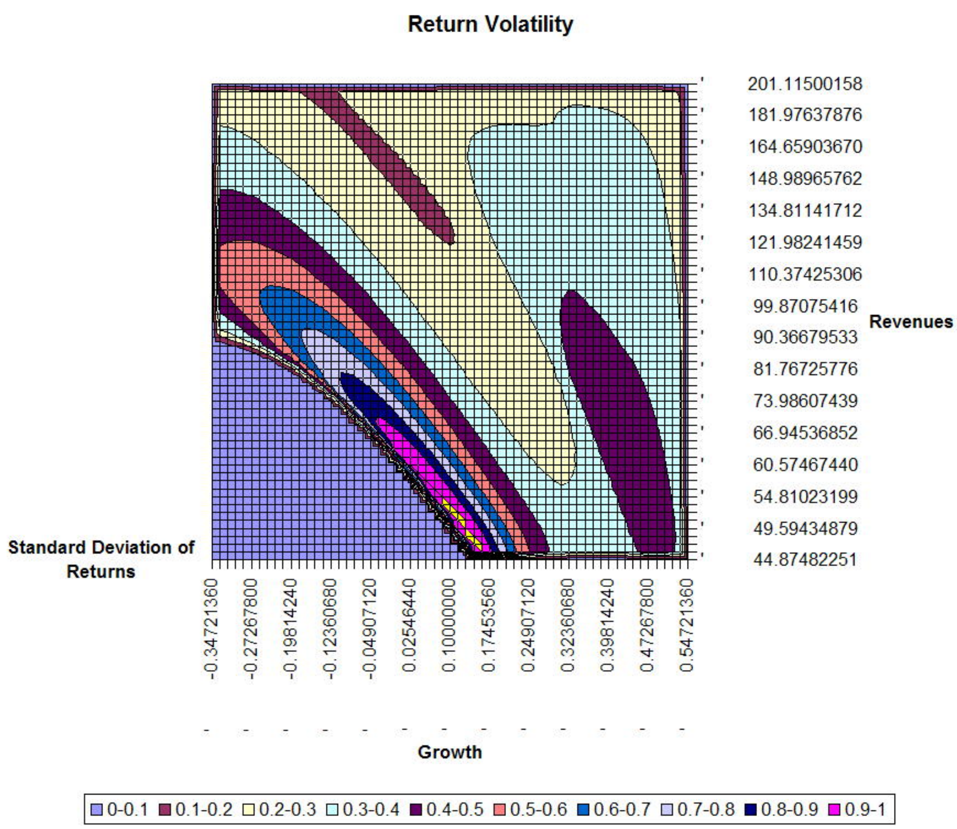
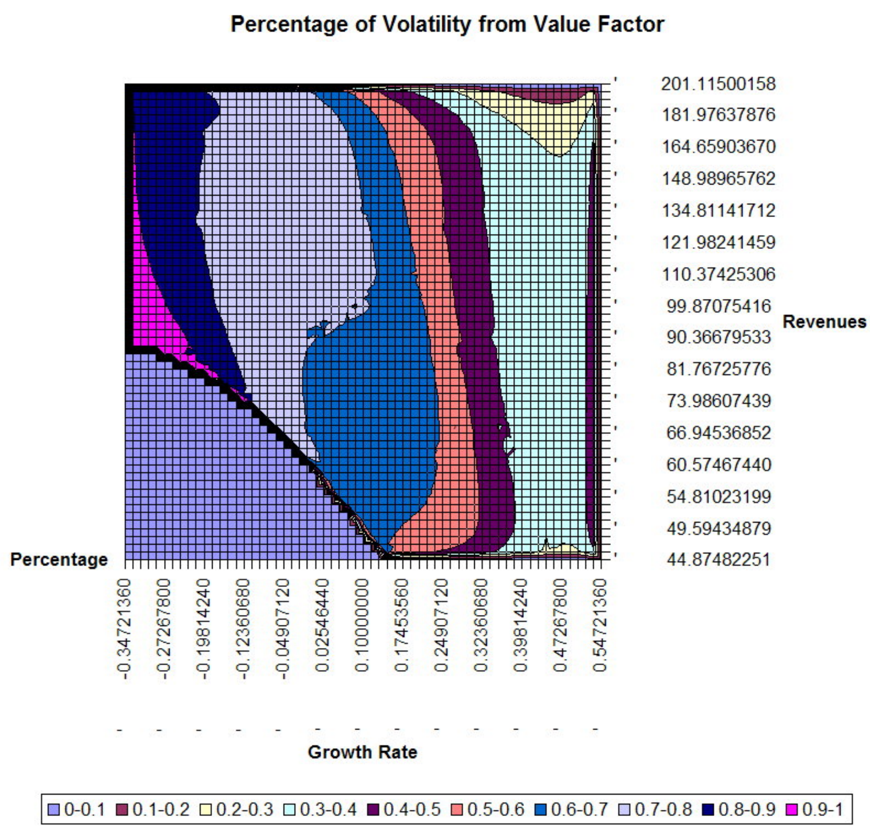

Value-Weighted True Distribution of Market Capitalization
201.11500158
181.97637876
164.65903670
148.98965762
134.81141712
121.98241459
110.37425306
99.87075416
90.36679533
81.76725776
73.98607439
66.94536852
60.57467440
54.81023199
49.59434879
44.87482251
Revenues
Density
-0.34721360
-0.27267800
-0.19814240
-0.12360680
-0.04907120
0.02546440
0.10000000
0.17453560
0.24907120
0.32360680
0.39814240
0.47267800
0.54721360
Growth
0-0.0005
0.0005-0.001
0.001-0.0015
0.0015-0.002
0.002-0.0025
0.0025-0.003

Figure 8:



Return Volatility
201.11500158
181.97637876
164.65903670
148.98965762
134.81141712
121.98241459
110.37425306
99.87075416
90.36679533
81.76725776
73.98607439
66.94536852
60.57467440
54.81023199
49.59434879
44.87482251
Revenues
Standard Deviation of Returns
-0.34721360
-0.27267800
-0.19814240
-0.12360680
-0.04907120
0.02546440
0.10000000
0.17453560
0.24907120
0.32360680
0.39814240
0.47267800
0.54721360
Growth
0-0.1
0.1-0.2
0.2-0.3
0.3-0.4
0.4-0.5
0.5-0.6
0.6-0.7
0.7-0.8
0.8-0.9
0.9-1

Figure 9:


Percentage of Volatility from Value Factor
201.11500158
181.97637876
164.65903670
148.98965762
134.81141712
121.98241459
110.37425306
99.87075416
90.36679533
81.76725776
73.98607439
66.94536852
60.57467440
54.81023199
49.59434879
44.87482251
Revenues
Percentage
-0.34721360
-0.27267800
-0.19814240
-0.12360680
-0.04907120
0.02546440
0.10000000
0.17453560
0.24907120
0.32360680
0.39814240
0.47267800
0.54721360
Growth Rate
0-0.1
0.1-0.2
0.2-0.3
0.3-0.4
0.4-0.5
0.5-0.6
0.6-0.7
0.7-0.8
0.8-0.9
0.9-1