

# Dynamic Security Design\*

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## Abstract

We analyze dynamic financial contracting under moral hazard. The ability to rely on future rewards relaxes the tension between incentive and participation constraints relative to the static case. Entrepreneurs are incited to effort by the promise of future payments after several successes and the threat of liquidation after several failures. The more severe the moral hazard problem, the greater the liquidation risk. The optimal contract can be implemented by holding cash reserves and issuing debt and equity. The firm is liquidated when it runs out of cash. Dividends are paid only when accumulated earnings reach a certain threshold. In the continuous-time limit of the model, stock prices follow a diffusion process, with a stochastic volatility that increases after price drops. In line with empirical findings, performance shocks induce long lasting changes in leverage.

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# 1 Introduction

While corporate finance theory has generated interesting insights within one or two-period models, financial data and business operations are inherently dynamic. How robust are the insights of the theoretical models in dynamic settings? Which new effects arise in such settings? How do the securities designed in a dynamic world differ from their static counterparts? Can theoretical models be used to impose restrictions on time series data? The present paper aims at answering these questions. We also take a first step towards bridging the gap between corporate finance and asset pricing. Our dynamic corporate finance model characterizes the process of endogenous securities payoffs and offers some insights into their dynamic pricing.

In our model, an entrepreneur has discovered an investment project, which only her can manage successfully. She has limited wealth and limited liability. She contacts financiers, who can be a group of dispersed investors, to fund the initial investment and operating costs. Both the entrepreneur and the financiers are risk-neutral. After the initial investment, the project yields random observable and contractible cash-flows at each period. These cash-flows can be positive, i.e., operating profits, or negative, i.e., operating losses. The net present value of the project is positive if the entrepreneur exerts effort at each period. Managerial effort is not observable, however. Thus, there is a moral hazard problem: the entrepreneur is the agent and the financiers the principal.<sup>1</sup> We analyze the optimal contract, specifying the payments to the entrepreneur and the financiers as well as the decision to liquidate the firm or to continue operating it, as a function of observable variables.

Our model can be viewed as a generalization of the unobservable effort model of Innes (1990) and Holmström and Tirole (1997) to the dynamic case. As in Spear and Srivastava (1987) and Thomas and Worall (1990), we use the continuation payoff of the entrepreneur as the state variable, upon which the optimal contract is contingent. The intuition is that, at each point in time, the (unobservable) action of the entrepreneur only depends on what she expects to earn in the future, as a function of her performance, as well as on current transfers. Thus, the optimal contract, regulating the incentives of the entrepreneur, is also a function of her expected gains in all the continuation subgames. Also, because we consider an infinite horizon problem with i.i.d. shocks, the optimal contract is stationary (rather than time dependent), which considerably simplifies the analysis. Using dynamic programming techniques to solve for the optimal contract, we write the social value of the firm as the solution to a Bellman equation. The command variables are the functions mapping the state variable into the payoffs to the entrepreneur and the financiers, and the liquidation decision. We show that there exists a unique solution to this problem and characterize this solution.

The optimal contract pins down the dynamics of the continuation payoff, and thus the incentives of the entrepreneur. It evolves stochastically with the cash-flows generated by the firm between an upper bound and a lower bound. After profits, it increases, while after losses it decreases. In the optimal contract, the entrepreneur is compensated by current transfers when the firm has accumulated a strong performance record and the upper bound is reached. On the other hand, after a series of losses, the continuation payoff of the entrepreneur goes down,

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<sup>1</sup>This bears similarities with a model without effort decision, but where cash-flows are privately observable by the entrepreneur, as in Diamond (1984) and Bolton and Scharfstein (1990). Dynamic versions of the privately observed cash-flow model are studied by Gromb (1999), Clementi and Hopenhayn (2002) and DeMarzo and Fishman (2002). In the same spirit, Thomas and Worall (1990) and Albuquerque and Hopenhayn (2004) assume that, after cash-flows have been generated, the entrepreneur can collect them and default on her obligations to the lenders. Similarly, Quadrini (2004) assumes that the entrepreneur can divert some of the resources available to the firm.

and the risk of liquidation increases. If the sequence of bad draws continues too long, the lower bound is reached and the firm is liquidated.

As in Holmström and Tirole (1997), the income which can be pledged to the outside financiers is the total discounted expected operating income from the project, minus the income which must be promised to the entrepreneur to incite her to exert effort, as specified in the optimal contract. When the moral hazard problem is too severe, the pledgeable income is lower than the need for outside funds, and the project cannot be funded.

Our analysis encompasses the case where the entrepreneur and the financiers are equally impatient and the case where the entrepreneur is more impatient than the financiers. The main qualitative difference between the two cases is that, when discount rates are equal, the upper bound on the state variable is an absorbing barrier, while it is reflecting when the entrepreneur is more impatient than the financiers. The state variable is the discounted sum of expected future transfers received by the entrepreneur. When the entrepreneur and the financiers have the same discount rate, financial frictions can be eliminated altogether if the state variable reaches the upper threshold. It then stays constant, and the entrepreneur is promised sustained future rents. When the entrepreneur is more impatient than the financiers, she prefers trading off some of these rents against current consumption.

To analyze further the implications of our theoretical analysis, we study in detail the implementation of the optimal contract. We focus on the case where the outside financiers hold securities, i.e., claims to non-negative cash-flows. Thus, the financiers can form a diffuse investor base, with limited liability. This narrows the set of implementation devices and enables one to then analyze the dynamic market pricing of these securities. Since operating cash-flows can be negative, and the entrepreneur has limited liability, non-negative payments to security holders require that the firm hoard cash reserves. Hence, in our implementation of the optimal contract, we determine the cash-flow statement of the firm: cash-flows from operations, plus interests earned on cash reserves, and changes in cash reserves equal payments to security holders. Thus, cash reserves can be interpreted as accumulated earnings. The firm is liquidated when cash reserves drop to zero. Intuitively enough, the securities implementing the optimal contract are debt and equity. Debt pays a non-negative coupon at each period until the firm is liquidated. Equity receives dividends when accumulated earnings are sufficiently large. A fraction of the dividends accrues to the entrepreneur (inside equity), while the other fraction accrues to the financiers (outside equity). As mentioned above, when the discount rate of the financiers is equal to that of the entrepreneur, the upper bound of the state variable is absorbing. In our implementation this has an intuitive interpretation. The upper bound corresponds to the case where so much cash has been accumulated that the interest on cash reserves is enough to finance the coupon and the operating costs. At this point, informational constraints do not bind any longer. We also find that the greater the magnitude of the moral hazard problem, the greater the need to provide incentives to the entrepreneur, the greater the fraction of the equity allocated to the entrepreneur, and the lower the initial feasible leverage. This, in turn, reduces the pledgeable income and makes credit rationing more likely.

The continuous-time limit of our analysis is sufficiently tractable that we can characterize the dynamics of the values of the securities arising endogenously in the model. When the firm generates a series of positive cash-flows, the market value of debt and that of equity rise. Simultaneously, the firm becomes more liquid, as the ratio of cash reserves to debt increases. While positive cash flows reduce the probability of liquidation, default risk is never eliminated when the entrepreneur is more impatient than the financiers. Even when the firm fares quite well, there remains a risk that it will decline in the future, and eventually be liquidated. Hence,

the bond price includes a default risk premium. Furthermore, leverage, measured as the ratio of the market value of debt to the market value of equity, decreases after the firm has performed well. While the volatility of operational cash-flows is constant in our model, the volatility of the stock price is stochastic, and larger for smaller firms. Finally, our theoretical model generates a rich set of implications on the relationships between moral hazard problems, firm performance, liquidity ratios, book to market ratios and price earning ratios.

The characterization of debt and equity as optimal contracts in our dynamic model differs from that arising in one period security design models. In the latter, different securities correspond to different functions (concave for debt and convex for equity) of the (single and final) cash-flow.<sup>2</sup> In our model different securities correspond to different time series of payoffs, more in line with stylized facts. Debt pays out a stream of coupons, until the firm is liquidated. Equity pays off less frequently and randomly, when accumulated earnings reach a prespecified threshold.

The present paper is in line with the insightful analyzes of dynamic financial contracting by Gromb (1999), Clementi and Hopenhayn (2002), and DeMarzo and Fishman (2002).<sup>3</sup> The nature of the moral hazard problem we consider is different, since they analyze the unobservable cash-flow case (ex-post moral hazard), while we focus on the unobservable effort case (ex-ante moral hazard). The optimal contract we obtain differs from that arising in Clementi and Hopenhayn (2002) and DeMarzo and Fishman (2002). In particular, we find different partitions of the state space, giving rise to different optimal policies. Also, because of the nature of the problem we analyze and the mathematical techniques we use, we are able to provide a more explicit characterization of the optimal policy and the associated social surplus. Furthermore, our implementation of the optimal contract via securities differs from the previous literature on dynamic financing. DeMarzo and Fishman (2002) rely on a credit line. Clementi and Hopenhayn (2002) do not analyze how the claims of the investors can be split between equity and debt. Another contribution of our analysis, relatively to these papers, lies in the implications generated by our model for the continuous-time dynamics of book to market ratios, leverage, liquidity, and default risk.<sup>4</sup>

Our analysis of the dynamics of bond prices in presence of imperfections complements the literature extending the bond pricing model of Merton (1974) to the cases where there are bankruptcy costs (Leland (1994), Leland and Toft (1996)) or where debtors behave strategically (Anderson and Sundaresan (1996), Acharya, Huang, Subrahmanyam and Sundaram (2002)). Our paper differs from these because the financial claims we study arise endogenously as optimal contracts. This enables us to endogenize certain features of their dynamics. For example, that liquidation occurs when the stock price reaches zero is an assumption in Leland (1994), while it is a result in our analysis.<sup>5</sup>

An interesting stream of papers studies dynamic moral hazard with a risk-averse agent (see for instance Rogerson (1985), Spear and Srivastava (1987), Fudenberg, Holmström and Milgrom

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<sup>2</sup>See, e.g., Gale and Hellwig (1985), and Innes (1990).

<sup>3</sup>Clementi and Hopenhayn (2002) offer an interesting analysis of the dynamics of firm size. This is beyond the scope of the present paper.

<sup>4</sup>In contemporaneous work, DeMarzo and Sannikov (2004) have analyzed a continuous-time model building on DeMarzo and Fishman (2002). Their approach is quite different from ours. While they directly analyze the continuous-time model, we first solve the discrete case and then prove convergence of the value function and the optimal contract. Thus the mathematical methods used in the two papers are different.

<sup>5</sup>Without information asymmetry, it would be conceivable to write a contract such that, for some paths, the stock price would go to zero, while the firm would be continued, and all its cash-flows would accrue to the investors holding the bonds issued by the firm. With moral hazard this is not feasible. Indeed, the entrepreneur must be promised dividends to incite her to exert effort. Without such effort it is suboptimal to continue the firm.

(1990), and Chiappori, Macho, Rey and Salanié (1994)). The present paper avoids the difficulties associated with the trade-off between the desire to insure the agent and the need to provide her incentives. The relative simplicity of the problem we analyze enables us to characterize more explicitly the optimal mechanism. Obviously, unlike in the risk-averse case, there is no intertemporal smoothing of the agent's consumption in the optimal contract we obtain.

The paper is organized as follows. Section 2 outlines the model, proves existence and uniqueness of the solution and establishes general properties. Section 3 derives the optimal contract and offers an implementation with securities—debt and inside and outside equity—in the simple case in which the discount rate of the entrepreneur is equal to that of the financiers. This gives us an opportunity to present some of the key economic intuitions of our results in a relatively tractable context. Section 4 analyzes the more difficult case where the entrepreneur is more impatient than the financiers. After characterizing the optimal policy in that case, we analyze the continuous-time limit of the model, and in particular the convergence of the value function. Section 5 concludes. All proofs are in the Appendices.

## 2 The Model

### 2.1 The Environment

We extend to the dynamic case the one-period analysis of corporate financing under moral hazard of Holmström and Tirole (1997). We first consider a discrete-time infinite-horizon model. Then we study its continuous-time limit.

There are two types of agents: an entrepreneur and financiers. For simplicity, all agents are assumed to be risk-neutral. Only the entrepreneur is endowed with the specific skills required to operate a given investment project. Undertaking the project requires an initial investment outlay  $I \geq 0$  at date 0. At each subsequent date, the project can be continued or liquidated. The liquidation value of the project is equal to 0.

At any date  $t = 0, 1, \dots$  prior to liquidation, the entrepreneur can decide to exert effort,  $e_t = 1$ . In this case, with probability  $p$ , the operations of the firm during the period are successful and the cash-flow generated by the project is  $R_t = R_+ > 0$ , while with probability  $1 - p$  there is a failure and the cash-flow is  $R_t = R_- < 0$ . If the entrepreneur does not exert effort,  $e_t = 0$ , she gets a private benefit  $B$ , but the probability of success is only  $p - \Delta p < p$ . Cash-flows are assumed to be independently distributed across periods. We denote:

$$\bar{R} = pR_+ + (1 - p)R_-,$$

$$\underline{R} = (p - \Delta p)R_+ + (1 - p + \Delta p)R_-.$$

Operational cash-flows are equal to revenues minus costs. The project is profitable if the entrepreneur exerts effort, while it has a negative social value if she does not:

$$\bar{R} > 0 > \underline{R} + B.$$

If the entrepreneur had deep pockets, she would be able to fund the initial investment  $I$  and the operating expenses  $R_-$ . Being the owner of the project, she would then exert the optimal level of effort,  $e = 1$ . In this paper, however, we assume the entrepreneur has a limited amount

of cash  $A < I - (1 + r)R_-/r$ , so she needs to contract with outside financiers.<sup>6</sup> Two key assumptions prevent from achieving the first-best outcome with such lending contracts. First, the entrepreneur has limited liability. Hence, contracts cannot stipulate negative payments to the entrepreneur. Second, managerial effort and the consumption of private benefits are not observable. Consequently, there is a moral hazard problem. We assume that:

$$\frac{pB}{\Delta p} > \bar{R}.$$

Under this assumption, the moral hazard problem is severe, as the temptation to shirk is greater when  $B$  is large and  $\Delta p$  is low. Indeed, in the static version of the model, this condition implies that even if  $I = 0$ , the project cannot be financed with certainty (or at full scale, see below) whenever the entrepreneur holds only a small amount of cash, see Subsection 2.2.

We denote by  $\rho$  the discount rate of the entrepreneur and by  $r$  that of the financiers, and we assume that  $\rho \geq r$ . A difference in discount rates may arise because of the desire of the entrepreneur to consume, and the desire of the financiers to save. With different discount rates, there are potential gains from trade between the entrepreneur and the financiers, as the former would like to borrow from the latter. It will turn out that the features of the optimal contract and the dynamics of the project depend crucially on whether  $\rho$  is strictly larger than or equal to  $r$ . Before dealing separately with each case, we first recall the properties of the optimal contract in the static model of Holmström and Tirole (1997). Then we move to the repeated version of this model, and we show that the dynamic program characterizing the optimal lending contract admits a unique solution and establish some general properties of this solution.

## 2.2 The Static Version of the Model

In this section, for simplicity, we assume that  $I = 0$ , so that the entrepreneur only needs to finance its operating cost  $-R_-$ . As above, let  $A$  denote its self-financing capacity. The contract specifies the promised repayments to the entrepreneur ( $u_+$  in case of success and  $u_-$  in case of failure) and the probability  $x$  that the firm is allowed to operate (as we shall see below,  $x$  can also be interpreted as a scaling factor). The optimal contract maximizes expected output  $x\bar{R}$  under the incentive compatibility constraint of the entrepreneur:

$$\Delta p(u_+ - u_-) \geq B, \tag{1}$$

the limited liability constraint:

$$(u_-, u_+) \in \mathbb{R}_+^2, \tag{2}$$

the feasibility constraint:

$$x \in [0, 1], \tag{3}$$

and the financiers' participation constraint:

$$x\bar{R} + A \geq x[pu_+ + (1 - p)u_-]. \tag{4}$$

The solution is then immediate.

- The entrepreneur is remunerated only in case of success:  $u_+ = B/\Delta p$ ,  $u_- = 0$ .

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<sup>6</sup>The amount  $I - (1 + r)R_-/r$  corresponds to the sum of the investment cost  $I$  and of the present value, actualized at rate  $r$ , of operating expenses  $-R_-$  for all future periods. If the initial cash holdings  $A$  of the entrepreneur exceed this amount, perfect self-financing is possible.

- The project is operated with certainty (or at full scale) only if the entrepreneur has enough cash:  $x = 1$  if  $A \geq pB/\Delta p - \bar{R} \equiv A^*$ .
- If the entrepreneur does not have enough cash, the probability (or scale) of operation is reduced proportionally to  $A$ :  $x = A/A^*$  if  $A < A^*$ .

To set the stage for the analysis of the dynamic case, we focus on the expected utility of the entrepreneur, denoted by  $w$ , and the expected surplus generated by the contract, denoted by  $V(w)$ . One has  $w = x\bar{R} + A$  and  $V(w) = x\bar{R}$ . The optimal contract is thus characterized by two regions:

- (i) For  $w \geq w^* = pB/\Delta p$ , the firm operates with probability one and the surplus is maximum:

$$V(w) = \bar{R}.$$

- (ii) For  $w < w^*$ , the probability of operation is less than 1 and the surplus is linear in  $w$ :

$$V(w) = \frac{\bar{R}w}{w^*}.$$

Notice that effort is always optimal, as for all  $w \geq 0$ ,  $V(w) \geq 0 > \underline{R} + B$ . Figure 1 represents this optimal contract.

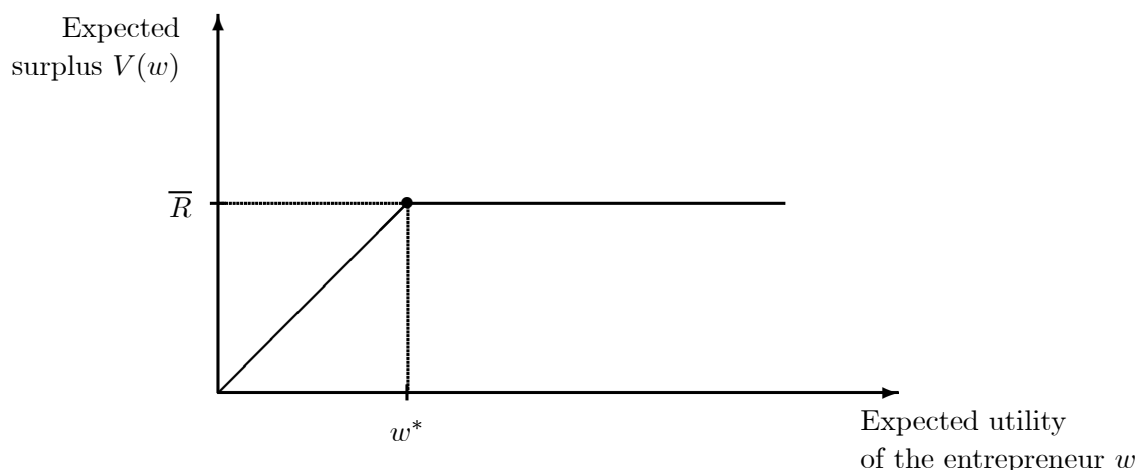


Figure 1: The optimal contract in the static case.

### 2.3 The Dynamic Programming Formulation of the Problem

To focus on a single source of market imperfection (the unobservability of effort), we assume that the entrepreneur and the financiers can fully commit to a long-term lending contract. Two contractual instruments can be used to cope with the moral hazard problem. First, the investment project can be liquidated if the entrepreneur does not perform well. Second, transfers can be made to the entrepreneur if she is successful. Denote by  $w$  the expected discounted utility of the entrepreneur. As in Spear and Srivastava (1987), this is the state variable upon which the contract is contingent. In any period prior to liquidation, a long-term contract specifies as a function of  $w$  a continuation probability  $x$  and, conditional on the project not being liquidated, an effort level  $e$ , transfers to the entrepreneur  $u_+$  and  $u_-$  contingent on success or failure,

and contingent continuation utilities for the entrepreneur,  $w_+$  and  $w_-$ .<sup>7</sup> Equivalently,  $x$  can be understood as a scaling factor. In that interpretation, when  $x < 1$ , the firm continues to operate, but on a smaller scale. Specifically, cash-flows are irreversibly downsized by a factor  $x$ , along with the expected utilities of the entrepreneur and of the financiers. We focus on contracts in which the entrepreneur always exerts effort, conditional on the project being continued. We show in Appendix 1 (Proposition 13) that this is indeed optimal when  $\overline{R} - \underline{R}$  is large enough.

At any date  $t = 0, 1, \dots$  prior to liquidation, the financiers receive the cash-flow from the project, net of the transfer to the entrepreneur,  $R_t - u_t$ . Denote by  $F(w)$  the expected discounted utility of the financiers given a promised expected discounted utility  $w$  for the entrepreneur. The function  $F$  satisfies the following Bellman equation:

$$F(w) = \max x \left[ \overline{R} - pu_+ - (1-p)u_- + \frac{pF(w_+) + (1-p)F(w_-)}{1+r} \right] \quad (5)$$

for all  $w \geq 0$ , subject to several constraints (constraints (6)–(9) stated below). First, the contract must satisfy a consistency condition, stating that the expected discounted utility of the entrepreneur at the beginning of any period must be equal to the expected payment made to her during the period plus the expected present value of her continuation utility:

$$w = x \left[ pu_+ + (1-p)u_- + \frac{pw_+ + (1-p)w_-}{1+\rho} \right]. \quad (6)$$

Second, we impose the condition that the entrepreneur must always prefer to exert effort. Effort is optimal for the agent if the increase in expected utility it generates exceeds the loss of private benefits it implies:

$$\Delta p \left( u_+ + \frac{w_+}{1+\rho} - u_- - \frac{w_-}{1+\rho} \right) \geq B. \quad (7)$$

Finally, the following limited liability constraints are required:

$$(u_-, u_+, w_-, w_+) \in \mathbb{R}_+^4, \quad (8)$$

as well as the feasibility constraint:

$$x \in [0, 1]. \quad (9)$$

Variables  $u_+$  and  $u_-$  (current payments to the agent) can be eliminated from the objective function by introducing an auxiliary function defined by  $V(w) = w + F(w)$ . The function  $V$  satisfies the following Bellman equation:

$$V(w) = \max x \left\{ \overline{R} + \frac{pV(w_+) + (1-p)V(w_-)}{1+r} - \frac{(\rho-r)[pw_+ + (1-p)w_-]}{(1+r)(1+\rho)} \right\} \quad (10)$$

for all  $w \geq 0$ , subject to constraints (6)–(9). This function measures the social surplus generated by the project, which is independent of transfers effected during the period. Since, as a result,  $u_-$  and  $u_+$  do not appear in (10), it will be useful to eliminate them from the constraints as well. Given a triple  $(x, w_-, w_+) \in [0, 1] \times \mathbb{R}_+^2$ , one can find a pair  $(u_-, u_+)$  satisfying (6)–(9) if and only if the set of  $(u_-, u_+) \in \mathbb{R}_+^2$  such that:

$$u_+ - u_- \geq \frac{B}{\Delta p} - \frac{w_+ - w_-}{1+\rho},$$

$$w - x \left[ \frac{pw_+ + (1-p)w_-}{1+\rho} \right] \geq x[pu_+ + (1-p)u_-]$$

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<sup>7</sup>In full generality, one should also allow for transfers to the entrepreneur in case of liquidation. However, promising positive transfers to the entrepreneur in case of liquidation would obviously be suboptimal, as this would weaken incentives. To ease the exposition, we therefore rule out such transfers from the outset.



is non-empty. The second constraint trivially implies that

$$w \geq \frac{x[pw_+ + (1-p)w_-]}{1+\rho}.$$

Now if the right-hand side of the first constraint is negative, then  $u_+ = u_- = 0$  is trivially feasible. It remains to see what happens when this right-hand side is non-negative, in which case  $u_+ \geq u_-$ . This case is represented in Figure 2 below.

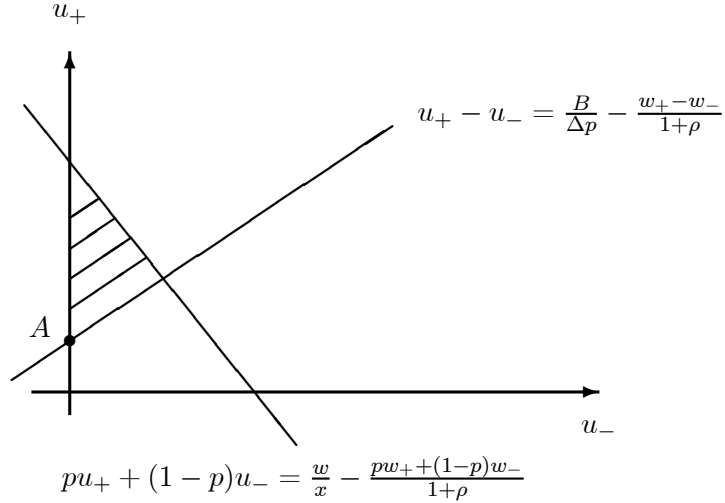


Figure 2: The feasible set of contemporary payments  $u_+$ ,  $u_-$ .

Thus the feasible set is not empty if and only if  $A = (0, B/\Delta p - (w_+ - w_-)/(1 + \rho))$  is feasible, which is equivalent to:

$$p\left(\frac{B}{\Delta p} - \frac{w_+ - w_-}{1 + \rho}\right) \leq \frac{w}{x} - \frac{pw_+ + (1-p)w_-}{1 + \rho}.$$

After simplifying this expression, we conclude that constraints (6)–(9) are equivalent to:

$$w \geq x\left(\frac{w_-}{1 + \rho} + \frac{pB}{\Delta p}\right), \quad (11)$$

$$w \geq x\left[\frac{pw_+ + (1-p)w_-}{1 + \rho}\right], \quad (12)$$

together with:

$$(x, w_-, w_+) \in [0, 1] \times \mathbb{R}_+^2. \quad (13)$$

It follows immediately from (11) and (13) that for  $w < pB/\Delta p$ , liquidation must occur with positive probability or, equivalently, downsizing must take place. The probability of liquidation goes to 1 as  $w$  goes to 0. This reflects the fact that the optimal contract relies on the threat of project termination as well as the promise of future rents to provide incentives to the entrepreneur. When rents are low, the threat of liquidation becomes the major instrument. Note that this threat is socially costly since, even at the time of liquidation, the project remains profitable.

Our first result shows existence and uniqueness of  $V$ , and establishes some crucial properties:

**Proposition 1.** *There exists a unique continuous and bounded solution  $V$  to (10) subject to (11)–(13).  $V$  is non-decreasing, concave,  $V(0) = 0$ , and there exists  $w^* > 0$  such that  $V$  is strictly increasing over  $[0, w^*)$  and constant over  $[w^*, \infty)$ .*

Proposition 1 states that the social surplus  $V(w)$  is increasing in the entrepreneur’s rent  $w$ . As the firm improves its performance record, the utility of the entrepreneur increases, and the probability of liquidation decreases. This decrease generates an increase in the social surplus from the project. The proposition also states that  $V(0) = 0$ . Indeed, under moral hazard, profitable operations are sustainable only if the entrepreneur is left a rent. Finally, the proposition states that  $V$  is concave. When  $w$  is small, liquidation risk is high. Good firm performance significantly enhances value by reducing this risk. By contrast, when  $w$  is very high, the liquidation risk is small. Hence, the impact of strong performance on liquidation risk and thus value is relatively small.

To gain further insights into the dynamics of the incentives provided to the entrepreneur and the value of the firm, consider  $\hat{w}$ , the smallest point at which the mapping  $w \mapsto V(w) - (\rho - r)w/(1 + \rho)$  reaches its maximum. To maximize the value function given by (10) one would like to set  $x = 1$  and  $w_+ = w_- = \hat{w}$ . This is possible, and the value function reaches its maximum  $V(w^*)$ , when, for this choice of the command variables, the constraints (11), (12) and (13) hold. When  $x = 1$  and  $w_+ = w_- = \hat{w}$ , the relevant constraint is (11). It holds when  $w$  is at least as large as  $\hat{w}/(1 + \rho) + pB/\Delta p$ . Hence, the thresholds  $\hat{w}$  and  $w^*$  are related by:

$$w^* = \frac{\hat{w}}{1 + \rho} + \frac{pB}{\Delta p}. \quad (14)$$

Whenever  $w \geq w^*$ , it is never optimal to liquidate the project,  $x = 1$ , and the continuation utility for the entrepreneur is the same after a success and a failure,  $w_- = w_+ = \hat{w}$ . Contingent transfers that implement the high effort decision are  $u_+ = w - w^* + B/\Delta p$  after a success and  $u_- = w - w^*$  after a failure.<sup>8</sup> Thus, when the expected discounted utility of the entrepreneur is high, she is incited to effort by contingent transfers within the period, and not through continuation utilities or current liquidation threats. Building on Proposition 1, the next proposition provides additional information on the upper bound for the continuation expected discounted utility of the entrepreneur,  $w^*$ .

**Proposition 2.**  *$\hat{w} = w^*$  if  $\rho = r$ , while  $\hat{w} < w^*$  if  $\rho > r$ .*

The proof that  $\hat{w} < w^*$  if  $\rho > r$  relies on a simple variational argument.<sup>9</sup> Given that when  $w \geq w^*$ ,  $w_- = w_+ = \hat{w}$ , a key consequence of Proposition 2 is that  $w^*$  is an absorbing barrier for the expected discounted utility of the entrepreneur when  $\rho = r$ , while it is a reflecting barrier when  $\rho > r$ . Hence, whether discount rates are different or identical for the entrepreneur and the financiers has major consequences for the dynamics of the lending relationship. We shall therefore address each case separately, starting with the simplest case in which  $\rho = r$ .

### 3 Identical Discount Rates

We first characterize the optimal contract in the case where  $\rho = r$ . We then discuss how to implement it with securities.

<sup>8</sup>Note that if  $w > w^*$ , there are other pairs  $(u_-, u_+)$  that implement  $e = 1$  given  $x = 1$  and  $w_- = w_+ = \hat{w}$ . For  $\rho > r$  and  $w = w^*$ , this is the only optimal contract.

<sup>9</sup>One may be tempted to argue that  $V'(\hat{w}) = (\rho - r)/(1 + \rho) > 0 = V'(w^*)$ , so that  $w^* > \hat{w}$  by concavity of  $V$ . This presumes, however, that  $V$  is differentiable at  $\hat{w}$  and  $w^*$ , which is not always true as we will see below.

### 3.1 The Optimal Contract

If  $\rho = r$ , the dynamic program defining  $V$  simplifies to:

$$V(w) = \max x \left[ \bar{R} + \frac{pV(w_+) + (1-p)V(w_-)}{1+r} \right] \quad (15)$$

for all  $w \geq 0$ , subject to:

$$w \geq x \left( \frac{w_-}{1+r} + \frac{pB}{\Delta p} \right), \quad (16)$$

$$w \geq x \left[ \frac{pw_+ + (1-p)w_-}{1+r} \right], \quad (17)$$

and (13). Under the mild additional restriction that  $p > r$ ,<sup>10</sup> it turns out that the optimal contract can be characterized in terms of three thresholds  $w^* > w^{**} > w^{***}$  for the expected discounted utility of the entrepreneur. These thresholds are defined as follows:

$$w^* = \frac{(1+r)pB}{r\Delta p}, \quad (18)$$

$$w^{**} = \frac{[(1+r)p - r]B}{r\Delta p} = w^* - \frac{B}{\Delta p}, \quad (19)$$

$$w^{***} = \frac{pB}{\Delta p}. \quad (20)$$

The next proposition gives a characterization of the optimal contract.

**Proposition 3.** *When  $\rho = r$  and  $p > r$ , the following contract is optimal:*

- (i)  $w^*$  is an absorbing state for the expected discounted utility of the entrepreneur. Once it is reached, the project is operated with certainty forever. In all periods, the entrepreneur receives a transfer  $u_+ = B/\Delta p$  in case of success and  $u_- = 0$  in case of failure;
- (ii) When  $w^{**} \leq w < w^*$ , the project is operated with certainty in the current period. The entrepreneur receives a transfer  $u_+ = w - w^{**}$  in case of success and  $u_- = 0$  in case of failure, and her continuation utility is  $w_+ = w^*$  in case of success and  $w_- = (1+r)(w - pB/\Delta p)$  in case of failure;
- (iii) When  $w^{***} \leq w < w^{**}$ , the project is operated with certainty in the current period. The entrepreneur receives no transfers,  $u_+ = u_- = 0$ , and her continuation utility is  $w_+ = (1+r)[w + (1-p)B/\Delta p]$  in case of success and  $w_- = (1+r)(w - pB/\Delta p)$  in case of failure;
- (iv) When  $0 \leq w < w^{***}$ , the project is operated with probability  $w/w^{***}$  and liquidated with probability  $1 - w/w^{***}$ . Equivalently, the project is downsized by a factor  $w/w^{***}$ . If the project is operated, the entrepreneur receives no transfers,  $u_+ = u_- = 0$ , and her continuation utility is  $w_+ = (1+r)[w^{***} + (1-p)B/\Delta p]$  in case of success and  $w_- = 0$  in case of failure.

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<sup>10</sup>Indeed, the probability of success in any given period,  $p$ , can be reasonably expected to be larger than 1/2, while the interest rate over the period,  $r$ , is likely to be less than 10%. This condition is automatically satisfied when the duration of each period goes to 0, a case we study below.

The four regions spelled out in the proposition differ in terms of which constraints bind. In the lower region  $[0, w^{***})$ , all constraints bind and  $x$  must be set strictly below 1. This corresponds to a financial distress region for the firm. In the next region  $[w^{***}, w^{**})$ ,  $x$  is equal to 1, but the incentive compatibility and limited liability conditions still bind, and the entrepreneur receives no transfers. This can be interpreted as a trial region for the firm. The following region  $[w^{**}, w^*)$  correspond to the case where the limited liability condition no longer binds, but the incentive compatibility condition does. In this region, the entrepreneur receives dividends in case of success. Finally, in the upper region  $[w^*, \infty)$ , the constraints are not binding any longer. This sequence reflects the fact that, as the accumulated performance of the firm increases, the constraints bind less and less, and, at the extreme of the spectrum, when  $w = w^*$ , we reach the unconstrained case, where information asymmetry and limited liability do not matter any longer. It should be noted that, for reasonable parametrization of the model, the trial region  $[w^*, w^{**})$  is likely to be large in comparison to the financial distress region  $[0, w^*]$  and the dividend region  $[w^{**}, w^{***})$ .<sup>11</sup>

Proposition 3 sheds light on the relative roles of promises and threats in coping with the moral hazard problem.

First, unless  $w \geq w^*$ , the entrepreneur receives transfers only in case of success and when  $w$  reaches the upper bound  $w^*$ , either because it was already there, or because it was in  $[w^{**}, w^*)$  and the project was successful. The reason why it is not optimal to reward the entrepreneur earlier, i.e., for lower values of  $w$  is that this would be dominated by promising the same expected discounted payment, contingent on  $w$  reaching  $w^*$ . Indeed, such a contract would provide better incentives to the entrepreneur, at the same cost for outside financiers.

Second, once  $w$  has reached  $w^*$ , the project is insulated from the threat of liquidation, and financial constraints cease to bind. The expected social surplus generated by the project is then given by the present value of a perpetual annuity of  $\bar{R}$ ,

$$V^* = \frac{(1+r)\bar{R}}{r}.$$

Third, over  $[0, w^*)$ , the expected discounted utility of the entrepreneur evolves randomly as a function of the performance of the project, increasing after successes and decreasing after failures. Consider for example the region  $[w^{***}, w^{**})$  of the state space. In this region, both constraints (16) and (17) are binding, and the continuation utility of the entrepreneur evolves according to:

$$w_{t+1} = (1+r)[w_t + k(R_t - \bar{R})],$$

where  $R_t$  is the cash-flow generated by the project at date  $t$ , and:

$$k = \frac{B}{(R_+ - R_-)\Delta p} < 1 \tag{21}$$

measures the sensitivity of the entrepreneur's reward to the performance of the project, which increases in the magnitude of the moral hazard problem, as measured by  $B/\Delta p$ . In this region, the entrepreneur is only compensated by the promise of future rents, and her continuation utility grows on average at a rate equal to her discount rate,

$$\mathbb{E}[w_{t+1}] = (1+r)w_t.$$

Deviations around this deterministic trend reflect the performance of the project.

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<sup>11</sup>For instance, if  $p = 1/2$  and  $r = 10\%$ ,  $w^{**} - w^{***} = 8w^{***}$  and  $w^* - w^{**} = 2w^{***}$ .

As pointed out above, it follows from (11) and (13) that for  $w < pB/\Delta p$ , the promise of future rents gives the entrepreneur insufficient incentives, so that incentives must be provided also by the threat of liquidation or downsizing. Note however that, in the optimal contract, immediate liquidation or downsizing never occurs as long as  $w$  is above  $w^{***} = pB/\Delta p$ . This reflects the fact that shutting down the project is a costly way to provide incentives to the entrepreneur, which is used only as a last resort. Indeed, the following result indicates that, at  $w^{***}$ , the agents are not indifferent at the margin between liquidating and continuing the project. Specifically, it illustrates the strong preference to avoid liquidation and the sharp decrease in social surplus incurred when  $w$  drops below  $w^{***}$  and liquidation must be resorted to.

**Corollary 1.**  *$V$  is not differentiable at  $w^{***}$ .*

Figure 3 represents the optimal contract.

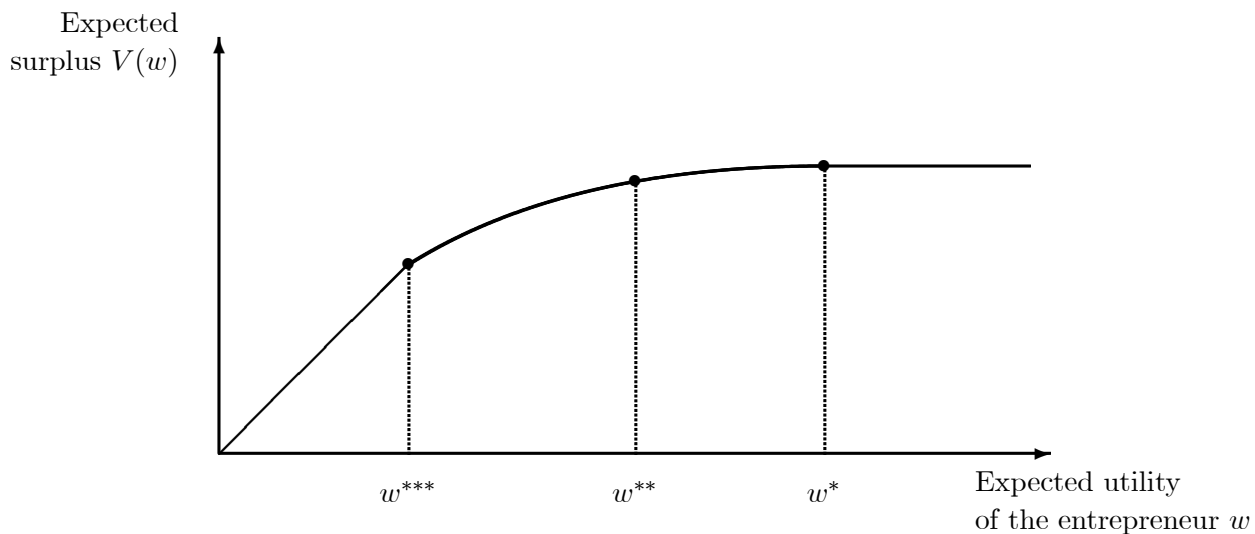


Figure 3: The optimal contract in the dynamic case.

Formulas (18)–(20) show that the greater  $B/\Delta p$ , the greater the thresholds  $w^{***}$ ,  $w^{**}$  and  $w^*$ . This reflects the fact that when the agency problem becomes more severe, the threat of liquidation is used more often, and it takes a larger number of successes for the entrepreneur to be compensated with immediate contingent transfers and eventually reach financial safety.

While the value function  $V$  is unique, there are multiple optimal policies solving the dynamic programming problem (13)–(17).<sup>12</sup> The intuition for this indeterminacy is as follows. The initial agency problem arises because the entrepreneur does not have enough wealth to fund the project by herself. When her expected discounted utility reaches the region  $[w^*, \infty)$ , the project has generated so much wealth that this problem does not arise any longer. Hence, decisions can be made without concern for informational asymmetry. In this case, however, the decision whether to save or to consume immediately is indeterminate, since the discount rate  $\rho$  of the entrepreneur is equal to the interest rate  $r$ , and both the entrepreneur and the financiers are risk-neutral. In the next section, we consider the case  $\rho > r$ . In that case, the solution is no longer indeterminate, since the entrepreneur prefers to consume immediately rather than to save at rate  $r$ . The optimal contract derived in Proposition 3 corresponds to the limit of the optimal contract when  $\rho$  converges to  $r$  from above. In this solution, the entrepreneur is

<sup>12</sup>Indeed, one can even construct a “bubble” solution in which (17) is always binding, so that the entrepreneur never actually receives any transfers, and her expected discounted utility grows at rate  $r$  on average.

rewarded by immediate contingent transfers whenever his expected discounted utility  $w$  reaches  $w^*$ . Specifically, when  $w \geq w^*$ , the entrepreneur first receives a special bonus  $w - w^*$ , and thereafter receives a transfer  $B/\Delta p$  whenever the project is successful and no transfer whenever the project fails. Her continuation utility then stays constant at  $w^*$ .

$V$  is the total economic surplus which can be generated by the project. At date 0, if the project is undertaken, the initial rent of the entrepreneur is  $w_0$ , while the initial expected discounted payoff to financiers is  $V(w_0) - w_0$ . Thus, the maximum initial pledgeable income is:

$$\max_{w \geq 0} \{V(w) - w\}.$$

Consider a situation where the entrepreneur has a cash endowment  $A$  that can be used to finance partially the investment  $I$ . Two cases arise. If the maximum pledgeable income is lower than the need for outside funds,  $I - A$ , then the project cannot be financed and there is credit rationing, although the project has positive net present value. This market breakdown problem is similar to that arising in the one-period model of Holmström and Tirole (1997). If, on the contrary,

$$\max_{w \geq 0} \{V(w) - w\} \geq I - A,$$

then the project can be financed. In this case, while the moral hazard problem would have precluded financing the project with a one-period horizon, the ability to use the promise of future rents to provide incentives to the entrepreneur relaxes the tension between incentive and participation constraints, thus providing access to external funding. If the entrepreneur has all the bargaining power, she chooses the initial rent to maximize her expected utility, subject to the financiers' participation constraint:

$$w_0^m = \max\{w \geq 0 \mid V(w) - w \geq I - A\}.$$

On the other hand, if the financiers have the bargaining power, they choose the initial rent to maximize their expected profit:

$$w_0^f \in \arg \max_{w \geq 0} \{V(w) - w\}.$$

Since  $V$  is concave and bounded above, this leads in general to a lower initial rent ( $w_0^f \leq w_0^m$ ), corresponding to lower social surplus and economic efficiency. This is in line with the rent-efficiency trade-off typically arising in principal-agent problems. An empirical implication is that financiers' market power generates greater liquidation risk. Note that since she is cash-constrained, the entrepreneur cannot generate a Pareto improvement by offering a monetary transfer in exchange for an increase in  $w$ .

### 3.2 Implementing the Optimal Contract with Cash and Securities

We now turn to the implementation with financial contracts of the optimal mechanism characterized above. In general there are multiple ways of implementing the optimal contract. As long as incentive compatibility, participation and feasibility constraints are satisfied, the Modigliani and Miller (1958) logic applies: slicing and dicing of cash-flows is irrelevant. To narrow down the set of implementations, we impose the palatable restriction that outside financiers hold securities, i.e., claims with limited liability, excluding negative payoffs. These claims can be held by a diffuse investor basis. Now, operating cash-flows can be negative, in case of failure. Hence,

to avoid negative payoffs to outside financiers, the firm must hold cash reserves. The change in the level of reserves is equal to the sum of the interest rate received on the reserves and of the operating cash-flow from the project minus the compensation of the entrepreneur and the payoff to financiers. Thus, the cash reserves correspond to accumulated earnings. That the firm is liquidated when cash reserves go to 0 is an implication of the limited liability of security holders and  $R_- < 0$ .<sup>13</sup>

For purposes of implementation, we shall interpret  $x$  as a scaling factor, and not as a liquidation probability. Thus, when  $x_t < 1$ , the firm continues to operate, but on a scale reduced by a factor  $x_t$ . The size of the firm at the beginning of date  $t$  is thus  $X_t = \prod_{s=0}^{t-1} x_s$ . The quantities  $w_t$  and  $V(w_t)$  must then be interpreted as the expected utility of the entrepreneur and the expected social surplus normalized by the size of the firm. Note that this interpretation is consistent with all the formal results hitherto obtained.

In the abstract implementation of the optimal contract, the state variable was the rent of the entrepreneur,  $w_t$ . In the implementation of the contract, payoffs and decisions are contingent on the size-adjusted accumulated earnings, that is the ratio  $m_t$  of accumulated earnings  $M_t$  at date  $t$  to the size of the firm  $X_t$ . The size-adjusted accumulated earnings  $m_t$  will be designed so that  $m_t = w_t/k$ , where  $k$  is defined by (21). Thus, to characterize the implementation of the optimal contract, we define three thresholds:

$$m^* = \frac{(1+r)p(R_+ - R_-)}{r},$$

$$m^{**} = \frac{[(1+r)p - r](R_+ - R_-)}{r},$$

$$m^{***} = p(R_+ - R_-),$$

corresponding respectively to  $w^*$ ,  $w^{**}$ , and  $w^{***}$ . The implementation of the optimal contract with securities and cash reserves is given in the next proposition.

**Proposition 4.** *The optimal contract can be implemented with stocks distributing dividends whenever  $m_t$  is larger or equal than  $m^*$  at the end of period  $t$ , and bonds distributing a coupon  $C_t = X_t \bar{R}$  per period. At date 0, the firm issues the stock and the bond, and grants a fraction  $k$  of the equity to the entrepreneur, where  $k$  is defined by (21). With the proceeds from the issuance, the firm acquires the asset and hoards an amount of cash  $M_0 = w_0/k$ . When  $m_t < m^{***}$ , cash reserves are insufficient to meet the firm's short-term commitments (operating cost  $-X_t R_-$  and coupon  $C_t$ ). Hence it enters financial distress. In that case, the firm is downsized by a factor  $m_t/m^{***}$ . Then, after failure the firm is liquidated, while after success the firm emerges from financial distress but keeps operating on a smaller scale.*

This implementation of the optimal contract with securities and cash reserves has several palatable features.

First, the endogenous securities we obtain fit several stylized facts. Unlike the optimal securities arising in one-period models, which are defined as functions of a single cash-flow, our securities are defined as streams of cash-flows. Debt is a claim on a steady stream of constant coupons. Equity is a claim on a more irregular stream of cash, paid only after accumulated

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<sup>13</sup>Note that these restrictions on the implementation of the optimal mechanism do not affect the efficiency of the outcome. We *are* implementing the optimal contract.

earnings have reached a threshold. These payments can be made under the form of dividends or share repurchases.

Second, both sides of the balance sheet matter, and in particular liquidity plays a crucial role. Liquidation occurs when the firm does not have enough cash to meet its obligations to bondholders and suppliers. At the other end of the spectrum, when the size-adjusted accumulated earnings have reached  $m^*$ , interest income on cash is enough to pay the debt coupon and the operating cost. At this point, financial constraints due to information asymmetry cease to bind.

Third, outside funding can be provided by a diffuse investor base. Each investor holds limited liability securities. That the investor base is diffuse precludes renegotiation of the financial contract.<sup>14</sup>

That payments can be made to shareholders only when accumulated reserves are sufficiently large is in line with observed legal and contractual restrictions. Young firms, especially in the high-tech industry, pay no cash to shareholders for long periods of time. This is in line with our analysis, where the firm distributes dividends only when it has grown relatively mature, i.e., after sufficient time for the size-adjusted accumulated earnings to reach  $m^*$ . While standard dividend processes tend to be smoother, it should be noted that firms increasingly tend to use share repurchases rather than regular dividends to pay off shareholders. Such repurchases are much more erratic than standard dividends. Our theoretical analysis generates implications for the dynamics of cash reserves, transfers to stockholders and profits which are in line with some of the empirical results of Grullon and Michaely (2002). They find no evidence that share repurchases are followed by improvements in profitability. While at odds with the signaling theory of share repurchases, this finding is consistent with our moral hazard theory. Also consistent with the implications from our analysis, Grullon and Michaely (2002) find that share repurchases reduce the firms' cash reserves.

## 4 Different Discount Rates

In this section, we first characterize the optimal contract in the case where  $\rho > r$ . We then examine the properties of this optimal contract when the time interval between periods becomes small.

### 4.1 The Optimal Contract

As was discussed above after Proposition 2 and will be confirmed by the propositions below, when  $\rho > r$ ,  $w^*$  is a reflecting barrier. This contrasts with the equal discount rates case, where the upper bound was an absorbing barrier. The intuition for this difference is the following: When  $w$  reaches an absorbing barrier, the expected utility of the entrepreneur is high, thanks to the promise of future transfers. When the entrepreneur is impatient, however, she prefers to lower these future transfers in exchange for current consumption. This enables the entrepreneur to raise  $u$ , at the expense of a drop of  $w$  below  $w^*$ . While intuitive, this tradeoff makes the analysis more complex than in the case where  $\rho = r$ . Depending on the parameters of the model, two different regimes can arise, spelled out in Propositions 5 and 6 below.

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<sup>14</sup>One could construct an alternative implementation of the optimal contract, relying on credit lines, as in DeMarzo and Fishman (2002). While the analysis in terms of credit lines generates interesting insights, one advantage of our alternative approach is that it emphasizes the role played by securities and cash reserves.



To identify the relevant regions in the parameter space we first need to define recursively the auxiliary sequence  $\{X_n\}$  by:

$$X_0 = 1, \tag{22}$$

$$X_{n+1} = \frac{(1+r)X_n}{(1-p)(1+\rho)} - \frac{p}{(1-p)(1+\rho)^{n+1}}.$$

It is easy to check that  $X_n \geq (1+r-p)^n / [(1-p)^n(1+\rho)^n] > 0$  for each  $n \in \mathbb{N}$ . A simple computation then implies that for  $n$  large enough, one has  $\sum_{k=0}^n X_k \geq (1+\rho)/(\rho-r)$ . Thus, since  $\bar{R} < pB/\Delta p$ , there exists a unique integer  $\bar{n} \in \{-1\} \cup \mathbb{N}$  such that:

$$\frac{\rho-r}{1+\rho} \sum_{n=0}^{\bar{n}} X_n < \frac{\bar{R}}{pB/\Delta p} \leq \frac{\rho-r}{1+\rho} \sum_{n=0}^{\bar{n}+1} X_n, \tag{23}$$

with the convention that  $\sum_{n=0}^{-1} = 0$ . It turns out that if:

$$p \sum_{n=1}^{\bar{n}+1} \frac{1}{(1+\rho)^n} \leq 1, \tag{24}$$

with the convention that  $\sum_{n=1}^0 = 0$ , the optimal contract can be characterized in terms of two thresholds  $w^* \geq w^{***}$  for the expected discounted utility of the entrepreneur, defined by:

$$w^* = \frac{pB}{\Delta p} \sum_{n=0}^{\bar{n}+1} \frac{1}{(1+\rho)^n}, \tag{25}$$

$$w^{***} = \frac{pB}{\Delta p}, \tag{26}$$

Note that from (14) and (25),

$$\hat{w} = \frac{pB}{\Delta p} \sum_{n=0}^{\bar{n}} \frac{1}{(1+\rho)^n}. \tag{27}$$

One has then the following result.

**Proposition 5.** *When (24) holds, the following contract is optimal:*

- (i) *When  $w \geq w^*$ , the project is operated with certainty in the current period. The entrepreneur receives a transfer  $u_+ = w - w^* + B/\Delta p$  in case of success and  $u_- = w - w^*$  in case of failure, and her continuation utility is  $w_+ = w_- = \hat{w}$  in both cases;*
- (ii) *When  $w^{***} \leq w < w^*$ , the project is operated with certainty in the current period. The entrepreneur receives a transfer  $u_+ = w - w^* + B/\Delta p$  in case of success and  $u_- = 0$  in case of failure, and her continuation utility is  $w_+ = \hat{w}$  in case of success and  $w_- = (1+r)(w - pB/\Delta p)$  in case of failure;*
- (iii) *When  $0 \leq w < w^{***}$ , the project is operated with probability  $w/w^{***}$  and liquidated with probability  $1 - w/w^{***}$ . If it is operated, the entrepreneur receives no transfers,  $u_+ = u_- = 0$ , and her continuation utility is  $w_+ = \hat{w}$  in case of success and  $w_- = 0$  in case of failure.*

When (24) holds, the optimal contract is characterized by at most three regions. The value function  $V$  is piecewise linear, and can be explicitly computed, see (64) and (65)–(67), which correspond respectively to the case  $\bar{n} = -1$  and  $\bar{n} \in \mathbb{N}$ .<sup>15</sup> Besides the fact that  $w^*$  is no longer an absorbing barrier for the expected discounted utility of the entrepreneur, the key difference with Proposition 3 is that, whenever  $w > w^{***}$ , constraint (12) is never binding, so that contingent transfers are always used as an incentive device, along with promised continuation utilities.<sup>16</sup> Thus, current transfers are used for a greater subset of the state space than in Proposition 3. This is in line with the fact that, when  $\rho > r$ , current consumption by the entrepreneur is more valuable than when  $\rho = r$ .

To verify that (12) holds, note that it is satisfied for  $w = w^{***} = pB/\Delta p$  as:

$$\frac{pB}{\Delta p} \geq \frac{p(pB/\Delta p) \sum_{n=0}^{\bar{n}} 1/(1+\rho)^n}{1+\rho} = \frac{p\hat{w}}{1+\rho} \quad (28)$$

because of (24) and (27). In particular, (12) is slack for  $w > w^{***}$ . The fact that  $w_+$  remains constant over  $[w^{***}, w^*]$  and that  $w_-$  is a linear function of  $w$  over this interval explains the piecewise linear structure of  $V$ .

There is a simple characterization of the circumstances under which the optimal contract is of the form outlined in Proposition 5. As in the previous section, define:

$$w^{**} = w^* - \frac{B}{\Delta p}. \quad (29)$$

From (14), one has:

$$\hat{w} = (1+\rho) \left[ w^{**} + \frac{(1-p)B}{\Delta p} \right]. \quad (30)$$

Under (24), it follows from (14) and (28) that  $w^{**} \leq pB/\Delta p$ . Conversely, suppose this holds. It is then clear from (30) that given  $w = w^{***} = pB/\Delta p$ , the choice  $(x, w_-, w_+) = (1, 0, \hat{w})$  satisfies (11)–(12). It turns out that this choice is indeed optimal, and one has the following result.

**Corollary 2.** *When  $w^{**} \leq pB/\Delta p$ , (24) holds and the optimal contract is as characterized in Proposition 5.*

Now suppose that (24) does not hold. Then by Corollary 2,  $w^{**} > pB/\Delta p$ , and the optimal contract can be characterized in terms of three thresholds  $w^* > w^{**} > w^{***}$  for the expected discounted utility of the entrepreneur, where  $w^*$  and  $w^{**}$  are related by (29).

**Proposition 6.** *When (24) does not hold, the following contract is optimal:*

- (i) *When  $w \geq w^*$ , the project is operated with certainty in the current period. The entrepreneur receives a transfer  $u_+ = w - w^* + B/\Delta p$  in case of success and  $u_- = w - w^*$  in case of failure, and her continuation utility is  $w_+ = w_- = \hat{w}$  in both cases;*

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<sup>15</sup>When  $\bar{n} = -1$ , one has  $w^* = w^{***}$ , so that the intermediary regime vanishes, and the financial relationship lasts for at most one period. In line with the fact that a one-period project is not viable by assumption, this case is degenerate in the sense that the investment capacity is 0. Indeed, since  $\bar{R} < pB/\Delta p$ , the slope of  $V$  is strictly less than 1 by (64), so that  $F(w) = V(w) - w < 0$  for all  $w > 0$ .

<sup>16</sup>This markedly differs from the optimal contract presented by DeMarzo and Fishman (2002). In particular, in their analysis, the entrepreneur receives transfers only when the state variable reaches the upper bound.

- (ii) When  $w^{**} \leq w < w^*$ , the project is operated with certainty in the current period. The entrepreneur receives a transfer  $u_+ = w - w^{**}$  in case of success and  $u_- = 0$  in case of failure, and her continuation utility is  $w_+ = w^*$  in case of success and  $w_- = (1+r)(w - pB/\Delta p)$  in case of failure;
- (iii) When  $w^{***} \leq w < w^{**}$ , the project is operated with certainty in the current period. The entrepreneur receives no transfers,  $u_+ = u_- = 0$ , and her continuation utility is  $w_+ = (1+r)[w + (1-p)B/\Delta p]$  in case of success and  $w_- = (1+r)(w - pB/\Delta p)$  in case of failure;
- (iv) When  $0 \leq w < w^{***}$ , the project is operated with probability  $w/w^{***}$  and liquidated with probability  $1 - w/w^{***}$ . If it is operated, the entrepreneur receives no transfers,  $u_+ = u_- = 0$ , and her continuation utility is  $w_+ = (1+r)[w^{***} + (1-p)B/\Delta p]$  in case of success and  $w_- = 0$  in case of failure.

While the thresholds  $w^{***}$ ,  $w^{**}$  and  $w^*$  cannot be analytically characterized, unlike in the case where  $\rho = r$ , Proposition 6 parallels Proposition 3.<sup>17</sup> As mentioned above, the main qualitative difference is that, when  $\rho > r$ ,  $w^*$  is a reflecting barrier.

## 4.2 The Case of a Small Time Interval between Periods

The goal of this subsection is to analyze the continuous-time limit of the model. To do so, denote by  $h$  be the time interval between periods. We hereafter focus on the case where  $h$  is small, the limit being the continuous-time case. We want to write the cash-flow process in such a way that it converges to a diffusion when  $h$  goes to 0. Thus we write the parameters of the model as functions of  $h$ , such that the mean and variance of returns per unit of time are preserved, i.e., means and variances are linear in time:

$$pR_+ + (1-p)R_- = \mu h,$$

$$\Delta p(R_+ - R_-) = \Delta \mu h,$$

$$p(1-p)(R_+ - R_-)^2 = \sigma^2 h.$$

We also impose that the variance of the cash-flows be unaffected by shirking:<sup>18</sup>

$$p(1-p)(R_+ - R_-)^2 = (p - \Delta p)(1 - p + \Delta p)(R_+ - R_-)^2$$

There is a unique solution to this system, namely:

$$p_h = \frac{1}{2} \left[ 1 + \frac{h\Delta\mu}{\sqrt{(h\Delta\mu)^2 + 4\sigma^2 h}} \right], \quad (31)$$

$$\Delta p_h = \frac{h\Delta\mu}{\sqrt{(h\Delta\mu)^2 + 4\sigma^2 h}}, \quad (32)$$

<sup>17</sup>Like Proposition 5, Proposition 6 differs from the result obtained by DeMarzo and Fishman (2002). In particular, the optimal policy we characterize varies across four regions, while the optimal contract in DeMarzo and Fishman (2002) involves only three regimes. Also, our analysis emphasizes that, depending on parameter values, two different situations can arise, spelled out in Propositions 5 and 6. This contrasts with the optimal contract arising in DeMarzo and Fishman (2002).

<sup>18</sup>If these variances were different, shirking could be detected instantaneously in the continuous-time limit of the model by computing the quadratic variation of the cash-flows.

$$R_{h,-} = \left( \mu - \frac{\Delta\mu}{2} \right) h - \frac{1}{2} \sqrt{(h\Delta\mu)^2 + 4\sigma^2 h}, \quad (33)$$

$$R_{h,+} = \left( \mu - \frac{\Delta\mu}{2} \right) h + \frac{1}{2} \sqrt{(h\Delta\mu)^2 + 4\sigma^2 h}. \quad (34)$$

Our objective is to examine the model in which  $p_h$ ,  $\Delta p_h$ ,  $R_{h,-}$  and  $R_{h,+}$  are given by (31)–(34),  $B$ ,  $r$  and  $\rho$  are multiplied by  $h$ , and  $h$  is close to 0. As  $h$  goes to 0, and conditional on high effort being exerted in all periods and no liquidation occurring, the cash-flow process converges in distribution to a Brownian motion with drift:

$$dR_t = \mu dt + \sigma dZ_t; \quad t \geq 0,$$

where  $Z = \{Z_t; t \geq 0\}$  is a standard Brownian motion. Within this framework, we first analyze the convergence of the optimal value function and the optimal contract. In Subsections 4.3 and 4.4, we discuss implications of our analysis for the dynamics of securities values.

Denote by  $V_h$  the associated value function solution to (10)–(13) for these values of the parameters. One has first the following result.

**Proposition 7.** *When  $h$  is close to 0, (24) does not hold and the optimal contract is as characterized in Proposition 5, with critical thresholds  $w_h^* > w_h^{**} > w_h^{***}$ . As  $h$  goes to 0, the liquidation region reduces to a point,  $\lim_{h \downarrow 0} w_h^{***} = 0$ .*

The following proposition characterizes the limit of the value functions  $V_h$  as  $h$  goes to 0.

**Proposition 8.** *As  $h$  goes to 0, the value function  $V_h$  converges uniformly to the unique solution  $V_0$  to the free boundary value problem:*

$$rV(w) = \mu - (\rho - r)w + \rho w V'(w) + \frac{B^2 \sigma^2}{2\Delta\mu^2} V''(w); \quad w \in [0, w^*), \quad (35)$$

$$V(w) = V(w^*); \quad w \in [w^*, \infty), \quad (36)$$

$$V(0) = 0, \quad (37)$$

$$V'(w^*) = 0, \quad (38)$$

$$V''(w^*) = 0. \quad (39)$$

The continuous-time Bellman equation (35) is obtained through a second order Taylor expansion of the discrete-time Bellman equation (10). The initial condition (37) together with the supercontact conditions (38)–(39) pin down a unique solution  $V_0$  to (35) and a unique value for the threshold  $w^*$ .

In the continuous-time limit of the model, the expected utility of the entrepreneur follows the process  $\{W_t; t \geq 0\}$ , which is the limit in distribution (as  $h$  goes to 0) of the process of the continuation utilities arising in the discrete-time model. Denote by  $\tau_0$  the first time  $t$  at which  $W_t = 0$ , and by  $L_t$  the cumulated payment to the entrepreneur.  $\{L_t; t \geq 0\}$  is a continuous non-negative and non-decreasing process that increases only when  $W_t = w^*$ .

In the continuous-time limit, the entrepreneur's rent evolves according to the stochastic differential equation:

$$dW_t = \rho W_t dt + \frac{B\sigma}{\Delta\mu} dZ_t - dL_t; \quad t \in [0, \tau_0],$$

where  $\{Z_t; t \geq 0\}$  is a standard Brownian motion. Whenever  $W_t$  reaches 0, the project is liquidated and the continuation payoffs are 0 for both the entrepreneur and the financiers. Whenever  $W_t$  reaches  $w^*$ , the entrepreneur receives a transfer  $dL_t$ , and  $W_t$  bounces back in  $[0, w^*]$ . The entrepreneur receives no transfers in the region  $[0, w^*]$ .

In this context, as shown in Appendix 4,  $V_0$  admits a natural probabilistic interpretation. It can be represented as follows:

$$V_0(w) = w + \mathbb{E} \left[ \int_0^{\tau_0} e^{-rt} (\mu dt - dL_t) \right]; \quad w \in [0, w^*].$$

That is, the social surplus generated by the project is equal to the sum of the expected utility of the entrepreneur and the present value of the expected cash-flows to be received by the financiers.

### 4.3 Implementation

In the continuous-time limit, the optimal contract can be implemented with cash reserves, stocks and bonds, as in Subsection 3.2. Again, the stock is a claim to dividends, paid out at the stochastic dates at which the firm's cash reserves hit a given threshold, while the bond is a claim to a continuous stream of coupons. Also, the firm is liquidated when it runs out of cash and thus defaults on the bond. This suggests imposing the cash reserves  $M_t$  to be a deterministic function  $g(W_t)$  of the entrepreneur's rent  $W_t$ , that satisfies  $g(0) = 0$ . For simplicity, we assume that  $g$  is twice continuously differentiable.

The dynamics of cash reserves  $\{M_t; t \geq 0\}$  is given by:

$$dM_t = (rM_t + \mu)dt + \sigma dZ_t - dL_t - dP_t; \quad t \in [0, \tau_0], \quad (40)$$

where  $dP_t$  denotes the flow payment to outside financiers, and  $dL_t$  is the payment to the entrepreneur, as characterized above. Implementing the contract with limited liability securities implies that  $\{P_t; t \geq 0\}$  must be a non-decreasing process. In analogy with (21), let:

$$k = \frac{B}{\Delta\mu} < 1, \quad (41)$$

which, as in the discrete-time version of the model, measures the sensitivity of the entrepreneur's reward to the performance of the project. We will assume hereafter that  $\mu \geq (\rho - r)w^*/k$ , which, as is easily seen from (35)–(39), is typically the case whenever  $k$  is close to 1. Hinging on the fact that liquidation occurs when the firm runs out of cash, the following lemma details the dynamics of  $M_t$  and  $P_t$ .

**Lemma 1.** *For any  $t \in [0, \tau_0]$ ,  $M_t$  and  $W_t$  are related by  $M_t = W_t/k$ . The dynamics of cash reserves in the continuous time limit of our model is:*

$$dM_t = \rho M_t dt + \sigma dZ_t - \frac{1}{k} dL_t, \quad t \in [0, \tau_0], \quad (42)$$

while the dynamics of the financier's payoff is:

$$dP_t = [\mu - (\rho - r)m_t]dt + \frac{1 - k}{k} dL_t, \quad t \in [0, \tau_0]. \quad (43)$$

A natural decomposition of  $dP_t$  is to allocate its first component (which is in  $dt$ ) to one security and the second (which is in  $dL_t$ ) to a second security. The former pays a continuous stream of cash flow, while the latter pays out only when accumulated earnings reach a threshold. We interpret these securities as bonds and stocks, respectively.

*Stocks.* Stocks pay cash to their shareholders via dividends or share repurchases, whenever accumulated earnings  $M_t$  reach the threshold  $m^* = w^*/k$ . More formally, the dividend and share repurchase process is  $\{1_{\{t < \tau_0\}} L_t/k; t \geq 0\}$ . Let  $S_t$  denote the market value of the stock:

$$S_t = \mathbb{E} \left[ \int_t^\tau \frac{e^{-r(s-t)}}{k} dL_s \right].$$

Then,  $S_t = S(M_t)$ , where  $S$  is the solution of the differential equation:

$$rS(m) = \rho m S'(m) + \frac{\sigma^2}{2} S''(m); \quad m \in [0, m^*];$$

$$S(0) = 0,$$

$$S'(m^*) = 1.$$

Thus  $S_t$  obeys:

$$dS_t = rS_t dt + S_t \sigma^S(S_t) dZ_t - \frac{1}{k} dL_t, \quad (44)$$

where  $\sigma^S(s) = S'(S^{-1}(s))/s$ . An alternative and more intuitive formulation of the dynamics of the stock is that:

$$\frac{dS_t}{S_t} = r dt + \sigma^S(S_t) dZ_t$$

when  $S_t < S(m^*)$  and  $S_t$  bounces back each time it hits  $S(m^*)$ . We therefore obtain a stock dynamics of the Black and Scholes (1973) type, with however three important differences. First, the market value of the stock is bounded above by  $S(m^*)$ , as dividends are distributed every time  $S_t = S(m^*)$  and the stock price then bounces back. Next,  $S_t$  can attain 0 with positive probability, because  $\lim_{s \downarrow 0} \sigma^S(s) = \lim_{s \downarrow 0} S'(0)/s = +\infty$ . Last, the volatility of the stock increases when the value of the stock decreases. This is akin to what financial econometricians have dubbed the “leverage effect.”

*Bonds.* The bond is a claim to an instantaneous payment equal to  $1_{\{t < \tau_0\}}[\mu - (\rho - r)m_t]$ .<sup>19</sup> Let  $D_t$  denote the market value of the bond:

$$D_t = \mathbb{E} \left[ \int_t^\tau e^{-r(s-t)} [\mu - (\rho - r)m_t] dt \right].$$

Then,  $D_t = D(M_t)$ , where  $D$  is the solution of the differential equation:

$$rD(m) = \mu - (\rho - r)m + \rho m D'(m) + \frac{\sigma^2}{2} D''(m); \quad m \in [0, m^*],$$

$$D(0) = 0,$$

$$D'(m^*) = 0.$$

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<sup>19</sup>Alternatively, this security could be split into two securities, namely a straight bond, with constant coupon  $\mu - (\rho - r)m^*$ , and a contingent claim with payment  $(\rho - r)(m^* - m_t)$ . Note that both have limited liability.

Thus  $D_t$  obeys:

$$dD_t = rD_t dt + D_t \sigma^D(D_t) dZ_t - [\mu - (\rho - r)M_t] dt, \quad (45)$$

where  $\sigma^D(d) = D'(D^{-1}(d))/d$ .

#### 4.4 Empirical Implications

We can now discuss the dynamics of security values in the continuous-time limit of our model. First, note that, unlike when  $\rho = r$ , default risk is never eliminated. Even when  $M_t$  reaches the upper bound  $m^*$ , it is reflected downwards, and the probability that it goes down to 0 is strictly positive. Second, the stock price increase with the level of cash reserves of the firm. Specifically, one has the following lemma.

**Lemma 2.** *The stock price  $S(m)$  is a non-decreasing and concave function of the level  $m$  of cash reserves.*

Lemma 2 and equation (44) imply that stock prices are more volatile than earnings and, unlike earnings, have a stochastic volatility that increases after stock prices drops. The economic interpretation of this pattern is the following. After a series of failures, the cash reserves of the firm run down, so that bankruptcy risk becomes a serious issue, and the firm reaches the area where its social value, which is a concave function of  $w$  or  $m$ , reacts strongly to current performance. In this region the stock is very volatile. In contrast, after a series of successes, the firm is cash rich. Because the value function is concave, at this stage it reaches the region where it reacts only weakly to current performance. Hence volatility is limited.

Our approach also enables us to discuss the evolution of the leverage ratio of the firm.

**Proposition 9.** *The leverage expressed in market values,  $D(m)/S(m)$ , is a decreasing function of the level  $m$  of cash reserves.*

Proposition 9 implies that performance shocks and stock price movements are followed by persistent changes in the leverage of the firm. Such persistent changes have been documented by recent empirical analyzes (see for instance Welch (2004)). In discussing these results, Welch (2004, page 107) questions why “issuing activities are not used to counterbalance stock return induced equity value changes.” This could indeed sound puzzling, if one relied upon a static model, such as for instance the tradeoff theory, according to which there exists an optimal leverage ratio, to which companies should endeavor to revert. By contrast, persistent changes in financial structure after stock price movements are an implication of our model, rather than a puzzle. We show indeed that the optimal contract can be implemented without further issuing of securities.

The empirical finance literature has emphasized the correlation between book to market ratios and key economic variables such as returns. Our model can be used to generate implications on the joint dynamics of the book and market values of balance sheet items. At time  $t = 0$ , the asset side of the balance sheet includes cash ( $m_0$ ), property, plant and equipment ( $I$ ), and goodwill ( $G$ ), reflecting the net present value created by the project. On the liability side it includes debt ( $D_0$ ) and equity ( $S_0$ ). Thus, the initial value of equity is:

$$S_0 = I + G + m_0 - D_0.$$

At time  $t > 0$ , the only item which has changed on the asset side of the balance sheet is cash ( $m_t$ ). Similarly, on the liability side, debt is still accounted for at its historical value. Consequently, the book value of equity is:

$$I + G + m_t - D_0 = S_0 + m_t - m_0.$$

Correspondingly, the book to market ratio is:

$$BM_t = \frac{S_0 - m_0 + m_t}{S_t}.$$

Note that in our implementation, the book to market ratio is a deterministic function  $BM(m)$  of the cash reserves  $m$ . Relying on the previous propositions, the dynamics of the book to market ratio can be characterized as follows.

**Proposition 10.** *The book to market ratio  $BM(m)$  is a U-shaped function of the level  $m$  of cash reserves, that is decreasing at  $m_0$ .*

Obviously, the book to market ratio goes to infinity when  $m_t$  goes to 0, since the book value of the assets is bounded away from 0 while their market value goes to 0. Hence, this ratio is decreasing at 0. The increase in the book to market ratio occurring when cash balances rise towards  $m^*$  reflects the concavity of  $S(m)$ . Proposition 10 generates the following empirical implications. For financially distressed firms and young, recently established firms, stock returns should be negatively correlated with changes in book to market ratios. For mature, cash rich firms, changes in book to market ratios and stock returns should be positively correlated. While, in line with Fama and French (1993), the asset pricing literature has studied the predictive power of the level of book to market ratios for subsequent price returns, our theory implies a non-monotonic relation between contemporaneous changes in book to market ratios and stock returns.

Our next result provides some comparative statics properties of the initial balance sheet. For simplicity, we assume that the entrepreneur has all the bargaining power and no initial cash,  $A = 0$ .

**Proposition 11.** *The maximal amount of funds that the firm can raise is increasing with respect to  $\mu$ , and decreasing with respect to  $\sigma$  and  $k$ .*

A large  $\mu$  entails that the project has a large net present value, while large  $\sigma$  and  $k$  imply a severe agency problem. Not surprisingly, in line with the static version of this model (Holmström and Tirole (1997)), we find that the investment capacity of the firm increases with respect to the profitability of the project and decreases with respect to the severity of the agency problem and the risk of the operating cash-flows.

Proposition 11 states that the initial amount raised from outside financiers decreases with the moral hazard parameter,  $k$ . Because funds used must equal funds provided, the initial amount raised from outside financiers is equal to the investment  $I$ , plus the initial cash reserves  $m_0$ , minus the funds  $A$  provided by the entrepreneur. Thus, the initial amount of cash reserves is decreasing in the magnitude of the moral hazard problem. Otherwise stated, young firms facing severe moral hazard problems should be relatively illiquid. To implement this empirical implication from our model, one needs to proxy for  $k$ . Greater moral hazard problems can be expected in industries where managers have more discretion, and thus greater opportunities to shirk, with relatively limited impact on cash-flows. This is likely to be the case when competition in the product market is limited. It can also be expected in innovative and service industries with high immaterial expenses, such as R&D or marketing.



Our characterization also allows us to derive some comparative statics properties of the stock price. Specifically, one has the following result.

**Proposition 12.** *The stock price  $S(m)$  given cash reserves  $m$  is increasing with respect to  $\mu$ , and decreasing with respect to  $k$ .*

Proposition 12 states that, controlling for cash reserves, firms with greater moral hazard problems have lower prices. This is because, to cope with greater moral hazard, the optimal contract raises the threshold  $w^*$  for the expected discounted utility of the entrepreneur at which dividends are paid.<sup>20</sup> This reduces the present value of the dividends, that is, the stock price. In addition, Proposition 11 implies that, other things equal, firms with greater moral hazard problems have lower cash reserves. This also depresses stock prices, which are increasing in the level of cash reserves. Thus our theoretical analysis implies that firms facing greater moral problems have lower stock price valuations. Empirically, the latter can be measured by the price earnings ratio.

## 5 Conclusion

This paper extends to the dynamic case the static analysis of corporate financing under ex-ante moral hazard (Innes (1990), Holmström and Tirole (1997)). Relying on dynamic programming techniques, we characterize the optimal contract in the infinitely repeated version of the model. We show that the optimal contract can be implemented with cash reserves and endogenous securities, in line with stylized observations. Stocks, held by the entrepreneur and financiers pay dividends when accumulated earnings reach a prespecified threshold. Bonds pay a continuous stream of coupons. The firm is liquidated when its cash reserves run down to zero.

Some important insights obtained by one- or two-periods corporate finance models still apply in a dynamic context. For example, as in Holmström and Tirole (1997), moral hazard limits the income which can be pledged to outside investors, and this can generate credit rationing. Some new insights are also obtained. Relatively to the static case, the ability to rely on future rewards relaxes the tension between incentive and participation constraints. Furthermore, contracting in our dynamic framework is a natural generalization of its static counterpart. For example, at a given point in time the entrepreneur is rewarded as a function of the accumulated performance of the firm, rather than as a function of the current performance. Furthermore, in line with stylized facts, securities are defined as streams of cash-flows, rather than as functions of a single cash-flow.

Our analysis also delivers new empirical implications. In line with recent empirical results (see for instance Welch (2004)), our theory implies that performance shocks and stock price changes should be followed by persistent changes in the leverage of the firm. While this empirical observation might sound puzzling if interpreted in light of a static model, it is a characteristic feature of the optimal contract in our dynamic model.

Our dynamic corporate finance analysis generates endogenous securities price processes. As we study the convergence of our model to its continuous-time limit, we obtain a Black and Scholes (1973) like equation for the dynamics of stock prices. Although the volatility of earnings is constant, that of the stock is stochastic and increases after price drops. We hope our analysis can serve as a first step towards bridging the gap between asset pricing and corporate finance.

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<sup>20</sup>This is easy to check along the lines of Lemma 3 in Appendix 4.

## Appendix 1

*Proof of Proposition 1.* Suppose first that  $\rho > r$ . Let  $T$  be the Bellman operator associated to (10)–(13), and let  $v \in C_b(\mathbb{R}_+)$ . Note that the mapping  $w \mapsto v(w) - (\rho - r)w/(1 + \rho)$  is coercive as  $v \in C_b(\mathbb{R}_+)$  and  $\rho > r$ . Accordingly, let  $M_v$  be the maximum value of this function, and  $\hat{w}_v$  the smallest point at which it reaches its maximum. Setting  $x = w_- = w_+ = 0$ , we obtain that  $Tv \geq 0$ , so  $Tv$  is bounded below. Similarly,  $Tv \leq \bar{R} + M_v/(1 + r)$ , so  $Tv$  is bounded above. It is clear that for any value of  $w$ , there is no loss of generality in restricting  $w_-$  and  $w_+$  to be in  $[0, \hat{w}_v]$ . Hence Berge’s Maximum Theorem applies, and thus  $T$  maps  $C_b(\mathbb{R}_+)$  into itself. Using Blackwell’s Theorem, it is immediate to check that  $T$  is a contraction, and thus it has a unique fixed point  $V \in C_b(\mathbb{R}_+)$  by the Contraction Mapping Theorem.

We now prove that  $V$  is non-decreasing and concave. Since the set of continuous, bounded, non-decreasing and concave functions that vanish at 0 is a closed subset of  $C_b(\mathbb{R}_+)$ , it is sufficient to prove that  $T$  maps this set into itself. Specifically, let  $v$  be such a function, and let  $w' \geq w \geq 0$ . Assume that  $(x, w_-, w_+)$  is an optimal choice in the program that defines  $Tv(w)$ . Since  $w' \geq w$ , it follows from (11)–(13) that  $(x, w_-, w_+)$  is a feasible choice in the program that defines  $Tv(w')$ , and it yields the same value as  $Tv(w)$ . Hence  $Tv(w') \geq Tv(w)$ , and  $Tv$  is non-decreasing. That  $Tv(0) = 0$  follows directly from (10) together with the fact that  $w = 0$  implies  $x = 0$  because of (11). We now prove that  $Tv$  is concave. In line with Clementi and Hopenhayn (2002), we decompose (10)–(13) into two subproblems. First, we consider the problem of maximizing the expected social surplus conditional on not liquidating the project:

$$T^c v(w) = \max \left\{ \bar{R} + \frac{pv(w_+) + (1-p)v(w_-)}{1+r} - \frac{(\rho-r)[pw_+ + (1-p)w_-]}{(1+r)(1+\rho)} \right\} \quad (46)$$

for all  $w \geq pB/\Delta p$ , subject to:

$$w \geq \frac{w_-}{1+\rho} + \frac{pB}{\Delta p}, \quad (47)$$

$$w \geq \frac{pw_+ + (1-p)w_-}{1+\rho}, \quad (48)$$

and:

$$(w_-, w_+) \in \mathbb{R}_+^2. \quad (49)$$

The value function  $Tv$ , taking into account the possibility of liquidation, is then given by:

$$Tv(w) = \max \{ xTv(w^c) \} \quad (50)$$

for each  $w \geq 0$ , subject to:

$$w = xw^c, \quad (51)$$

and:

$$(x, w^c) \in [0, 1] \times \left[ \frac{pB}{\Delta p}, \infty \right). \quad (52)$$

By the same argument used to show that  $Tv$  is non-decreasing over  $\mathbb{R}_+$ , it follows that  $T^c v$  is non-decreasing over  $[pB/\Delta p, \infty)$ . Moreover, it is concave over this interval. Indeed, let  $w, w' \geq pB/\Delta p$ ,  $\lambda \in [0, 1]$  and  $w_\lambda = \lambda w + (1 - \lambda)w'$ . Assume that  $(w_-, w_+)$  is optimal in the program that defines  $T^c v(w)$ , that  $(w'_-, w'_+)$  is optimal in the program that defines  $T^c v(w')$ , and let  $(w_{\lambda-}, w_{\lambda+}) = \lambda(w_-, w_+) + (1 - \lambda)(w'_-, w'_+)$ . Since the constraints (47)–(49) are linear, it follows that  $(w_{\lambda-}, w_{\lambda+})$  is a feasible choice in the program that defines  $T^c v(w_\lambda)$ . Since  $v$  is concave, it follows that  $T^c v(w_\lambda) \geq \lambda T^c v(w) + (1 - \lambda)T^c v(w')$ , and thus  $T^c v$  is concave. Next, recall that  $\hat{w}_v$  is the smallest point at which the coercive mapping

$w \mapsto v(w) - (\rho - r)w/(1 + \rho)$  reaches its maximum, and let  $w_v^* = \hat{w}_v/(1 + \rho) + pB/\Delta p$ . Then, for any  $w \geq w_v^*$ ,  $(\hat{w}_v, \hat{w}_v)$  is a feasible choice in the program that defines  $T^c v(w)$ , and it yields the maximum utility in (10) and (46), namely  $\bar{R} + [v(\hat{w}_v) - (\rho - r)\hat{w}_v/(1 + \rho)]/(1 + r)$ , which implies that  $T^c v$  is constant over  $[w_v^*, \infty)$ . Next, (50)–(52) can be rewritten as:

$$Tv(w) = \max \left\{ \frac{T^c v(w^c)}{w^c} \right\} w \quad (53)$$

for each  $w \geq 0$ , subject to:

$$w^c \geq \max \left\{ \frac{pB}{\Delta p}, w \right\}. \quad (54)$$

Since  $T^c v$  is continuous over  $[pB/\Delta p, \infty)$  and constant over  $[w_v^*, \infty)$ , it is clear that the mapping  $w^c \mapsto T^c v(w^c)/w^c$  reaches its maximum in  $[pB/\Delta p, w_v^*]$ . Moreover, since  $T^c v$  is concave, the set  $\arg \max_{w^c \in [pB/\Delta p, w_v^*]} \{T^c v(w^c)/w^c\}$  is an interval  $[\underline{w}_v^c, \bar{w}_v^c]$ , possibly reduced to a point, and the mapping  $w^c \mapsto T^c v(w^c)/w^c$  is non-decreasing over  $[pB/\Delta p, \underline{w}_v^c]$  and non-increasing over  $[\bar{w}_v^c, w_v^*]$ . It follows from (53)–(54) that:

$$Tv(w) = \begin{cases} \max_{w^c \in [pB/\Delta p, w_v^*]} \left\{ \frac{T^c v(w^c)}{w^c} \right\} w & \text{if } w \leq \bar{w}_v^c, \\ T^c v(w) & \text{if } w \geq \bar{w}_v^c, \end{cases} \quad (55)$$

and  $Tv$  is concave by construction, which implies the result. We let  $\hat{w} = \hat{w}_V$  and  $w^* = w_V^*$ . Observe that  $V$  is constant over  $[w^*, \infty)$  by construction.

Finally, suppose that for some  $w < w' < w^*$ , one has  $V(w) = V(w')$ . Then, by concavity of  $V$ , one must have  $V(w) = V(w') = V(w^*)$ . Therefore, the optimal choice in (10)–(13) given  $w$  must be  $(1, \hat{w}, \hat{w})$ , which violates (11), a contradiction. This implies that  $V$  is strictly increasing over  $[0, w^*)$ , as claimed. This concludes the proof of the result for  $\rho > r$ .

Suppose next that  $\rho = r$ . We need only to prove that  $T$  has a fixed point  $V \in C_b(\mathbb{R}_+)$  that has a maximum, since the above proof that  $V$  is non-decreasing and concave does not rely on  $\rho > r$ . Let  $C'_b(\mathbb{R}_+)$  be the set of functions  $v \in C_b(\mathbb{R}_+)$  that are bounded above by  $(1 + r)\bar{R}/r$  and that are constant and equal to  $(1 + r)\bar{R}/r$  over  $[w^*, \infty)$ , where:

$$w^* = \frac{(1 + r)pB}{r\Delta p}.$$

Since the set of such functions is a closed subset of  $C_b(\mathbb{R}_+)$ , it is sufficient to prove that  $T$  maps this set into itself. Let  $v$  be any such function. Setting  $x = w_- = w_+ = 0$ , we obtain that  $Tv \geq 0$ , so  $Tv$  is bounded below. Similarly,  $Tv \leq \bar{R} + \bar{R}/r$ , so  $Tv$  is bounded above by  $(1 + r)\bar{R}/r$ . Now let  $w \geq w^*$ . It is immediate to check from (11)–(12) that  $x = 1$  and  $w_- = w_+ = w^*$  are feasible choices in the program that defines  $Tv(w)$ , which yield the maximum value, namely  $(1 + r)\bar{R}/r$ . Hence  $Tv$  is constant and equal to  $(1 + r)\bar{R}/r$  over  $[w^*, \infty)$ . It is clear that for any value of  $w$ , there is no loss of generality in restricting  $w_-$  and  $w_+$  to be in  $[0, w^*]$ . Hence Berge's Maximum Theorem applies, and thus  $T$  maps  $C'_b(\mathbb{R}_+)$  into itself. Since  $T$  is a contraction, it has a unique fixed point  $V \in C'_b(\mathbb{R}_+)$ , which implies the result. Note that  $\hat{w} = w^*$  by construction. ■

*Proof of Proposition 2.* If  $\rho = r$ , the result follows immediately from the characterization of  $V$  given in Proposition 1. Suppose now that  $\rho > r$ . Since  $\hat{w} \in \arg \max_w \{V(w) - (\rho - r)w/(1 + \rho)\}$ ,  $w^* \in \arg \max_w V(w)$  and  $\rho > r$ , it follows that  $\hat{w} \leq w^*$ . Suppose by way of contradiction that  $\hat{w} = w^* = (1 + \rho)pB/(\rho\Delta p)$ . For any  $\varepsilon > 0$  close enough to 0, the contract  $(x, u_-, u_+, w_-, w_+) = (1, 0, B/\Delta p, w^* -$

$(1 + \rho)\varepsilon, w^* - (1 + \rho)\varepsilon$ ) satisfies (11)–(12) and delivers a utility  $w^* - \varepsilon$  to the entrepreneur. By (10), one must then have:

$$V(w^* - \varepsilon) \geq \bar{R} + \frac{V(w^* - (1 + \rho)\varepsilon)}{1 + r} - \frac{(\rho - r)[w^* - (1 + \rho)\varepsilon]}{(1 + r)(1 + \rho)}. \quad (56)$$

Moreover, since  $\hat{w} = w^*$ , one has:

$$V(w^*) = \bar{R} + \frac{V(w^*)}{1 + r} - \frac{(\rho - r)w^*}{(1 + r)(1 + \rho)}, \quad (57)$$

so rearranging (56) yields:

$$\frac{1 + \rho}{1 + r} \left[ \frac{V(w^*) - V(w^* - (1 + \rho)\varepsilon)}{(1 + \rho)\varepsilon} \right] - \frac{V(w^*) - V(w^* - \varepsilon)}{\varepsilon} \geq \frac{\rho - r}{1 + r}.$$

Taking limits as  $\varepsilon$  goes to 0, and using the fact that  $\rho > r$ , it follows that  $V'_-(w^*) \geq 1$ . Since  $V$  is concave and  $V(0) = 0$ , one thus gets that  $V(w^*) \geq w^*$ . However, from  $\bar{R} < pB/\Delta p = \rho w^*(1 + \rho)$  and (57), it is straightforward to check that  $V(w^*) < w^*$ , a contradiction. Hence  $\hat{w} < w^*$ , as claimed.  $\blacksquare$

**Proposition 13.** *When  $\bar{R} - \underline{R}$  is large enough, it is always optimal that the entrepreneur exerts effort.*

*Proof.* By dynamic programming, we need only to check that the optimal contract under effort is robust to one-shot deviations. For any  $w \geq 0$ , the optimal one-shot deviation is characterized by:

$$\underline{V}(w) = \max x \left\{ \underline{R} + \frac{(p - \Delta p)V(w_+) + (1 - p + \Delta p)V(w_-)}{1 + r} - \frac{(\rho - r)[(1 - \Delta p)w_+ + (1 - p + \Delta p)w_-]}{(1 + r)(1 + \rho)} \right\}$$

subject to:

$$w = x \left[ B + (p - \Delta p) \left( u_+ + \frac{w_+}{1 + \rho} \right) + (1 - p + \Delta p) \left( u_- + \frac{w_-}{1 - \rho} \right) \right],$$

and:

$$\Delta p \left( u_+ + \frac{w_+}{1 + \rho} - u_- - \frac{w_-}{1 + \rho} \right) \leq B,$$

together with the limited liability and feasibility constraints (8)–(9). It is easy to see that the second constraint will be slack at the optimal one-shot deviation. Eliminating  $u_+$  and  $u_-$ , and using the concavity of  $V$ , one obtains that:

$$\underline{V}(w) = \max x \left\{ \underline{R} + \frac{V(w_-)}{1 + r} - \frac{(\rho - r)w_-}{(1 + r)(1 + \rho)} \right\}$$

subject to:

$$w \geq x \left( B + \frac{w_-}{1 + \rho} \right)$$

and the limited liability and feasibility constraints. Equivalently, this may be rewritten as:

$$\underline{V}(w) = \max x \left\{ \underline{R} + \frac{pV(w_+) + (1 - p)V(w_-)}{1 + r} - \frac{(\rho - r)[pw_+ + (1 - p)w_-]}{(1 + r)(1 + \rho)} \right\}$$

subject to:

$$w \geq x \left[ B + \frac{pw_+ + (1 - p)w_-}{1 + \rho} \right] \quad (58)$$

and the limited liability and feasibility constraints. The differences between  $V$  and  $\underline{V}$  are that the return is reduced from  $\bar{R}$  to  $\underline{R}$ , the incentives constraint is removed, and the private benefit  $B$  is received by the entrepreneur. In analogy with (46)–(49), it is natural to define:

$$\underline{V}^e V(w) = \max \left\{ \bar{R} + \frac{pv(w_+) + (1 - p)v(w_-)}{1 + r} - \frac{(\rho - r)[pw_+ + (1 - p)w_-]}{(1 + r)(1 + \rho)} \right\}$$

for all  $w \geq B$ , subject to (49) and (58). Then, in analogy with (50)–(52), one has:

$$\underline{V}(w) = \max x [\underline{T}^c V(w^c) - \bar{R} + \underline{R}]$$

for each  $w \geq 0$ , subject to (51) and:

$$(x, w^c) \in [0, 1] \times [B, \infty).$$

It should be noted that  $V$  and  $\underline{T}^c V$  do not depend on the difference  $\bar{R} - \underline{R}$ , which can be treated as a free parameter. Note first that for  $w$  large enough,  $V(w) = T^c V(w)$  and  $\underline{V}(w) = \underline{T}^c V(w) - \bar{R} + \underline{R}$ . Thus a necessary condition for  $V \geq \underline{V}$  is that:

$$\bar{R} - \underline{R} \geq \max_{w \in [\max\{B, pB/\Delta p\}, \infty)} \{\underline{T}^c V(w) - T^c V(w)\}.$$

This condition is not sufficient, however, in particular because  $T^c V$  and  $\underline{T}^c V$  do not have the same domain. Nevertheless, to establish the result, it suffices to notice that  $V(w) > 0$  for all  $w > 0$ , while  $\underline{V} \equiv 0$  whenever  $\bar{R} - \underline{R} \geq \max T^c V$ . In the continuous-time limit of the model, one can show that the condition ensuring that effort is always optimal is:

$$\Delta\mu \geq \max_{w \geq 0} \left\{ B[1 - V_0'(w)] - \frac{B^2 \sigma^2}{2\Delta\mu^2} V_0''(w) \right\},$$

where  $\Delta\mu$  and  $V_0$  are defined as in Subsection 4.2. Here the ratio  $B/\Delta\mu$  is treated as fixed, while  $\Delta\mu$  is a free parameter. ■

## Appendix 2

*Proof of Proposition 3.* First, we construct a solution to the problem of maximizing the expected surplus conditional on not liquidating the project, that is problem (46)–(49) given  $v = V$  and  $\rho = r$ . For any  $w \geq pB/\Delta p$ , the Lagrangian for this problem can be written as:

$$\begin{aligned} L^c(w, w_+, w_-, \lambda_+, \lambda_-) = & \bar{R} + \frac{pV(w_+) + (1-p)V(w_-)}{1+r} + (1-p)\lambda_- \left[ w - \left( \frac{w_-}{1+r} + \frac{pB}{\Delta p} \right) \right] \\ & + \lambda_+ \left[ w - \frac{pw_+ + (1-p)w_-}{1+r} \right], \end{aligned} \tag{59}$$

where  $(1-p)\lambda_-$  and  $\lambda_+$  are the Lagrange multipliers for (47) and (48), respectively. Three cases must be distinguished, according to the position of  $w$  with respect to the thresholds  $w^*$ ,  $w^{**}$  and  $w^{***}$  given by (18)–(20).

Assume first that  $w \geq w^*$ . It then follows from (18) that:

$$\begin{aligned} w & \geq \frac{w^*}{1+r} + \frac{pB}{\Delta p}, \\ w & > \frac{w^*}{1+r}. \end{aligned}$$

This implies that (47)–(48) are satisfied for  $w_+ = w_- = w^*$ . Since  $V$  attains its maximum at  $w^*$ , this is clearly an optimal choice.

Assume next that  $w^* > w \geq w^{**}$ . One cannot have  $\lambda_+ = \lambda_- = 0$  at the optimum, since otherwise  $w_- \geq w^*$  by (59) and thus, from (18):

$$\frac{w_-}{1+r} + \frac{pB}{\Delta p} \geq w^* > w,$$

so that (47) is violated. Hence one must have  $\lambda_+ + \lambda_- > 0$  at the optimum. Suppose by way of contradiction that (47) is slack, so that  $\lambda_- = 0$ . Then  $\lambda_+ > 0$ , so that (48) is binding. Using (18)–(19), it follows that:

$$w_+ > (1+r) \left[ w + \frac{(1-p)B}{\Delta p} \right] \geq (1+r) \left[ w^{**} + \frac{(1-p)B}{\Delta p} \right] = w^*.$$

However,  $w_+ \in \arg \max_w \{V(w) - \lambda_+ w\}$  and  $w^* \in \arg \max_w V(w)$ . Since  $\lambda_+ > 0$ , it follows that  $w_+ \leq w^*$ , a contradiction. Therefore (47) is binding, which gives  $w_- = (1+r)(w - B/\Delta p)$ . Using (18)–(19), it is straightforward to check that  $w \geq w^{**}$  if and only if:

$$w \geq \frac{pw^* + (1-p)(1+r)(w - pB/\Delta p)}{1+r}.$$

This implies that (48) is satisfied for  $w_+ = w^*$  given that  $w_- = (1+r)(w - pB/\Delta p)$ . Since  $V$  attains its maximum at  $w^*$ , this is clearly an optimal choice.

Assume finally that  $w^{**} > w \geq w^{***}$ . By the same argument as above, one must have  $\lambda_+ + \lambda_- > 0$  at the optimum. Suppose by way of contradiction that  $\lambda_+ = 0$ . Then  $\lambda_- > 0$ , so that (47) is binding, which gives  $w_- = (1+r)(w - pB/\Delta p)$ , and  $w_+ \geq w^*$  by (59). Since  $w < w^{**}$ , however, it follows that:

$$\frac{pw_+ + (1-p)w_-}{1+r} \geq \frac{pw^* + (1-p)(1+r)(w - pB/\Delta p)}{1+r} > w,$$

and (48) is violated, a contradiction. Therefore  $\lambda_+ > 0$ , and thus (48) is binding, so that the problem becomes simply to maximize:

$$pV\left(\frac{(1+r)w - (1-p)w_-}{p}\right) + (1-p)V(w_-)$$

with respect to  $w_- \in [0, (1+r)(w - pB/\Delta p)]$ . By concavity of  $V$ , this is a non-decreasing function of  $w_-$  over this range, and therefore  $w_+ = (1+r)[w + (1-p)B/\Delta p]$  and  $w_- = (1+r)(w - pB/\Delta p)$  form an optimal choice.

This completely characterizes a solution to (46)–(49) given  $v = V$  and  $\rho = r$ . It is easy to verify that the corresponding transfers coincide with those given in items (i)–(iii) of Proposition 2. Next, we show that  $T^c V = V$  over  $[w^{***}, \infty)$ , so that no liquidation is required on this region of the state space. To prove this, note that both  $V$  and  $T^c V$  are concave, respectively over  $\mathbb{R}_+$  and  $[w^{***}, \infty)$ , hence almost everywhere differentiable. In particular, it follows from the construction of  $V$  that  $V'_+(0)$  is well-defined and finite. The above characterization then implies that for almost every  $w$  in a right-neighborhood of  $w^{***}$ , the first-order condition with respect to  $w_-$  and the envelope condition for (46)–(49) given  $v = V$  can be written as:

$$V'\left((1+r)\left(w - \frac{pB}{\Delta p}\right)\right) = \lambda_+(w) + \lambda_-(w), \quad (60)$$

$$T^c V'(w) = \lambda_+(w) + (1-p)\lambda_-(w), \quad (61)$$

where we have made explicit the dependence of  $\lambda_+$  and  $\lambda_-$  upon  $w$ . Note that for  $w$  close enough to  $w^{***} = pB/\Delta p$ , the derivative in (60) is simply  $V'_+(0)$ . Taking limits in (60)–(61) as  $w$  converges to  $w^{***}$ , it follows that  $V'_+(0) \geq T^c V'_+(w^{***})$ , which implies the result given the characterization (55) of the Bellman operator  $T$ . ■

*Proof of Corollary 1.* For  $w = w^{***}$ , one has  $w_- = 0$  and  $w_+ = (1+r)B/\Delta p$ , which yields:

$$V(w^{***}) = \bar{R} + \frac{pV((1+r)B/\Delta p)}{1+r}.$$

Taking advantage of  $w^{***} = pB/\Delta p$  and  $\bar{R} > 0$ , it follows that:

$$\frac{V(w^{***})}{w^{***}} > \frac{V((1+r)B/\Delta p)}{(1+r)B/\Delta p}.$$

Thus, as  $V$  is linear over  $[0, w^{***}]$  and concave,

$$V'_-(w^{***}) = V'_+(0) > V'_-\left(\frac{(1+r)B}{\Delta p}\right). \quad (62)$$

Note that for  $w$  in a right-neighborhood of  $w^{***}$ ,

$$V(w) = \bar{R} + \frac{pV((1+r)[w + (1-p)B/\Delta p])}{1+r} + \frac{(1-p)V((1+r)(w - pB/\Delta p))}{1+r}.$$

Subtracting  $V(w^{***})$  from both sides of this inequality, dividing by  $w - w^{***}$  and letting  $w$  go to  $w^{***}$  from above, it follows in particular that:

$$V'_+(w^{***}) = pV'_+\left(\frac{(1+r)B}{\Delta p}\right) + (1-p)V'_+(0). \quad (63)$$

Since  $V$  is concave and  $V'_+(0) = V'_-(w^{***})$ , (63) implies that:

$$V'_+(w^{***}) \leq pV'_-\left(\frac{(1+r)B}{\Delta p}\right) + (1-p)V'_-(w^{***}) < V'_-(w^{***}),$$

where the second inequality derives from (62). Hence  $V$  is not differentiable at  $w^{***}$ . ■

*Proof of Proposition 4.* We must consider four cases, corresponding to the regions characterized in Proposition 3.

*Case 1.* Consider first the region  $[w^*, \infty)$ . In this case, the optimal contract requires that  $w_+ = w_- = w^*$ , irrespective of failure or success, and that  $u_+ = w - w^* + B/\Delta p$  and  $u_- = w - w^*$ . Normalizing by the size of the firm, we set dividends  $d_+ = m - m^* + R_+ - R_-$  and  $d_- = m - m^*$ , and a coupon rate  $C = \bar{R}$ . Using (21) together with  $w = km$  and  $w^* = km^*$ , we obtain that  $kd_+ = u_+$  and  $kd_- = u_-$  as required. Using the fact that  $C = \bar{R}$  together with the definition of  $m^*$ , it is immediate to check that the size-adjusted accumulated earnings evolve according to:

$$m_+ = (1+r)(m + R_+ - C - d_+) = m^*,$$

and:

$$m_- = (1+r)(m + R_- - C - d_-) = m^*,$$

so that  $km_+ = km_- = km^* = w^*$  as required.

*Case 2.* Consider next the region  $[w^{**}, w^*)$ . In this case, the optimal contract requires that  $w_+ = w^*$  and  $w_- = (1+r)(w - pB/\Delta p)$ , and that  $u_+ = w - w^{**} = w - w^* + B/\Delta p$  and  $u_- = 0$ . Normalizing by the size of the firm, we set dividends  $d_+ = m - m^* + R_+ - R_-$  and  $d_- = 0$ , and a coupon rate  $C = \bar{R}$ . Proceeding as in Case 1, we obtain that  $kd_+ = u_+$  and  $kd_- = u_-$  as required. Similarly,  $m_+ = m^*$  and therefore  $km_+ = w^*$ . In case of failure, the size-adjusted accumulated earnings evolve according to:

$$m_- = (1+r)(m + R_- - C) = (1+r)[m - p(R_+ - R_-)].$$

Using (21), it is immediate to check that this implies that  $km_- = (1+r)(w - pB/\Delta p)$  as required.

*Case 3.* Consider next the region  $[w^{***}, w^{**})$ . In this case, the optimal contract requires that  $w_+ = (1+r)[w + (1-p)B/\Delta p]$  and  $w_- = (1+r)(w - pB/\Delta p)$ , and that  $u_+ = u_- = 0$ . We set dividends

$d_+ = d_- = 0$  so that  $kd_+ = u_+$  and  $kd_- = u_-$  as required, and a coupon rate  $C$ . Proceeding as in Case 2, we obtain that  $km_- = (1+r)(w - pB/\Delta p)$ . In case of success, the size-adjusted accumulated earnings evolve according to:

$$m_+ = (1+r)(m + R_+ - C) = (1+r)[m + (1-p)(R_+ - R_-)].$$

Using (21), it is immediate to check that this implies that  $km_+ = (1+r)[w + (1-p)B/\Delta p]$  as required.

*Case 4.* Consider finally the region  $[0, w^{***})$ . In this case, the optimal contract requires that  $w_+ = (1+r)[w^{***} + (1-p)B/\Delta p]$  and  $w_- = 0$ , and that  $u_+ = u_- = 0$ . We set dividends  $d_+ = d_- = 0$  so that  $kd_+ = u_+$  and  $kd_- = u_-$  as required, and a coupon rate  $C$ . Since the project is downsized by a factor  $x = m/m^{***}$ , the size-adjusted accumulated earnings evolve according to:

$$m_+ = (1+r)\left(\frac{m}{x} + R_+ - C\right) = (1+r)[m^{***} + (1-p)(R_+ - R_-)],$$

and:

$$m_- = (1+r)\left(\frac{m}{x} + R_- - C\right) = (1+r)[m^{***} - p(R_+ - R_-)].$$

Using (21) and the explicit expression for  $m^{***}$ , it is immediate to check that this implies that  $km_+ = (1+r)[w^{***} + (1-p)B/\Delta p]$  and  $km_- = 0$  as required.  $\blacksquare$

### Appendix 3

*Proof of Proposition 5.* Suppose first that  $\bar{n} = -1$ , so that  $\bar{R}/(pB/\Delta p) \leq (\rho - r)/(1 + \rho)$ . We show that, in this case,

$$V(w) = \begin{cases} \frac{\bar{R}w}{w^*} & \text{if } w \leq w^*, \\ \bar{R} & \text{if } w \geq w^*, \end{cases} \quad (64)$$

where  $w^* = pB/\Delta p$ . We simply have to prove that  $TV = V$ . From (64), the slope of  $V$ , whenever defined, is less or equal than  $(\rho - r)/(1 + \rho)$ . Hence, setting  $w_- = w_+ = 0$  in (10) is optimal for each  $w \geq 0$ . For  $w \geq w^* = pB/\Delta p$ , it is optimal to set  $x = 1$  and thus  $TV(w) = \bar{R} = V(w)$ . For  $w < w^* = pB/\Delta p$ , it is optimal to set  $x = w/(pB/\Delta p) = w/w^*$  and thus  $TV(w) = \bar{R}w/(pB/\Delta p) = \bar{R}w/w^* = V(w)$  as well. It is immediate to verify that  $\hat{w} = 0$  and  $w^* = pB/\Delta p$  are consistent with their definition. Note that region (ii) is degenerate in this case as  $w^* = pB/\Delta p = w^{***}$ . It is easy to verify that the corresponding transfers coincide with those given in items (i) and (iii) of Proposition 5.

Suppose next that  $\bar{n} \in \mathbb{N}$ , and that (24) holds. We show that, in this case,

$$V(w) = \begin{cases} \alpha_0 w & \text{if } w \leq W_0, \\ V(W_n) + \alpha_{n+1}(w - W_n) & \text{if } W_n \leq w \leq W_{n+1}, \quad n = 0, \dots, \bar{n}, \\ V(W_{\bar{n}+1}) & \text{if } w \geq W_{\bar{n}+1}, \end{cases} \quad (65)$$

where:

$$W_n = \frac{pB}{\Delta p} \sum_{k=0}^n \frac{1}{(1+\rho)^k},$$

for each  $n = 0, \dots, \bar{n} + 1$ ,

$$\alpha_{n+1} = \frac{(1-p)(1+\rho)[\alpha_n - (\rho - r)/(1+\rho)]}{1+r}, \quad (66)$$



for each  $n = 0, \dots, \bar{n}$ , and  $\alpha_0$  and  $V(W_{\bar{n}})$  are related by:

$$\alpha_0 W_0 = \bar{R} + \frac{pV(W_{\bar{n}})}{1+r} - \frac{p(\rho-r)W_{\bar{n}}}{(1+r)(1+\rho)}. \quad (67)$$

It follows in particular that  $w^* = W_0$ ,  $\hat{w} = W_{\bar{n}}$  and  $w^* = W_{\bar{n}+1}$ . To establish this claim, it is convenient to fix all the parameters of the model except  $\bar{R}$ , and to search parameter restrictions on  $\bar{R}$  such that  $V$  is of the form (65). A necessary condition is that  $0 < \alpha_{\bar{n}+1} \leq (\rho-r)/(1+\rho)$ , otherwise one would have  $\hat{w} = W_{\bar{n}+1} > W_{\bar{n}}$ . Fix any value of  $\alpha_{\bar{n}+1} \in (0, (\rho-r)/(1+\rho)]$ , and construct recursively a sequence  $\alpha_{\bar{n}}, \dots, \alpha_0$  using (66). It is straightforward to check that  $\alpha_{\bar{n}+1} \leq (\rho-r)/(1+\rho) < \alpha_{\bar{n}} < \dots < \alpha_0$ , so  $V$  as constructed in (65) is increasing, concave, and  $\hat{w} = W_{\bar{n}}$  is the smallest point at which the mapping  $w \mapsto V(w) - (\rho-r)w/(1+\rho)$  reaches its maximum. A necessary condition for  $TV = V$  is that:

$$V(W_0) = \bar{R} + \frac{pV(W_{\bar{n}})}{1+r} - \frac{p(\rho-r)W_{\bar{n}}}{(1+r)(1+\rho)}, \quad (68)$$

or, equivalently:

$$\alpha_0 W_0 = \bar{R} + \frac{p \left[ \alpha_0 W_0 + \sum_{n=1}^{\bar{n}} \alpha_n (W_n - W_{n-1}) \right]}{1+r} - \frac{p(\rho-r)W_{\bar{n}}}{(1+r)(1+\rho)}. \quad (69)$$

Simple computations using the definition of the sequences  $\{X_n\}$ ,  $\{W_n\}$  and  $\{\alpha_n\}$  yield the parameter restrictions (23). Conversely, if (23) holds, there exists a unique  $\alpha_0$  solution to (69), where  $\alpha_1, \dots, \alpha_{\bar{n}+1}$  are defined by (66), and such that  $\alpha_{\bar{n}+1} \leq (\rho-r)/(1+\rho) < \alpha_{\bar{n}}$ . To conclude, we simply have to prove that  $TV = V$ . It follows from (24) that for each  $w \in [W_0, W_{\bar{n}+1}]$ , the choices  $w_+ = W_n$  and  $w_- = (1+\rho)(w - pB/\Delta p)$  make (11) binding while preserving (12), and are clearly optimal. By (68), we only need to prove that for each  $w \in [W_0, W_{\bar{n}+1}]$ ,

$$V(w) = V(W_0) + \frac{(1-p)V(w_-)}{1+r} - \frac{(\rho-r)(1-p)w_-}{(1+r)(1+\rho)}. \quad (70)$$

By construction, if  $W_n \leq w \leq W_{n+1}$  for  $n = 0, \dots, \bar{n}$ , then  $W_{n-1} \leq (1+\rho)(w - pB/\Delta p) \leq W_n$ , with the convention that  $W_{-1} = 0$ . It is immediate to check from the definition of  $\alpha_1$  that (70) holds for each  $w \in [W_0, W_1]$ . Now suppose that it holds for each  $w \in [W_n, W_{n+1}]$ ,  $n = 0, \dots, \bar{n}-1$ , and let  $w \in [W_{n+1}, W_{n+2}]$ . Then  $V(w) = V(W_n) + \alpha_{n+1}(w - W_n)$ , or, equivalently:

$$\begin{aligned} V(w) &= V(W_0) + \frac{(1-p)V(W_{n-1})}{1+r} - \frac{(\rho-r)(1-p)W_{n-1}}{(1+r)(1+\rho)} + \alpha_{n+1}(w - W_n) \\ &= V(W_0) + \frac{(1-p)[V(W_{n-1}) + \alpha_n(w_- - W_{n-1})]}{1+r} - \frac{(\rho-r)(1-p)w_-}{(1+r)(1+\rho)} \\ &= V(W_0) + \frac{(1-p)V(w_-)}{1+r} - \frac{(\rho-r)(1-p)w_-}{(1+r)(1+\rho)}, \end{aligned}$$

where the first equality follows from the induction hypothesis, the second from (66) and  $w_n = w_{n-1} + 1/(1+\rho)^n$ , and the third from (65). Hence (70) holds for each  $w \in [W_0, W_{\bar{n}+1}]$  by induction. It follows that  $V = T^e V$  on  $[W_0, \infty)$ . Since  $\alpha_0 > \alpha_1$ , the region in which liquidation occurs with positive probability is  $[0, W_0) = [0, pB/\Delta p)$ , and for  $w < W_0$ , it is optimal to set  $x = w/W_0$  and thus  $TV(w) = V(W_0)w/W_0 = \alpha_0 w = V(w)$  as well. It is easy to verify that the corresponding transfers coincide with those given in items (i)–(iii) of Proposition 5.  $\blacksquare$

*Proof of Corollary 2.* As mentioned in the text,  $w^{**} \leq pB/\Delta p$  implies that given  $w = pB/\Delta p$ , the choice  $(x, w_-, w_+) = (1, 0, \hat{w})$  satisfies (11)–(12). We thus need only to prove that it is optimal. If this is not

the case, liquidation must occur with positive probability at  $w = pB/\Delta p$ . Let  $w^{***} > pB/\Delta p$  be then the upper bound of the interval over which liquidation occurs with positive probability in the optimal contract. Clearly  $w^{***} \leq w^*$ . If  $w^{***} = w^*$ , then the constant slope of  $V$  over  $(0, w^{***})$  must be smaller or equal to  $(\rho - r)/(1 + \rho)$ , otherwise one would have  $\hat{w} = w^*$  which contradicts Proposition 2. It follows that  $\hat{w} = 0$  so that  $w^* = pB/\Delta p$  by (14), a contradiction. Hence one must have  $w^{***} < w^*$ . This implies that for  $w = w^{***}$ , the optimal choice is  $(x^{***}, w_+^{***}, w_-^{***}) = (1, \hat{w}, (1 + \rho)(w^{***} - pB/\Delta p))$ , while for  $w = pB/\Delta p$  it must be  $(x, w_+, w_-) = (pB/(\Delta p w^{***}), \hat{w}, (1 + \rho)(w^{***} - pB/\Delta p))$ . Note that (11) is binding at  $w = pB/\Delta p$ , so that  $w_- = (1 + \rho)(1/x - 1)pB/\Delta p$ . On the other hand, since  $w^{**} \leq pB/\Delta p$ , one has  $\hat{w} \leq (1 + \rho)B/\Delta p$  by (30), hence:

$$\begin{aligned} \frac{x[pw_+ + (1 - p)w_-]}{1 + \rho} &= (1 - p) \frac{pB}{\Delta p} + x \left[ \frac{p\hat{w}}{1 + \rho} - (1 - p) \frac{pB}{\Delta p} \right] \\ &\leq (1 - p) \frac{pB}{\Delta p} + xp \frac{pB}{\Delta p} \\ &< \frac{pB}{\Delta p} \end{aligned}$$

as  $x < 1$ , so that (12) is slack at  $w = pB/\Delta p$ . Now, for any  $w \geq 0$ , the Lagrangian corresponding to (10) can be written as:

$$\begin{aligned} L(w, x, w_+, w_-, \lambda_+, \lambda_-) &= x \left\{ \bar{R} + \frac{pV(w_+) + (1 - p)V(w_-)}{1 + r} - \frac{(\rho - r)[pw_+ + (1 - p)w_-]}{(1 + r)(1 + \rho)} \right. \\ &\quad \left. - (1 - p)\lambda_- \left( \frac{w_-}{1 + \rho} + \frac{pB}{\Delta p} \right) - \lambda_+ \left[ \frac{pw_+ + (1 - p)w_-}{1 + \rho} \right] \right\} \quad (71) \\ &\quad + [(1 - p)\lambda_- + \lambda_+]w, \end{aligned}$$

where  $(1 - p)\lambda_-$  and  $\lambda_+$  are respectively the Lagrange multipliers associated to (11) and (12). For  $w = pB/\Delta p$ , one has  $w_+ = \hat{w}$  and  $\lambda_+ = 0$  from the above discussion. Let  $\alpha$  be the constant slope of  $V$  over  $(0, w^{***})$ . Since  $pB/\Delta p \in (0, w^{***})$ ,  $V$  is differentiable at  $pB/\Delta p$ , and  $\alpha = (1 - p)\lambda_-$  from the Envelope Theorem. Moreover,  $w_- = (1 + \rho)(w^{***} - pB/\Delta p) < w^{***}$  for  $w = pB/\Delta p$ . Hence  $V(w_-) = \alpha w_-$ , which implies, after some simple computations, that the Lagrangian at  $w = pB/\Delta p$  can be written as:

$$\begin{aligned} x \left( \bar{R} + \frac{pV(\hat{w})}{1 + r} - \frac{p(\rho - r)\hat{w}}{(1 + r)(1 + \rho)} \right. \\ \left. + \frac{\{[(1 - p)(1 + \rho) - (1 + r)]\alpha - (1 - p)(\rho - r)\}w_- - \frac{\alpha pB}{\Delta p}}{(1 + r)(1 + \rho)} + \frac{\alpha pB}{\Delta p} \right). \end{aligned}$$

Since  $x \in (0, 1)$  at the optimum, one must have:

$$\bar{R} + \frac{pV(\hat{w})}{1 + r} - \frac{p(\rho - r)\hat{w}}{(1 + r)(1 + \rho)} + \frac{\{[(1 - p)(1 + \rho) - (1 + r)]\alpha - (1 - p)(\rho - r)\}w_-}{(1 + r)(1 + \rho)} = \frac{\alpha pB}{\Delta p}.$$

Moreover, since  $w_- > 0$  at the optimum as  $w^{***} > pB/\Delta p$ , one must have:

$$[(1 - p)(1 + \rho) - (1 + r)]\alpha = (1 - p)(\rho - r).$$

These two observations imply that:

$$\bar{R} + \frac{pV(\hat{w})}{1 + r} - \frac{p(\rho - r)\hat{w}}{(1 + r)(1 + \rho)} = \frac{\alpha pB}{\Delta p} = V\left(\frac{pB}{\Delta p}\right),$$

which implies that the choice  $(x, w_+, w_-) = (1, \hat{w}, 0)$  is optimal at  $w = pB/\Delta p$ , a contradiction. One can then construct the value function  $V$  as in Proposition 5. In particular, one will have  $\hat{w} = W_{\bar{n}}$  and therefore, by (14) and (29),

$$w^{**} = \frac{pB}{\Delta p} \sum_{k=0}^{\bar{n}+1} \frac{1}{(1+\rho)^k} - \frac{B}{\Delta p}.$$

The requirement that  $w^{**} \leq pB/\Delta p$  then implies (24) as expected.  $\blacksquare$

*Proof of Proposition 6.* First, we construct a solution to the problem of maximizing the expected surplus conditional on not liquidating the project, that is problem (46)–(49) given  $v = V$ . Three cases must be distinguished, according to the position of  $w$  with respect to the thresholds  $pB/\Delta p$ ,  $w^{**}$  and  $w^{***}$ . (Note that by Corollary 2,  $w^{**} > pB/\Delta p$  as (24) does not hold.)

*Case 1.* Consider first the region  $[w^*, \infty)$ . It is easy to check that (47)–(48) are satisfied for  $w_+ = w_- = \hat{w}$ . Since the mapping  $w \mapsto V(w) - (\rho - r)w/(1 + \rho)$  attains its maximum at  $\hat{w}$ , this is clearly an optimal choice.

*Case 2.* Consider next the region  $[w^{**}, w^*)$ . Proceeding along the lines of the proof of Proposition 3, one obtains that (47) is binding, which gives  $w_- = (1 + \rho)(w - B/\Delta p)$ . Using (14) and (29), it is straightforward to check that  $w \geq w^{**}$  if and only if:

$$w \geq \frac{p\hat{w} + (1-p)(1+\rho)(w - pB/\Delta p)}{1 + \rho}.$$

This implies that (48) is satisfied for  $w_+ = \hat{w}$  given that  $w_- = (1 + \rho)(w - pB/\Delta p)$ . Since the mapping  $w \mapsto V(w) - (\rho - r)w/(1 + \rho)$  attains its maximum at  $\hat{w}$ , this is clearly an optimal choice.

*Case 3.* Consider finally the region  $[pB/\Delta p, w^{**})$ . Proceeding along the lines of the proof of Proposition 3, one obtains that (48) is binding, so that the problem becomes simply to maximize:

$$\frac{pV[(1+\rho)w - (1-p)w_-]/p}{1+r} + \frac{(1-p)V(w_-)}{1+r} - \frac{(\rho-r)w}{1+r}$$

with respect to  $w_- \in [0, (1 + \rho)(w - pB/\Delta p)]$ . By concavity of  $V$ , this is a non-decreasing function of  $w_-$  over this range, and therefore  $w_+ = (1 + \rho)[w + (1 - p)B/\Delta p]$  and  $w_- = (1 + \rho)(w - pB/\Delta p)$  form an optimal choice.

This completely characterizes a solution to (46)–(49) given  $v = V$ . Let  $w^{***}$  be the upper bound of the interval over which liquidation occurs with positive probability in the optimal contract. We first prove that  $w^{**} \geq w^{***}$ . Arguing as in the proof of Corollary 2, it is easy to show that  $w^{**} > pB/\Delta p$  implies that  $w^{***} < w^*$ . It follows that for  $w = w^{***}$ , the optimal choice is  $(x^{***}, w_+^{***}, w_-^{***}) = (1, \hat{w}, (1 + \rho)(w^{***} - pB/\Delta p))$ , while for  $w = w^{**}$  it must be  $(x, w_+, w_-) = (w^{**}/w^{***}, \hat{w}, (1 + \rho)(w^{**} - pB/\Delta p))$ . Since  $w^{***} > w^{**}$ , (12) is slack at  $w = w^{***}$ , which yields:

$$\frac{x[pw_+ + (1-p)w_-]}{1+\rho} = \frac{w^{**}[p\hat{w} + (1-p)(1+\rho)(w^{***} - pB/\Delta p)]}{w^{***}(1+\rho)} < w^{**},$$

so that (12) is slack at  $w = w^{**}$  as well. Let  $\alpha$  be the constant slope of  $V$  over  $(0, w^{***})$ . Arguing as in the proof of Corollary 2, we obtain that  $[(1-p)(1+\rho) - (1+r)]\alpha = (1-p)(\rho - r)$ . Since  $a > 0$  and  $\rho > r$ , it follows that  $(1-p)(1+\rho) > (1+r)$ , and:

$$\alpha = \frac{(1-p)(\rho - r)}{(1-p)(1+\rho) - (1+r)}. \quad (72)$$

For  $w$  in a right-neighborhood of  $w^{***}$ , the optimal continuation values are  $\hat{w}$  in case of success and  $(1 + \rho)(w - pB/\Delta p) < w^{***}$  in case of failure. Thus:

$$\begin{aligned} V(w) &= \bar{R} + \frac{pV(\hat{w})}{1+r} - \frac{p(\rho-r)\hat{w}}{(1+r)(1+\rho)} + \frac{(1-p)(1+\rho)[\alpha - (\rho-r)/(1+\rho)](w - pB/\Delta p)}{1+r} \\ &= \bar{R} + \frac{pV(\hat{w})}{1+r} - \frac{p(\rho-r)\hat{w}}{(1+r)(1+\rho)} + \alpha \left( w - \frac{pB}{\Delta p} \right), \end{aligned} \quad (73)$$

where the second inequality follows from (72). It is then easy to check that the slope of  $V$  is constant over  $(0, w^*)$  and equal to  $\alpha$ . However, from (72),  $\alpha > (\rho - r)/(1 + \rho)$ . This implies that  $w^*$  is the smallest point at which the mapping  $w \mapsto V(w) - (\rho - r)w/(1 + \rho)$  reaches its maximum. Thus  $w^* = \hat{w}$ , which contradicts Proposition 2. Hence  $w^{**} \geq w^{***}$ , as claimed. It remains to prove that  $w^{**} > w^{***}$ . The above argument does not allow to conclude, since for  $w < w^{**}$ , (12) will be binding. Suppose by way of contradiction that  $w^{**} = w^{***}$ . For any  $w \in (0, w^{**})$ , the optimal choice is then given by  $(x, w_+, w_-) = (w/w^{**}, \hat{w}, (1 + \rho)(w^{**} - pB/\Delta p))$  and the corresponding Lagrangian is given by (71). Let  $\alpha$  be the constant slope of  $V$  over  $(0, w^{**})$ . Since  $w \in (0, w^{***})$ ,  $V$  is differentiable at  $w$ , and  $\alpha = (1 - p)\lambda_- + \lambda_+$  from the Envelope Theorem. Moreover,  $w_- = (1 + \rho)(w^{**} - pB/\Delta p) < w^{**}$ . Hence  $V(w_-) = \alpha w_-$ , and since  $w_- > 0$  at the optimum as  $w^{**} > pB/\Delta p$ , it is easy to check that one must have:

$$\lambda_- + \lambda_+ = \frac{(1 + \rho)\alpha - (\rho - r)}{1 + r}.$$

Combining this with the envelope condition, we obtain that:

$$\begin{aligned} \lambda_- &= \frac{(\rho - r)(\alpha - 1)}{p(1 + r)}, \\ \lambda_+ &= \alpha - \frac{(1 - p)(\rho - r)(\alpha - 1)}{p(1 + r)}. \end{aligned}$$

Inserting back the value of  $\lambda_+$  in (71), and using the fact that  $w_+ = \hat{w}$  at the optimum, we easily obtain that:

$$V'_-(\hat{w}) \geq \alpha - \frac{(\rho - r)(\alpha - 1)}{p(1 + \rho)}. \quad (74)$$

For  $w \geq w^{**}$ , the optimal continuation values are  $\hat{w}$  in case of success and  $(1 + \rho)(w - pB/\Delta p)$  in case of failure. It is easy to see that this implies that  $V$  is piecewise linear, and given by:

$$V(w) = \begin{cases} \tilde{\alpha}_0 w & \text{if } w \leq \tilde{W}_0, \\ V(\tilde{W}_n) + \tilde{\alpha}_{n+1}(w - \tilde{W}_n) & \text{if } \tilde{W}_n \leq w \leq \tilde{W}_{n+1}, \quad n = 0, \dots, \tilde{n}, \\ V(W_{\tilde{n}+1}) & \text{if } w \geq W_{\tilde{n}+1}, \end{cases} \quad (75)$$

where:

$$\tilde{W}_n = \frac{w^{**}}{(1 + \rho)^n} + \frac{pB}{\Delta p} \sum_{k=0}^{n-1} \frac{1}{(1 + \rho)^k},$$

for each  $n = 0, \dots, \tilde{n} + 1$ , with the convention that  $\sum_{k=0}^{-1} = 0$ ,  $\tilde{\alpha}_0 = \alpha$  and:

$$\tilde{\alpha}_{n+1} = \frac{(1 - p)(1 + \rho)[\tilde{\alpha}_n - (\rho - r)/(1 + \rho)]}{1 + r} \quad (76)$$

for each  $n = 0, \dots, \tilde{n}$ , and  $\tilde{n}$  is the unique integer such that  $\tilde{\alpha}_{\tilde{n}} \geq (\rho - r)/(1 + \rho) > \tilde{\alpha}_{\tilde{n}+1}$ , which is well-defined since, as (24) does not hold, one has  $p > \rho$  so that  $(1 - p)(1 + \rho)/(1 + r) < 1$ . In particular,

one must have  $\hat{w} = \tilde{W}_{\tilde{n}}$ ,  $V'_-(\hat{w}) = \tilde{\alpha}_{\tilde{n}}$  and:

$$V'_+(w^{**}) = \tilde{\alpha}_1 = \frac{(1-p)(1+\rho)[\tilde{\alpha}_0 - (\rho-r)/(1+\rho)]}{1+r}. \quad (77)$$

Since  $V$  is concave and  $\hat{w} > w^{**}$  by (30), one must have  $V'_+(w^{**}) \geq V'_-(\hat{w})$ . Hence, by (74) and (77), it follows that:

$$\tilde{\alpha}_0 \left[ \frac{\rho-r}{p(1+\rho)} + \frac{(1-p)(1+\rho)}{1+r} - 1 \right] \geq \frac{\rho-r}{p(1+\rho)} + \frac{(1-p)(\rho-r)}{1+r}.$$

As  $\rho > r$ , the bracketed term on the left-hand of this expression must be strictly positive, so that we obtain a lower bound on  $\tilde{\alpha}_0$ . However, by (74) and (76), one also has:

$$\tilde{\alpha}_{\tilde{n}+1} \geq \frac{(1-p)(1+\rho)}{1+r} \left[ \tilde{\alpha}_0 - \frac{(\rho-r)(\tilde{\alpha}_0-1)}{p(1+\rho)} - \frac{\rho-r}{1+\rho} \right].$$

Combining this with the lower bound on  $\tilde{\alpha}_0$  and using the fact that  $p > \rho > r$ , it is easy to verify that  $\tilde{\alpha}_{\tilde{n}+1} > (\rho-r)/(1+\rho)$ , which contradicts the definition of  $\tilde{n}$ . Hence  $w^{**} > w^{***}$ , as claimed.  $\blacksquare$

## Appendix 4

*Proof of Proposition 7.* Suppose that (24) holds for  $h$  arbitrarily close to 0. Define a sequence  $\{X_{n,h}\}$  as in (22), with  $rh$ ,  $\rho h$  and  $p_h$  instead of  $r$ ,  $\rho$  and  $p$ . Note first that for each  $n \in \mathbb{N}$ ,  $X_{n,h} < (1+rh)^n / [(1-p_h)^n(1+\rho h)^n]$ . Hence, by Proposition 4, if:

$$\frac{\mu}{p_h B / \Delta p_h} \geq \frac{(\rho-r)h}{1+\rho h} \sum_{k=0}^n \frac{(1+rh)^k}{(1-p_h)^k(1+\rho h)^k}, \quad (78)$$

one will have:

$$\hat{w}_h \geq \frac{p_h B h}{\Delta p_h} \sum_{k=0}^n \frac{1}{(1+\rho h)^k}, \quad (79)$$

provided that (24) holds. We shall show that (24) cannot hold when  $n$  is chosen to be the largest integer compatible with (78), that is:

$$n_h = \left\lfloor \frac{\ln \left( 1 + \frac{\mu(1+\rho h)}{p_h B (\rho-r)h / \Delta p_h} \left[ \frac{1+rh}{(1+\rho h)(1-p_h)} - 1 \right] \right)}{\ln \left( \frac{1+rh}{(1-p_h)(1+\rho h)} \right)} \right\rfloor - 1 \quad (80)$$

for  $h$  close enough to 0, where  $\lfloor x \rfloor$  denotes the integer part of  $x$  for any real number  $x$ . From (31)–(32) and (80), one has:

$$\lim_{h \downarrow 0} n_h = \lim_{h \downarrow 0} \left\lfloor \frac{\ln \left( 1 + \frac{2\mu\Delta\mu}{B(\rho-r) \left[ h\Delta\mu + \sqrt{(h\Delta\mu)^2 + 4\sigma^2 h} \right]} \right)}{\ln 2} \right\rfloor - 1 = \infty.$$

It follows that  $n_h \geq 3$  for  $h$  close enough to 0. Since  $p_h \geq 1/2$ , one will have:

$$p_h \sum_{k=1}^{n_h+1} \frac{1}{(1+\rho h)^k} \geq \frac{1}{2} \left[ \frac{1}{1+\rho h} + \frac{1}{(1+\rho h)^2} + \frac{1}{(1+\rho h)^3} \right] > 1$$

for  $h$  close enough to 0, which implies the claim. It follows from Corollary 2 that  $w_h^{**} > p_h Bh/\Delta p_h$  for  $h$  close enough to 0.

We now prove that  $w_h^{***} < (1 + \rho h)Bh/\Delta p_h$  for any  $h$  close enough to 0, which implies the weaker claim that  $\lim_{h \downarrow 0} w_h^{***} = 0$ . Extracting a subsequence if necessary, suppose that  $w_h^{***} \geq (1 + \rho h)Bh/\Delta p_h$  for any  $h$  close enough to 0. For any such  $h$ , consider the choice  $(x, w_-, w_+) = (1, 0, (1 + \rho h)Bh/\Delta p_h)$ , which clearly satisfies (11)–(12) for  $w = p_h Bh/\Delta p_h$ . For  $h$  close enough to 0,  $w_h^{**} > p_h Bh/\Delta p_h$ , and thus  $\hat{w}_h > (1 + \rho h)Bh/\Delta p_h$  by (30), where, for any  $h > 0$ ,  $\hat{w}_h$  is defined as the smallest point at which the mapping  $w \mapsto V_h(w) - (\rho - r)hw/(1 + \rho h)$  reaches its maximum. Defining the operator  $T_h^c$  as in (46)–(49), it follows that, for  $h$  close enough to 0,

$$T_h^c V_h \left( \frac{p_h Bh}{\Delta p_h} \right) = \mu h + \frac{p_h V_h((1 + \rho h)Bh/\Delta p_h)}{1 + rh} - \frac{p_h(\rho - r)h^2 B/\Delta p_h}{1 + rh}. \quad (81)$$

For any  $h > 0$ , let  $\alpha_h$  be the slope of  $V_h$  over  $(0, w_h^{***})$ . Since  $w_h^{***} \geq (1 + \rho h)Bh/\Delta p_h$ , (81) can be rewritten as:

$$T_h^c V_h \left( \frac{p_h Bh}{\Delta p_h} \right) = \mu h + \frac{p_h \alpha_h (1 + \rho h)Bh/\Delta p_h}{1 + rh} - \frac{p_h(\rho - r)h^2 B/\Delta p_h}{1 + rh},$$

or, equivalently:

$$\frac{T_h^c V_h(p_h Bh/\Delta p_h)}{p_h Bh/\Delta p_h} = \frac{\mu \Delta p_h}{p_h B} + \frac{\alpha_h (1 + \rho h)}{1 + rh} - \frac{(\rho - r)h}{1 + rh}. \quad (82)$$

For  $h$  close to 0, one has, by (32):

$$\frac{\mu \Delta p_h}{p_h B} = \frac{\mu \Delta \mu \sqrt{h}}{B \sigma} + o(\sqrt{h}) > \frac{(\rho - r)h}{1 + rh}.$$

Thus, by (82) and the fact that  $\rho > r$ , one has, for  $h$  close enough to 0:

$$\frac{T_h^c V_h(p_h Bh/\Delta p_h)}{p_h Bh/\Delta p_h} > \frac{\alpha_h (1 + \rho h)}{1 + rh} > \alpha_h,$$

which contradicts the fact that  $V_h(p_h Bh/\Delta p_h) = \alpha_h p_h Bh/\Delta p_h$  for  $h$  close enough to 0 as  $p_h Bh/\Delta p_h \in (0, w_h^{***})$ . Hence one cannot have  $w_h^{***} \geq (1 + \rho h)Bh/\Delta p_h$  for  $h$  arbitrarily close to 0, which implies the result.  $\blacksquare$

*Proof of Proposition 8.* First, we establish existence and concavity properties for  $V_0$ .

**Lemma 3.** *There exists a unique solution  $V_0$  to the boundary value problem (35)–(39).*

*Proof.* The proof goes through a number of steps.

*Step 1.* On  $[0, w^*]$ ,  $V_0$  can be written as the sum of a particular solution to (35), namely  $w + \mu/r$ , and the general solution to the homogenous equation:

$$rH(w) = \rho w H'(w) + \frac{B^2 \sigma^2}{2 \Delta \mu^2} H''(w). \quad (83)$$

A basis  $(H_0, H_1)$  of the space of solutions to this equation, called normalized hypergeometric functions, is characterized by the following initial conditions:

$$H_0(0) = 1, \quad H_0'(0) = 0,$$

$$H_1(0) = 0, \quad H_1'(0) = 1,$$

where  $H_0$  is even and  $H_1$  is odd. One can thus write:

$$V_0(w) = w + \frac{\mu}{r} + \alpha_0 H_0(w) + \alpha_1 H_1(w)$$

for some coefficients  $\alpha_0$  and  $\alpha_1$  that can be deduced from the boundary conditions  $V_0(0) = 0$  and  $V_0'(w^*) = 0$ . We obtain:

$$V_0(w) = w + \frac{\mu}{r} [1 - H_0(w)] + \frac{H_1(w)}{H_1'(w^*)} \left[ \frac{\mu}{r} H_0'(w^*) - 1 \right]. \quad (84)$$

Our objective is to show that there exists a unique  $w^*$  and associated function  $V_0$  such that  $V_0''(w^*) = 0$ .

*Step 2.* Next, we show that  $H_1'(w) \geq 0$  for all  $w \geq 0$ . Suppose the contrary, and let  $w_1 = \inf\{w \geq 0 \mid H_1'(w) < 0\}$ . Then  $H_1'(w_1) = 0$  and  $H_1''(w_1) \leq 0$ . By (83), it follows that  $H_1(w_1) \leq 0$ . However, since  $H_1'(0) = 1$ ,  $H_1$  must be increasing on  $[0, w_1]$ . Thus  $H_1(w_1) > H_1(0) = 0$ , a contradiction.

*Step 3.* By (35), one has:

$$\frac{B^2 \sigma^2}{2 \Delta \mu^2} V_0''(w^*) = r V_0(w^*) - \mu + (\rho - r) w^*.$$

Using the expression of  $V_0(w)$  given by (84), we obtain:

$$\frac{B^2 \sigma^2}{2 \Delta \mu^2} V_0''(w^*) = \rho w^* - \frac{\mu(H_0 H_1' - H_1 H_0')(w^*) + r H_1(w^*)}{H_1'(w^*)}.$$

We are thus left to prove that the function:

$$\rho w H_1'(w) - r H_1(w) - \mu(H_0 H_1' - H_1 H_0')(w)$$

has a unique zero. We first simplify this expression. Let  $D = H_0 H_1' - H_1 H_0'$ . Then by (83),

$$\begin{aligned} D'(w) &= (H_0 H_1'' - H_1 H_0'')(w) \\ &= \frac{2 \Delta \mu^2}{B^2 \sigma^2} \{H_0(w)[r H_1(w) - \rho w H_1'(w)] - H_1(w)[r H_0(w) - \rho w H_0'(w)]\} \\ &= -\frac{2 \rho \Delta \mu^2 w}{B^2 \sigma^2} (H_0 H_1' - H_1 H_0')(w) \\ &= -\frac{2 \rho \Delta \mu^2 w}{B^2 \sigma^2} D(w). \end{aligned}$$

Since  $D(0) = 1$ , one has  $D(w) = \exp(-\rho \Delta \mu^2 w^2 / (B^2 \sigma^2))$  for all  $w \geq 0$ . Define:

$$\varphi(w) = [\rho w H_1'(w) - r H_1(w)] \exp\left(\frac{\rho \Delta \mu^2 w^2}{B^2 \sigma^2}\right).$$

We must show that there exists a unique  $w^*$  such that  $\varphi(w^*) = \mu$ . By (83), one has:

$$\begin{aligned} \varphi'(w) &= \left\{ (\rho - r) H_1'(w) + \rho w H_1''(w) + \frac{2 \rho \Delta \mu^2 w}{B^2 \sigma^2} [\rho w H_1'(w) - r H_1(w)] \right\} \exp\left(\frac{\rho \Delta \mu^2 w^2}{B^2 \sigma^2}\right) \\ &= (\rho - r) H_1'(w) \exp\left(\frac{\rho \Delta \mu^2 w^2}{B^2 \sigma^2}\right), \end{aligned}$$

which is non-negative by Step 2. Hence  $\varphi$  is non-decreasing. Moreover, by (83) again, one has:

$$\begin{aligned} \varphi''(w) &= (\rho - r) \left[ H_1''(w) + \frac{2 \Delta \mu^2 \rho w}{B^2 \sigma^2} H_1'(w) \right] \exp\left(\frac{\rho \Delta \mu^2 w^2}{B^2 \sigma^2}\right) \\ &= \frac{2 r \Delta \mu^2}{B^2 \sigma^2} (\rho - r) H_1(w) \exp\left(\frac{\rho \Delta \mu^2 w^2}{B^2 \sigma^2}\right), \end{aligned}$$

which is strictly positive. Thus  $\varphi$  is strictly convex, and since it is non-decreasing,  $\lim_{w \rightarrow \infty} \varphi(w) = \infty$ . Since  $\varphi(0) = 0$ , this establishes the existence and uniqueness of  $w^*$ .  $\square$

**Lemma 4.**  $V_0$  is strictly concave on  $[0, w^*]$ .

*Proof.* The solution  $V_0$  to (35)–(39) admits a useful probabilistic interpretation. From Lions and Sznitman (1984), there exists a unique pair  $(W, L)$  that solves the Skorokhod equation:

$$dW_t = \rho W_t dt + \frac{B\sigma}{\Delta\mu} dZ_t - dL_t; \quad t \geq 0, \quad (85)$$

$$W_t \leq w^*; \quad t \geq 0,$$

$$W_0 = w,$$

where  $w \leq w^*$ ,  $Z = \{Z_t; t \geq 0\}$  is a standard Brownian motion and  $L = \{L_t; t \geq 0\}$  is a continuous non-negative and non-decreasing process that increases only when  $W_t = w^*$ , that is,  $L_t = \int_0^t 1_{\{W_s = w^*\}} dL_s$  for each  $t \geq 0$ . Define the stopping time  $\tau_0 = \inf\{t \geq 0 \mid W_t \leq 0\}$ . Then since  $V_0$  solves the Neumann problem given by (35) and (37)–(38) on  $[0, w^*]$ , it satisfies:

$$V_0(w) = w + \mathbb{E} \left[ \int_0^{\tau_0} e^{-rt} (\mu dt - dL_t) \right]; \quad w \in [0, w^*], \quad (86)$$

see Bass (1998, Theorem II.6.1). We use this characterization to prove that  $V_0$  is strictly concave on  $[0, w^*]$ . By (35),  $V_{0+}''(0) = -2\mu\Delta\mu^2/(B^2\sigma^2) < 0$ . Moreover, using the fact that any solution to (35) is smooth over  $(0, w^*)$ , one can differentiate (35) to obtain:

$$rV_0'(w) = -\rho + r + \rho V_0'(w) + \rho w V_0''(w) + \frac{B^2\sigma^2}{2\Delta\mu^2} V_0'''(w) \quad (87)$$

for all  $w \in (0, w^*)$ , from which it follows that  $V_{0-}'''(w^*) = 2(\rho - r)\Delta\mu^2/(B^2\sigma^2) > 0$  by (38)–(39), and thus  $V_0'' < 0$  in a left-neighborhood of  $w^*$  as  $V_0''(w^*) = 0$ . Now suppose that there exists some  $w \in (0, w^*)$  such that  $V_0''(w) = 0$  and  $V_0'''(w) \leq 0$ . Then, from (87), it is easy to verify that  $V_0'(w) \geq 1$ , and from (35), we obtain that  $rV_0(w) \geq \mu + rw$ , which contradicts (86). Hence  $V_0'' < 0$  on  $(0, w^*)$ , so that  $V_0$  is strictly concave on  $[0, w^*]$ .  $\square$

Next, we examine what happens when one applies the operator  $T_h$  to  $V_0$  for  $h$  close enough to 0. Since  $V_0$  is strictly concave on  $[0, w^*]$  by Lemma 4, there exists a unique point  $\hat{w}_{h, V_0} \in (0, w^*)$  at which the coercive mapping  $w \mapsto V_0(w) - (\rho - r)hw/(1 + \rho h)$  reaches its maximum, and it is characterized by the first-order condition  $V_0'(\hat{w}_{h, V_0}) = (\rho - r)h/(1 + \rho h)$ . Note that since  $V_0'(w^*) = 0$  and  $V_0$  is continuously differentiable,  $\lim_{h \downarrow 0} \hat{w}_{h, V_0} = w^*$ . Proceeding along the lines of Proposition 6, it follows that for any  $h$  close enough to 0, the solution to the problem of applying  $T_h$  to  $V_0$  is characterized by three thresholds  $w_{h, V_0}^* > w_{h, V_0}^{**} > w_{h, V_0}^{***}$  that satisfy:

$$w_{h, V_0}^* = \frac{\hat{w}_{h, V_0}}{1 + \rho h} + \frac{p_h B h}{\Delta p_h}, \quad (88)$$

$$w_{h, V_0}^{**} = w_{h, V_0}^* - \frac{B h}{\Delta p_h}. \quad (89)$$

The optimal choices in the regions  $[w_{h, V_0}^*, \infty)$ ,  $[w_{h, V_0}^{**}, w_{h, V_0}^*)$ ,  $[w_{h, V_0}^{***}, w_{h, V_0}^{**})$  and  $[0, w_{h, V_0}^{***})$  are as characterized in items (i)–(iv) of Proposition 6, with obvious notational adjustments. A key observation is that,



by (32) and (88)–(89), one has, for  $h$  close to 0:

$$w_{h,V_0}^* = \hat{w}_{h,V_0} + \frac{B\sigma\sqrt{h}}{\Delta\mu} + o(\sqrt{h}) > \hat{w}_{h,V_0}, \quad (90)$$

$$w_{h,V_0}^{**} = \hat{w}_{h,V_0} - \frac{B\sigma\sqrt{h}}{\Delta\mu} + o(\sqrt{h}) < \hat{w}_{h,V_0}. \quad (91)$$

Moreover, using the fact that  $V_0'(w^*) = V_0''(w^*) = 0$ ,  $V_0'''(w^*) = 2(\rho - r)\Delta\mu^2/(B^2\sigma^2)$  and  $V_0'(\hat{w}_{h,V_0}) = (\rho - r)h/(1 + \rho h)$ , a Taylor–Young expansion yields:

$$w^* = \hat{w}_{h,V_0} + \frac{B\sigma\sqrt{h}}{\Delta\mu} + o(\sqrt{h}). \quad (92)$$

Note that, as  $h$  goes to 0, the operator  $T_h$  converges to the identity, and the same is true of the operator  $T_h^c$ . Heuristically, our next result says that the derivative at  $h = 0$  of the operator  $T_h^c$  applied to  $V_0$  is 0.

**Lemma 5.**  $\sup_{w \geq p_h B h / \Delta p_h} |T_h^c V_0(w) - V_0(w)| = o(h)$ .

*Proof.* For each  $(w, h) \in \mathbb{R}_+^2$ , denote  $\Delta_h(w) = (1 + rh)[T_h^c V_0(w) - V_0(w)]$ .

*Case 1.* Consider first the region  $[w_{h,V_0}^*, \infty)$ . For  $w$  in this region, one has:

$$T_h^c V_0(w) = \mu h + \frac{V_0(\hat{w}_{h,V_0})}{1 + rh} - \frac{(\rho - r)h\hat{w}_{h,V_0}}{(1 + rh)(1 + \rho h)}.$$

Multiplying by  $1 + rh$  and subtracting  $(1 + rh)V_0(w)$ , we obtain that:

$$\Delta_h(w) = \mu h + V_0(\hat{w}_{h,V_0}) - (1 + rh)V_0(w) - (\rho - r)h\hat{w}_{h,V_0} + o(h).$$

First, note that we have the following lower bound on  $\Delta_h(w)$ :

$$\Delta_h(w) \geq h[\mu - rV(\hat{w}_{h,V_0}) - (\rho - r)\hat{w}_{h,V_0}] + (1 + rh)[V_0(\hat{w}_{h,V_0}) - V_0(w^*)] + o(h). \quad (93)$$

We show that the right-hand side of (93) is a  $o(h)$ . For the first term, this follows at once from the fact that  $\lim_{h \downarrow 0} \hat{w}_{h,V_0} = w^*$  and  $rV(w^*) = \mu - (\rho - r)w^*$  by (35) and (38)–(39). For the second term, the concavity of  $V_0$  implies that:

$$\frac{|V_0(\hat{w}_{h,V_0}) - V_0(w^*)|}{h} \leq \frac{V_0'(\hat{w}_{h,V_0})(w^* - \hat{w}_{h,V_0})}{h} = \frac{(\rho - r)(w^* - \hat{w}_{h,V_0})}{1 + \rho h},$$

and the result follows since  $\lim_{h \downarrow 0} \hat{w}_{h,V_0} = w^*$ . Next, from (90),  $w_{h,V_0}^* > \hat{w}_{h,V_0}$  for  $h$  close enough to 0, and thus we have the following upper bound on  $\Delta_h(w)$ :

$$\Delta_h(w) \leq \mu h - rhV_0(\hat{w}_{h,V_0}) - (\rho - r)h\hat{w}_{h,V_0} + o(h). \quad (94)$$

Proceeding as for (93), it is immediate to check that the right-hand side of (94) is a  $o(h)$ . It follows that  $\sup_{w \in [w_{h,V_0}^*, \infty)} |\Delta_h(w)| = o(h)$ .

*Case 2.* Consider next the region  $[w_{h,V_0}^{**}, w_{h,V_0}^*)$ . For  $w$  in this region, one has:

$$T_h^c V_0(w) = \mu h + \frac{p_h V_0(\hat{w}_{h,V_0}) + (1 - p_h)V_0(w_{h,-}(w))}{1 + rh} - \frac{(\rho - r)h[p_h\hat{w}_{h,V_0} + (1 - p_h)w_{h,-}(w)]}{(1 + rh)(1 + \rho h)},$$

where  $w_{h,-}(w) = (1 + \rho h)(w - p_h B h / \Delta p_h)$ . Multiplying by  $1 + rh$  and subtracting  $(1 + rh)V_0(w)$ , we obtain that:

$$\begin{aligned} \Delta_h(w) &= \mu h - rhV_0(w) - \frac{(\rho - r)h[p_h\hat{w}_{h,V_0} + (1 - p_h)w_{h,-}(w)]}{1 + \rho h} \\ &\quad + p_h[V_0(\hat{w}_{h,V_0}) - V_0(w)] + (1 - p_h)[V_0(w_{h,-}(w)) - V_0(w)]. \end{aligned}$$

Proceeding as in Case 1, it is easy to check that:

$$\sup_{w \in [w_{h,V_0}^{**}, w_{h,V_0}^*]} \left| \mu h - rhV_0(w) - \frac{(\rho - r)h[p_h \hat{w}_{h,V_0} + (1 - p_h)w_{h,-}(w)]}{1 + \rho h} \right| = o(h).$$

We now consider the two remaining terms in  $V_0(\hat{w}_{h,V_0}) - V_0(w)$  and  $V_0(w_{h,-}(w)) - V_0(w)$  in the expression of  $\Delta_h(w)$ . The following argument is valid for both terms, so we shall only consider the first. A Taylor–Young expansion yields:

$$V_0(\hat{w}_{h,V_0}) - V_0(w) = V_0'(w)(\hat{w}_{h,V_0} - w) + \frac{V_0''(w)}{2}(\hat{w}_{h,V_0} - w)^2 + o((\hat{w}_{h,V_0} - w)^2). \quad (95)$$

We treat each term on the right-hand side of (95) separately. If  $w \in (w_{h,V_0}^{**}, w_{h,V_0}^*)$ , (90)–(92) imply that:

$$|\hat{w}_{h,V_0} - w| \leq \frac{B\sigma\sqrt{h}}{\Delta\mu} + o(\sqrt{h}), \quad (96)$$

$$|w - w^*| \leq \frac{2B\sigma\sqrt{h}}{\Delta\mu} + o(\sqrt{h}). \quad (97)$$

Since  $V_0'''$ , whenever defined, is bounded above by a positive constant  $K$ , and since  $V_0''(w^*) = 0$ , (97) implies that:

$$|V_0''(w)| \leq K|w - w^*| \leq \frac{2B\sigma K\sqrt{h}}{\Delta\mu} + o(\sqrt{h}). \quad (98)$$

for any  $w \in (w_{h,V_0}^{**}, w_{h,V_0}^*)$ . Finally, since  $V_0'(w^*) = 0$ , (97)–(98) imply that:

$$V_0'(w) = - \int_w^{w^*} V_0''(x) dx \leq \left[ \frac{2B\sigma K\sqrt{h}}{\Delta\mu} + o(\sqrt{h}) \right] |w - w^*| \leq \frac{4B^2\sigma^2 K h}{\Delta\mu^2} + o(h) \quad (99)$$

for any  $w \in (w_{h,V_0}^{**}, w_{h,V_0}^*)$ . From (95)–(99), one has:

$$\sup_{w \in [w_{h,V_0}^{**}, w_{h,V_0}^*]} |V_0(\hat{w}_{h,V_0}) - V_0(w)| = o(h),$$

and a similar result holds for  $V_0(w_{h,-}(w)) - V_0(w)$ . It follows that  $\sup_{w \in [w_{h,V_0}^{**}, w_{h,V_0}^*]} |\Delta_h(w)| = o(h)$ .

*Case 3.* Consider finally the region  $[pBh/\Delta p_h, w_{h,V_0}^{**})$ . For  $w$  in this region, one has:

$$\begin{aligned} T_h^c V_0(w) &= \mu h + \frac{p_h V_0(w_{h,+}(w)) + (1 - p_h) V_0(w_{h,-}(w))}{1 + rh} \\ &\quad - \frac{(\rho - r)h \{ p_h w_{h,+}(w) + (1 - p_h) w_{h,-}(w) \}}{(1 + rh)(1 + \rho h)}. \end{aligned}$$

Multiplying by  $1 + rh$  and subtracting  $(1 + rh)V_0(w)$ , we obtain that:

$$\begin{aligned} \Delta_h(w) &= \mu h - rhV_0(w) - (\rho - r)hw \\ &\quad + p_h[V_0(w_{h,+}(w)) - V_0(w)] + (1 - p_h)[V_0(w_{h,-}(w)) - V_0(w)] + o(h). \end{aligned}$$

A Taylor–Young expansion yields:

$$\begin{aligned} \Delta_h(w) &= \mu h - rhV_0(w) - (\rho - r)hw + \rho h w V_0'(w) \\ &\quad + \frac{V_0''(w)}{2} \{ p_h [w_{h,+}(w) - w]^2 + (1 - p_h) [w_{h,-}(w) - w]^2 \} \\ &\quad + o([w_{h,+}(w) - w]^2) + o([w_{h,-}(w) - w]^2) + o(h). \end{aligned} \quad (100)$$

By (91)  $w_{h,V_0}^{**} < \hat{w}_{h,V_0} < w^*$  for  $h$  close enough to 0. Hence  $V_0$  satisfies the differential equation (35) at  $w$ . One can therefore rewrite (100) as:

$$\begin{aligned} \Delta_h(w) &= \frac{V_0''(w)}{2} \left\{ p_h [w_{h,+}(w) - w]^2 + (1 - p_h) [w_{h,-}(w) - w]^2 - \frac{B^2 \sigma^2 h}{\Delta \mu^2} \right\} \\ &\quad + o([w_{h,+}(w) - w]^2) + o([w_{h,-}(w) - w]^2) + o(h). \end{aligned} \quad (101)$$

Using (32) and the definitions of  $w_{h,+}(w)$  and  $w_{h,-}(w)$ , one has:

$$w_{h,+}(w) - w = \rho h w + \frac{B\sigma\sqrt{h}}{\Delta\mu} + o(\sqrt{h}) \quad (102)$$

$$w_{h,-}(w) - w = \rho h w - \frac{B\sigma\sqrt{h}}{\Delta\mu} + o(\sqrt{h}) \quad (103)$$

Since  $w \leq w^*$ , and  $V_0''$  is bounded on  $[0, w^*]$  as  $V_0$  is twice continuously differentiable, it follows from (101)–(103) that  $\sup_{w \in [p_h B h / \Delta p_h, w_{h,V_0}^{**}]} |\Delta_h(w)| = o(h)$ . This completes the proof.  $\square$

We are now ready to prove that  $V_h$  converges uniformly to  $V_0$  as  $h$  goes to 0. For any  $w \geq w_h^{***}$ , one has  $V_h(w) - V_0(w) = T_h^c V_h(w) - T_h^c V_0(w) + T_h^c V_0(w) - V_0(w)$ , so that:

$$\begin{aligned} \sup_{w \geq w_h^{***}} |V_h(w) - V_0(w)| &\leq \sup_{w \geq p_h B h / \Delta p_h} |T_h^c V_h(w) - T_h^c V_0(w)| + \sup_{w \geq p_h B h / \Delta p_h} |T_h^c V_0(w) - V_0(w)| \\ &\leq \frac{1}{1 + r h} \sup_{w \geq 0} |V_h(w) - V_0(w)| + \sup_{w \geq p_h B h / \Delta p_h} |T_h^c V_0(w) - V_0(w)|, \end{aligned} \quad (104)$$

where the first inequality follows from the fact that  $w_h^{***} \geq p_h B h / \Delta p_h$ , and the second from a straightforward adaptation of Blackwell's Theorem. Three cases must be distinguished.

*Case 1.* Suppose first that along a subsequence  $h$  converging to 0, one has  $T_h^c(w_h^{***})/w_h^{***} \geq V_{0+}'(0)$ . Then the function  $V_h - V_0$  is convex and non-negative on  $[0, w_h^{***}]$ , and it reaches its maximum at  $w_h^{***}$ . It follows that  $\sup_{w \geq 0} |V_h(w) - V_0(w)| = \sup_{w \geq w_h^{***}} |V_h(w) - V_0(w)|$ , so that (104) implies that:

$$\sup_{w \geq w_h^{***}} |V_h(w) - V_0(w)| \leq \frac{1 + r h}{r h} \sup_{w \geq p_h B h / \Delta p_h} |T_h^c V_0(w) - V_0(w)|,$$

and it follows from Lemma 5 that  $\lim_{h \downarrow 0} \sup_{w \geq w_h^{***}} |V_h(w) - V_0(w)| = 0$ . In particular,  $\lim_{h \downarrow 0} |V_h(w_h^{***}) - V_0(w_h^{***})| = 0$ , and since  $\lim_{h \downarrow 0} w_h^{***} = 0$  by Proposition 7,  $\lim_{h \downarrow 0} V_h(w_h^{***}) = 0$ . Since  $V_h(0) = V_0(0)$  and both  $V_h$  and  $V_0$  are increasing over  $[0, w_h^{***}]$ , one has:

$$\sup_{w \in [0, w_h^{***}]} |V_h(w) - V_0(w)| \leq V_h(w_h^{***}) + |V_h(w_h^{***}) - V_0(w_h^{***})| + V_0(w_h^{***}),$$

and thus  $\lim_{h \downarrow 0} \sup_{w \in [0, w_h^{***}]} |V_h(w) - V_0(w)| = 0$ . Hence  $V_h$  converges uniformly to  $V_0$  along this subsequence, as claimed.

*Case 2.* Suppose next that along a subsequence  $h$  converging to 0, one has  $V_0'(w_h^{***}) \geq T_h^c(w_h^{***})/w_h^{***}$ . Then the function  $V_0 - V_h$  is convex and non-negative on  $[0, w_h^{***}]$ , and it reaches its maximum at  $w_h^{***}$ . It follows that  $\sup_{w \geq 0} |V_h(w) - V_0(w)| = \sup_{w \geq w_h^{***}} |V_h(w) - V_0(w)|$ , and the rest of the proof follows as in Case 2.

*Case 3.* Suppose finally that along a subsequence  $h$  converging to 0,  $T_h^c(w_h^{***})/w_h^{***} < V_{0+}'(0)$  and  $V_0'(w_h^{***}) < T_h^c(w_h^{***})/w_h^{***}$ . Therefore  $\lim_{h \downarrow 0} T_h^c(w_h^{***})/w_h^{***} = V_{0+}'(0)$  as  $V_0$  is continuously differentiable.

Since  $w_h^{***} \leq (1 + \rho h)Bh/\Delta p_h$  for  $h$  close enough to 0 as shown in the proof of Proposition 7, one has  $\sup_{w \in [0, w_h^{***}]} |V_h(w) - V_0(w)| = o(\sqrt{h})$ . Now two cases can occur. Either  $\lim_{h \downarrow 0} \sup_{w \geq 0} |V_h(w) - V_0(w)| = 0$ , in which case  $V_h$  converges uniformly to  $V_0$  along this subsequence. Or  $\lim_{h \downarrow 0} \sup_{w \geq 0} |V_h(w) - V_0(w)| > 0$ , in which case  $\sup_{w \geq 0} |V_h(w) - V_0(w)| = \sup_{w \geq w_h^{***}} |V_h(w) - V_0(w)|$  for  $h$  close enough to 0, and one can show as in Case 1 that  $V_h$  converges uniformly to  $V_0$  along this subsequence, a contradiction. In all cases, we obtain that  $V_h$  converges uniformly to  $V_0$ , and the result follows.  $\blacksquare$

## Appendix 5

*Proof of Lemma 1.* In our implementation, cash reserves  $M_t$  are a deterministic function  $g(W_t)$  of the entrepreneur's rent  $W_t$ , with  $g(0) = 0$ . Using (85) and applying Itô's lemma, we obtain:

$$dM_t = \left[ \rho W_t g'(W_t) + \frac{k^2 \sigma^2}{2} g''(W_t) \right] dt + k \sigma g'(W_t) dZ_t - g'(w^*) dL_t. \quad (105)$$

Comparing with (40), we can identify the martingale parts of  $dM_t$  in (40) and (105). Since  $\{P_t; t \geq 0\}$  is an increasing process by assumption, it cannot contain a martingale term. Hence,  $kg' = 1$ . Since  $g(0) = 0$ , it must be that  $g(w) = w/k$  and therefore that  $m_0 = w_0/k$ . From (105), we obtain (42). Equating (40) and (42) then yields (43). Note that since  $\mu \geq (\rho - r)w^*/k$ , the process  $\{P_t; t \geq 0\}$  thereby obtained is indeed an increasing process.  $\blacksquare$

*Proof of Lemma 2.* Recall that, over  $[0, m^*]$ ,  $S$  solves the homogenous equation:

$$rh(m) = \rho wh'(m) + \frac{\sigma^2}{2} h''(m), \quad (106)$$

which is simply (83) with  $B/\Delta\mu = 1$ . A basis  $(h_0, h_1)$  of the space of solutions to this equation is characterized by the following initial conditions:

$$h_0(0) = 1, \quad h_0'(0) = 0,$$

$$h_1(0) = 0, \quad h_1'(0) = 1,$$

where  $h_0$  is even and  $h_1$  is odd. Proceeding as in the proof of Lemma 3, one has  $h_1'(m) \geq 0$  for all  $m \geq 0$ . Since  $S(0) = 0$ ,  $S$  must be a positive multiple of  $h_1$  over  $[0, m^*]$ , specifically  $S(m) = h_1(m)/h_1'(m^*)$ , and is thus non-decreasing over this interval. We now prove that  $h_1$  is strictly concave, and thus strictly increasing. Differentiating (106), we get that at any point  $m > 0$  where  $h_1'(m) = 0$ ,  $h_1'''(m)$  and  $(r - \rho)h_1'(m)$  have the same sign, which is  $-1$  as  $h_1'(m) = rh(m)/(\rho m) > 0$  by (106). Thus,  $h_1''$  is decreasing at its zeros over  $(0, m^*)$ , if any. From (106),  $h_1''(m) = 2(r - \rho)h_1'(m)/\sigma^2 + o(m)$  and is thus negative at  $0^+$ , since  $h_1'(0) = 1$ . It follows that  $h_1'' < 0$  over  $\mathbb{R}_+$  which implies the claim. Since  $S$  is a positive multiple of  $h_1$  over  $[0, m^*]$ , the result follows. Note that  $S(m)$  is strictly increasing and strictly concave with respect to  $m$  over  $[0, w^*]$ , and affine and equal to  $S(m^*) + m - m^*$  over  $[m^*, \infty)$ .  $\blacksquare$

*Proof of Proposition 9.* Let  $N(m) = D'(m)S(m) - D(m)S'(m)$ . It is straightforward to verify that:

$$\rho m N(m) = -\frac{\sigma^2}{2} N'(m) - [\mu - (\rho - r)m]S(m).$$

Thus,  $N'(m) < 0$  whenever  $N(m) = 0$ , which shows that  $N$  admits at most one zero. Note that:

$$S_+''(0) = 0,$$

$$S_+'''(0) = \frac{2(r - \rho)}{\sigma^2} S_+'(0),$$

and:

$$D_+''(0) = -\frac{2\mu}{\sigma^2},$$

$$D_+'''(0) = \frac{2(r-\rho)}{\sigma^2} [D_+'(0) - 1].$$

As a result, one has:

$$S(m) = S_+'(0)m + o(m^2),$$

$$S'(m) = S_+'(0) + \frac{r-\rho}{\sigma^2} S_+'(0)m^2 + o(m^2)$$

and:

$$D(m) = D_+'(0)m - \frac{\mu}{\sigma^2} m^2 + o(m^2)$$

$$D'(m) = D_+'(0) - \frac{2\mu}{\sigma^2} m + \frac{(r-\rho)}{\sigma^2} [D_+'(0) - 1]m^2 + o(m^2).$$

It follows that  $N(m) = -\mu S_+'(0)m^2/\sigma^2 + o(m^2)$ , which is negative at  $0^+$ . This implies the result.  $\blacksquare$

*Proof of Proposition 10.* To focus on an interesting case, assume that  $m^* > m_0$ . The derivative of the book to market ratio with respect to  $m$  is:

$$BM'(m) = \frac{S(m) - S'(m)(S_0 - m_0 + m)}{S(m)^2} = \frac{\phi(m)}{S(m)^2}.$$

While  $\phi(0) < 0$ ,  $\phi(m^*) > 0$ . Indeed,  $\phi(m^*) > 0$  is equivalent to  $S(m^*) > S'(m^*)(S_0 - m_0 + m^*)$ . Since  $S'(m^*) = 1$ , this amounts to  $S(m^*) - m^* > S_0 - m_0$ , which holds as  $S$  is strictly concave over  $[0, m^*]$  and  $S'(m^*) = 1$ . Furthermore,  $\phi$  is increasing as  $\phi'(m) = -S''(m)(S_0 - m_0 + m) > 0$ . Hence, there exists a unique  $m \in (0, m^*)$  such that  $\phi(m) = 0$ . It follows that  $BM$  is U-shaped, as claimed. Finally, note that  $\phi(m_0) = S(m_0)[1 - S'(m_0)]$  is negative as  $S$  is strictly concave over  $[0, m^*]$  and  $S'(m^*) = 1$ . Hence  $S'(m_0) < 0$ , as claimed.  $\blacksquare$

*Proof of Proposition 11.* The present value of the expected payoff to the investors is  $F(w) = V(w) - w$ . Since  $w = km$ , we shall hereafter focus on  $m$ , rather than on  $w$ , as the state variable. The expected revenue of the financiers is  $\hat{F}(m) = V(km) - km$ . The maximum amount that can be obtained from the financiers,  $F^*$ , is simply the maximum value of  $\hat{F}$ . From (35)–(39),  $\hat{F}$  is concave and solves:

$$r\hat{F}(m) = \mu + \rho m\hat{F}'(m) + \frac{\sigma^2}{2} \hat{F}''(m); \quad m \in [0, m^*),$$

$$\hat{F}(0) = 0,$$

$$\hat{F}'(m^*) = -k,$$

$$\hat{F}''(m^*) = 0.$$

We study the variations of  $\hat{F}$  with respect to  $\mu$ ,  $\sigma^2$ , and  $k$ .

Case 1. Let  $\hat{F}_\mu = \partial\hat{F}/\partial\mu$ .  $\hat{F}_\mu$  solves:

$$r\hat{F}_\mu(m) = 1 + \rho m\hat{F}'_\mu(m) + \frac{\sigma^2}{2}\hat{F}''_\mu(m); \quad m \in [0, m^*],$$

$$\hat{F}_\mu(0) = 0,$$

$$\hat{F}'_\mu(m^*) = 0.$$

Thus,  $F_\mu(m) = \mathbb{E}[\int_0^{\tau_0} e^{-rt} dt] > 0$ . It follows that  $\hat{F}$  increases with  $\mu$ , and so does  $F^*$ .

Case 2. Let  $\hat{F}_k = \partial\hat{F}/\partial k$ .  $\hat{F}_k$  solves:

$$r\hat{F}_k(m) = \rho m\hat{F}'_k(m) + \frac{\sigma^2}{2}\hat{F}''_k(m); \quad m \in [0, m^*],$$

$$\hat{F}_k(0) = 0,$$

$$\hat{F}'_k(m^*) = -1.$$

This is the same equation as that regulating the dynamics of  $S$ , except that the boundary condition at  $m^*$  has the opposite sign, as  $S'(m^*) = 1$ . Thus,  $\hat{F}_k(m) = -S(m) < 0$  for all  $m > 0$ . It follows that  $\hat{F}$  decreases with  $k$ , and so does  $F^*$ .

Case 3. Let  $\hat{F}_{\sigma^2} = \partial\hat{F}/\partial\sigma^2$ .  $\hat{F}_{\sigma^2}$  solves:

$$r\hat{F}_{\sigma^2}(m) = \frac{1}{2}\hat{F}''(m) + \rho m\hat{F}'_{\sigma^2}(m) + \frac{\sigma^2}{2}\hat{F}''_{\sigma^2}(m); \quad m \in [0, m^*],$$

$$\hat{F}_{\sigma^2}(0) = 0,$$

$$\hat{F}'_{\sigma^2}(m^*) = 0.$$

Thus,  $H(m) = \mathbb{E}[\int_0^\tau \hat{F}''(m_t)e^{-rt} dt]/2$ . Since  $\hat{F}$  is concave this is negative. It follows that  $\hat{F}$  decreases with  $\sigma^2$ , and so does  $F^*$ . ■

*Proof of Proposition 12.* From Lemma 2, one has:

$$S(m) = \begin{cases} h_1(m)/h'_1(m^*) & \text{if } m \leq m^*, \\ S(m^*) + m - m^* & \text{if } m \geq m^*. \end{cases} \quad (107)$$

Proceeding along the lines of Lemma 3, it is easy to establish that  $m^*$  is implicitly defined by  $\phi(m^*) = \mu/k$ , where:

$$\phi(m) = [\rho mh'_1(m) - rh_1(m)] \exp\left(\frac{\rho m^2}{\sigma^2}\right)$$

is a convex and increasing function such that  $\phi(0) = 0$ . This implies that  $m^*$  is increasing with respect to  $\mu$  and decreasing with respect to  $k$ . The result then follows immediately from (107) together with the concavity of  $h_1$ . ■

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