Belief Heterogeneity and Survival in Incomplete Markets*

Tarek Coury
Department of Economics
University of Oxford

Emanuela Sciubba
Faculty of Economics
University of Cambridge

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Abstract

In complete markets economies (Sandroni [15]), or in economies with Pareto optimal outcomes (Blume and Easley [9]), the market selection hypothesis holds, as long as traders have identical discount factors. Traders who survive must have beliefs that are close to the truth in the sense that the true probability distribution is absolutely continuous with respect to surviving traders' beliefs. We show that in incomplete markets, regardless of traders' discount factors, the set of beliefs that are consistent with the survival of a trader contains beliefs that are not equivalent to the true probability distribution. The market selection hypothesis does not hold in incomplete markets.

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Address for Correspondence: Tarek Coury, University of Oxford, Department of Economics, Manor Road Building, Manor Road, Oxford, OX1 3UQ, United Kingdom. Email: tarek.coury@economics.oxford.ac.uk; Emanuela Sciubba, University of Cambridge, Faculty of Economics, Sidgwick Avenue, Cambridge, CB3 9DD, UK. Email: es204@econ.cam.ac.uk.

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Whenever this determinant [of business behaviour] happens to lead to behaviour consistent with rational and informed maximisation of returns, the business will prosper and acquire resources with which to expand; whenever it does not the business will tend to lose resources and can be kept in existence only by the addition of resources from the outside. The process of natural selection thus helps to validate the hypothesis [of profit maximisation] or, rather, given natural selection, acceptance of the hypothesis can be based largely on the judgement that it summarises appropriately the conditions for survival.

[Friedman, Essays in Positive Economics, 1953]

1 Introduction

Do markets select for correct expectations? The market selection hypothesis (Alchian [1], Friedman [12]) is one of the longest standing conjectures in economics. Traders who form more accurate predictions about future returns make more money at the expense of those who don't. In the long run all traders with inaccurate beliefs are driven out of the market and the only surviving ones have correct expectations. This hypothesis has a strong intuitive appeal and, if true, provides a robust justification to the assumption of rational expectations in both microeconomic and macroeconomic models. Given that long run market outcomes only reflect correct expectations, economists interested in the long run may as well assume rational expectations from the outset.

To test the validity of this conjecture, suppose that two traders disagree on the probability with which a particular state of nature occurs. If this disagreement does not have an impact on asymptotic wealth accumulation and survival, then Friedman's conjecture does not hold. Hence the market selection hypothesis requires that the trader with correct expectations is able to accumulate wealth at the other trader's expense by betting against him on the future realisation of that particular state of nature. It is only when there is a market that allows the two traders to make these bets that the trader with correct beliefs can actually accumulate more wealth than the other trader and drive him out of the market. When that market is missing, the link between accuracy of beliefs and survival becomes weaker.

We know that when markets are complete [15], or when the allocation is Pareto optimal [9] correct beliefs are selected for by market forces[1]. In particular, heterogeneity of beliefs does not persist, and all surviving traders have either correct beliefs or beliefs which merge with the true probability distribution. Blume and Easley [9] argue by providing counterexamples that the same need not hold when markets are incomplete. In this paper we show that in incomplete markets economies, regardless of traders' discount factors, the set of beliefs which are consistent with traders' survival contains beliefs that are not equivalent to the true probability distribution. So the market selection hypothesis does not hold in incomplete markets.

We consider an economy with an open ended future and a finite number of traders. Every period, traders trade securities to hedge their stochastic endowment risk. Preferences are of the expected utility form and utility from future consumption is discounted at a rate that is allowed to differ across traders. There are many consumption goods each period, but the securities pay off only in terms of a numeraire good. Also, securities are short-lived. These last two assumptions do not affect the intuition of the result but considerably simplify the analysis and guarantee existence of an equilibrium. Otherwise, the asset structure is rather general in that the payoff matrix may change from period to period.

The infinite horizon economy that we model satisfies conditions for existence

[1]This assumes that traders discount future consumption at the same rate so that their degree of impatience does not affect their survival.

of an equilibrium with a transversality condition. This requires traders not to borrow and roll over their debt *ad infinitum*. Our main result is that traders who survive admit beliefs that are not equivalent to the true probability distribution. To prove our result, we introduce the notion of *effectively identical beliefs* as the set of probability distributions for some trader that are consistent with the same overall equilibrium. Given an initial economy and its corresponding equilibrium, if some trader were to adopt beliefs that are effectively identical to his original beliefs, then the new equilibrium outcome would remain unchanged.

We then show that the set of effectively identical beliefs is a singleton under complete markets. By contrast, that set is not a singleton in incomplete markets. Moreover, there exists a probability distribution that belongs to this set that is not equivalent to the truth.

This has straightforward and important consequences for belief selection in incomplete markets. Suppose that a trader survives, our main result shows that there are probability distributions that are not equivalent to the truth which are consistent with his survival. Hence incomplete markets fail to select for traders with correct expectations.

The structure of the paper is as follows. In section 2 we summarise the existing literature. In section 3, we present the model: we start by providing the intuition for our main results in a simple two-period model (subsection 3.1), then we go on to describe the infinite horizon economy and we show that it always admits an equilibrium with transversality condition. Section 4 contains our main results: we introduce the notion of effectively identical beliefs and we show that, when markets are incomplete, the set of effectively identical beliefs is not a singleton. Section 5 draws the implications for survival and belief selection of our main results. Section 6 concludes the paper. For ease of exposition, all proofs are in the appendix.

2 Related Literature

The first attempts to validate the market selection hypothesis date back to the early 90s and address the related issue of whether markets select for rationality, with particular focus on the survival of noise traders. Shefrin and Statman [17] ask whether noise traders survive in financial markets by developing a model where rational and informed Bayesian traders interact with traders that make systematic cognitive errors. They show that, provided that noise traders are patient enough and that they do not commit errors that are "too serious", they will not be driven to extinction by informed traders. De Long et al. [10] and [11] prove that noise traders can eventually come to dominate the market, if they unwillingly happen to make "good" cognitive mistakes. Biais and Shadur [5] consider a market where non-overlapping generations of buyers and sellers trade to share risk. They show that irrational traders, who misperceive the risk but enjoy a higher bargaining power, might outperform rational traders who correctly assess the distribution of the risk.

While this literature assumes asset prices to be exogenous, the paper by Blume and Easley [8] addresses the same problem in a market model, where asset prices are determined endogenously and reflect the dynamics of the wealth shares of the different types of traders, each represented by a portfolio rule. They find that, as long as traders save at the same rate, markets do not select for rationality, but rather for a specific attitude towards risk. In particular logarithmic utility maximisers with accurate beliefs accumulate wealth at a faster rate than any other trader. As a result, they determine asset prices asymptotically and drive to extinction any other trader. Hence within this framework markets do not select necessarily for rationality, but rather for a specific portfolio rule. Irrational traders, or traders with inaccurate expectations, may well survive if their mistakes or irrationality imply that their portfolio rules are closer to the portfolio rule of a log maximiser. On the other hand, rational traders with correct expectations may well vanish, if they happen to have the wrong attitude towards risk.

The results from this early literature are important in that they formalise through wealth dynamics what one might mean by market selection. They are also quite provocative because they make it very clear that expected utility maximisation and survival are distinct objectives. Hence rational behaviour is not necessarily selected for by market forces and the market selection hypothesis need not hold within this setting.

Sandroni [15] adopts the same notion of market selection and survival as in Blume and Easley [8], but differs from the earlier contributions in that he considers not only portfolio decisions but also savings decisions to be endogenous. In a Lucas trees complete markets economy where traders are expected utility maximisers and discount the future at the same rate, he finds that under mild conditions on traders' utility functions, only traders with correct beliefs survive. Hence, among rational traders, complete markets select for correct beliefs.

Blume and Easley [9] generalise Sandroni's result to any Pareto optimal allocation. For any optimal allocation, survival of traders is determined entirely by beliefs and discount factors; in contrast with [8], risk attitudes do not matter for survival. Among traders who discount future consumption at the same rate, it is those with most accurate beliefs that survive, irrespective of their utility function. In particular, if there are traders whose beliefs merge with the truth, they will be the only survivors. Blume and Easley [9] provide two interesting counterexamples that show that the same results need not carry through under market incompleteness, where in general allocations are not Pareto optimal.

In this paper we prove for a large class of incomplete markets economies that surviving traders need not have beliefs that merge with the true probability distribution. The fact that, under incomplete markets, opportunities to trade are restricted implies that traders with incorrect beliefs are not wiped out by market forces. As a result surviving traders' beliefs do not necessarily merge either with the truth or with other traders' beliefs, and so beliefs' heterogeneity is persistent.

3 The Model

3.1 A Two-Period Example

Consider a two period economy with a unique consumption good and where there are S possible states of the world tomorrow. Time is indexed by $t = 0, 1$. Traders can trade $J \leq S$ securities whose period 1 payoff is the full rank $S \times J$ matrix A. They trade these securities to hedge against their period 1 stochastic endowment $\omega \in \mathbb{R}^S_{++}$. Consumption takes place in period 1 only. In an equilibrium, period 0 asset prices $q \in \mathbb{R}^J_{++}$ have to satisfy the no arbitrage equation:

$$q = \frac{\pi_1^i}{\pi_0^i} A \tag{1}$$

where $\pi_1^i \in \mathbb{R}^S_{++}$ is trader i's utility gradient and where $\pi_0^i \in \mathbb{R}_{++}$ is a Lagrange multiplier. The resulting ratio $\frac{\pi_1^i}{\pi_0^i}$ is trader i's normalized utility gradient or his state price vector. In the complete markets case, we get the usual condition that this ratio is equated across traders.[2] Note that $\pi_1^i(s) = \rho^i(s) v^{i\prime}(x^i(s))$ when traders have preferences of the expected utility form and their beliefs are represented by the probability distribution ρ^i. In the complete markets

[2] Equation (1) can be given the familiar form $q = \psi \mathbb{E}(V)$ where the expectation is taken with respect to some probability distribution. In the complete markets case, this probability distribution is unique. The scalar ψ represents the price of a bond that pays off one unit of consumption in each state (if that bond exists).

case, given an equilibrium outcome (x^*, q^*), there exists only one set of beliefs (an $S-$dimensional normalized vector ρ^i) such that $q^* \pi_0^i = \pi_1^i A$ where $\pi_1^i(s) = \rho^i(s) v^i(x^{*i}(s))$. This is because there are S equations in S unknowns. The only solution is the original normalized vector $\rho^i \in \mathbb{R}^S$.

In the incomplete markets case, $J < S$ and this system of equations may have multiple solutions. To guarantee multiple solutions to the no-arbitrage equation, one needs to assume that $J < S-1$. The additional degree of freedom is used to ensure that the resulting solution is a probability distribution. So, given an economy and a resulting equilibrium outcome, and for any trader $i \in \mathbb{I}$, there exist many probability distributions that are consistent with the original equilibrium: The latter is also an equilibrium of any economy where all traders' preferences remain unchanged, except for trader i. His beliefs can be any $\lambda^i \neq \rho^i$ such that $q = \frac{\pi_1^{i\prime}}{\pi_0^i} A$ where $\pi_1^{i\prime}$ now represents trader i's utility gradient under the new beliefs λ^i. When $J < S-1$, these beliefs exist. The intuition of this analysis is essentially the same in infinite horizon economies and ultimately drives our main result.

Turning to the case of infinite horizon economies, suppose that traders trade the same set of short-lived securities whose payoff next period is the matrix A. The no arbitrage equation takes the form:

$$q(s_t) = \frac{\pi_{t+1}^i(s_t)}{\pi^i(s_t)} A \tag{2}$$

where $\pi_{t+1}^i(s_t) \in \mathbb{R}_{++}^S$ is trader i's utility gradient for period $t+1$ when the current state of the world is s_t and where $\pi^i(s_t) \in \mathbb{R}_{++}$ is the marginal utility of consumption in node s_t of the date-event tree. Again, we consider the case of expected utility maximizers. Consider a particular economy and the resulting equilibrium outcome. Suppose that trader $i \in \mathbb{I}$ has beliefs represented by a probability distribution ρ^i. We wish to construct a probability distribution $\lambda^i \neq \rho^i$ such that the original equilibrium is still an equilibrium when trader i adopts beliefs λ^i. We do this by rewriting the no arbitrage equation (2):

$$q_t = \rho^i(t+1|s_t)V(s_t) \tag{3}$$

Where $\rho^i(t+1|s_t) \in \mathbb{R}^S_{++}$ is the conditional probability distribution of period $t+1$ events, conditioning on the current state of the world s_t and where $V(s_t)$ is an $S \times J$ matrix determined in equilibrium. We show that there exists a unique probability distribution ρ^i which satisfies equation (3) in the complete markets case. When $J < S - 1$, one can choose conditional probabilities $\lambda^i(t+1|s_t) \neq \rho^i(t+1|s_t)$ for each node in the date-event tree. Then one can construct a probability distribution λ^i over infinite events by using Kolmogorov's existence theorem. This implies that in the incomplete markets case, one can choose a probability distribution λ^i that is *effectively identical* to ρ^i but such that $\lambda^i \neq \rho^i$.

One can also choose λ^i such that λ^i and ρ^i are not equivalent. This requires that the marginals $\rho^i(t+1|s_t)$ are uniformly bounded away from the edges of the unit simplex. We can then choose λ^i uniformly bounded away from ρ^i. The theorem of Blackwell and Dubins [7] then implies that these distributions cannot be equivalent.

It follows that in an incomplete markets economy, observing a trader survive does not imply that his beliefs are equivalent to the truth. The above procedure can be used to construct beliefs for this trader that are not equivalent to the truth but that guarantee his survival in a way that is identical to the original economy. This is in contrast to the complete markets or Pareto efficient economy. In these economies and controlling for discount factors, traders who survive must have the truth be absolutely continuous with respect to their beliefs.

3.2 The Infinite Horizon Economy

The economy we model is a special case of the economy analyzed by Magill and Quinzii [14]. Our notation combines elements of [14], Araujo and Sandroni [3] and Sandroni [15]. Let $\mathbb{T} = \{0, 1, ..\}$ denote the set of time periods. Every period, the set of possible states is $T = \{1, .., S\}$, $S \in \mathbb{N}$. T^t is the $t-$Cartesian product of T. Let $\mathbb{S} = \{s^0\} \times T^\infty$ be the set of all possible infinite sequences of T where $s^0 \in T$ acts as the root element. Throughout, we use the notation $s_t = (s^0, s^1, .., s^t)$ for an element $s_t \in T^t$. All elements are taken to have $\{s^0\}$ as root so $s_t \in T^t$ necessarily means $s_t = \{s^0\} \times h_{t-1}$ where $h_{t-1} \in T^{t-1}$.

We can represent the information revelation process in this economy through a sequence of finite partitions of the state space $\mathbb{S}$. In particular, define the cylinder with base on $s_t \in T^t$, $t \in \mathbb{T}$:

$$C(s_t) = \{s \in T^\infty | s = (s_t, ..)\}$$

Let $\mathbb{F}_t = \{C(s_t) : s_t \in T^t\}$ be a partition of the set $\mathbb{S}$. Clearly, $\mathbb{F} = (\mathbb{F}_0, .., \mathbb{F}_t, ..)$ denotes a sequence of finite partitions of $\mathbb{S}$ such that $\mathbb{F}_0 = \{\mathbb{S}\}$ and $\mathbb{F}_t$ is finer[3] than $\mathbb{F}_{t-1}$. We assume that all traders have identical information and that the information revelation process is represented by the sequence $\mathbb{F}$.

Let $\mathbb{D} = \cup_{t \in \mathbb{T}, \sigma_t \in \mathbb{F}_t} (t, \sigma_t)$ denote the date-event tree and $\mathbb{D}^+ = \mathbb{D} - \{(0, \sigma_0)\} = \mathbb{D} - \{s^0\}$. We use the short-hand notation $s_t \in \mathbb{D}$, meaning $(t, \sigma_t) \in \mathbb{D}$ where $\sigma_t = C(s_t)$. $\mathbb{D}_T(s_t)$ denotes the subset of successor nodes of s_t at date T, i.e. all elements $s^T \in T^T$ such that $s^T = (s_t, ..)$.

Let $\mathcal{F}_t$ be the set consisting of all finite unions of cylinders with base on T^t. It is easily shown that $\mathcal{F}_t$ is a $\sigma-$field. Note that $\mathcal{F}_t = \sigma(\mathbb{F}_t)$. Define $\mathcal{F}_0$ as the trivial $\sigma-$field. Let $\mathcal{F} = \sigma(\cup_{t \in \mathbb{N}} \mathcal{F}_t)$. It can be shown that $\{\mathcal{F}_t\}_{t \in \mathbb{N}}$ is a filtration. Let ρ^i be trader i's beliefs on $\mathbb{S}$ represented by a probability

[3] $\sigma_t \in \mathbb{F}_t, \sigma_{t-1} \in \mathbb{F}_{t-1}$ implies that either $\sigma_t \subset \sigma_{t-1}$ or $\sigma_t \cap \sigma_{t-1} = \varnothing$.

measure on $(T^{\infty}, \mathcal{F})$. Let $\mathbb{E}^{\rho^i}$ be the expectation operator associated with ρ^i. Let $\mathbb{E}^{\rho^i}(.|\mathcal{F}_t)(s) = \mathbb{E}_t^{\rho^i}(.)(s)$ be the expectation operator associated with $\rho^i_{s_t}$ when $s = (s_t, ..)$ and where:

$$\rho^i_{s_t}(K) = \frac{\rho^i((T^t \times K) \cap C(s_t))}{\rho^i(C(s_t))} \text{ for any } K \in \mathbb{S} \text{ such that } T^t \times K \in \mathcal{F}$$

There are I$=\{1, .., I\}$ infinitely lived traders, $\mathbb{L} = \{1, .., L\}$ goods at each node. So $\mathbb{D} \times \mathbb{L}$ is the set of all goods over all nodes. Let $\mathbb{R}^{\mathbb{D}\times\mathbb{L}}$ denote the vector space of all maps $x : \mathbb{D} \times \mathbb{L} \to \mathbb{R}$. Let:

$$l_{\infty}(\mathbb{D} \times \mathbb{L}) = \left\{ x \in \mathbb{R}^{\mathbb{D}\times\mathbb{L}} : \sup_{(s_t,l)\in\mathbb{D}\times\mathbb{L}} |x_l(s_t)| < \infty \right\}$$

denote the subspace of bounded maps. Let $\|x\|_{\infty} = \sup_{(s_t,l)\in\mathbb{D}\times\mathbb{L}} |x_l(s_t)|$ denote the sup-norm of $l_{\infty}(\mathbb{D} \times \mathbb{L})$. Also, let:

$$l_1(\mathbb{D} \times \mathbb{L}) = \left\{ x \in \mathbb{R}^{\mathbb{D}\times\mathbb{L}} : \sum_{(s_t,l)\in\mathbb{D}\times\mathbb{L}} |x_l(s_t)| < \infty \right\}$$

Agent i has endowment $\omega \in l^+_{\infty}(\mathbb{D} \times \mathbb{L}) = \{x \in l_{\infty}(\mathbb{D} \times \mathbb{L}) : x_l(s_t) \geq 0 \text{ for all } \xi, l\}$.[4] Let $X^i = l^+_{\infty}(\mathbb{D} \times \mathbb{L})$ denote trader i's consumption set. Let $p \in \mathbb{R}^{\mathbb{D}\times\mathbb{L}}$ be the spot price process and set $p(s_t, 1) = 1$ for all $s_t \in \mathbb{D}$ so 1 is the numeraire good.[5]

Further, we consider only short-lived numeraire securities. Let $J(s_t)$ be the set of securities issued at node $s_t \in T^t$. $j(s_t) = \#J(s_t) < \infty$ is the number of securities. $A_j(s_t, s)$ is the payoff of security $j \in J(s_t)$ in the immediate successor node $(s_t, s) \in T^{t+1}$. $A(s_t, s) = \left[A_1(s_t, s), .., A_{j(s_t)}(s_t, s)\right]$ is the $1 \times j(s_t)$ vector of security payoffs in immediate successor node $(s_t, s) \in T^{t+1}$. Finally, let $A_{t+1}(s_t)$ denote the $S \times j(s_t)$ matrix of payoffs in period $t+1$. Also,

[4] Bewley [4] and subsequently Magill and Quinzii [14] impose the condition of Mackey contituity on traders' preferences. The Mackey topology on $l_{\infty}(\mathbb{D} \times \mathbb{L})$ is described in [4].

[5] We can do this because securities in this economy pay only in terms of the numeraire good.

$A = (A(s_t, s) : (s_t, s) \in \mathbb{D}^+, t \in \mathbb{T}) \in \Pi_{s_t \in \mathbb{D}} \mathbb{R}^{S \times j(s_t)}$ is the process of security payoffs. We assume that all securities pay off in terms of the numeraire good. Let $q(s_t) = (q_j(s_t) : j \in J(s_t))$ be the $1 \times j(s_t)$ vector of node s_t security prices. $q = (q(s_t) : s_t \in \mathbb{D}) \in \Pi_{s_t \in \mathbb{D}} \mathbb{R}^{J(s_t)} = Q$ be the security price process, an element of the security price space. $z^i = (z^i(s_t) : s_t \in \mathbb{D}) \in \Pi_{s_t \in \mathbb{D}} \mathbb{R}^{J(s_t)} = Z$ be the portfolio process for trader i, an element of the portfolio space, where $z^i(s_t) = \left(z^i_j(s_t) : j \in J(s_t)\right)$ is the $j(s_t) \times 1$ portfolio vector of trader i at node s_t.

Let $\succeq_i$ represent trader i's preference ordering over X^i. Preferences $\succeq_i$ are represented by an additively separable utility function:

$$\begin{aligned} u^i(x^i) &= \sum_{t \in \mathbb{T}} \sum_{s_t \in T^t} \rho^i(C(s_t)) \delta_i^{t(s)} v^i(x^i(s_t)) \\ u^i(x^i) &= \mathbb{E}^{\rho^i} \left[\sum_{t \in \mathbb{T}} \delta_i^t v^i(x_t^i) \right] \end{aligned}$$

Where $\rho^i(C(s_t))$ is the probability of $s_t \in T^t$, $\delta_i \in (0,1)$ is an intertemporal discount factor and $v^i : \mathbb{R}^L_+ \to \mathbb{R}$ is a continuous, increasing and concave function with $v^i(0) = 0$. These assumptions on the utility function satisfy Mackey continuity (as shown in [4]).[6]

Let $\succeq = (\succeq_1, .., \succeq_I)$, $\omega = \left(\omega^1, .., \omega^I\right)$. Finally, let $\mathcal{E}_\infty(\mathbb{D}, \succeq, \omega, A)$ denote the economy.

Assumption A Endowments are uniformly bounded away from zero and aggregate endowments are uniformly bounded. Formally, there is an $m > 0$ such that $\omega^i_l(s_t) > m$ for all i, s_t, l; moreover there is an $m' > m > 0$ such that $\sum_i \omega^i_l(s_t) < m'$ for all s_t, l.

Assumption B There exists a riskless bond at every node $s_t \in D$. Formally,

[6] [2] shows that Mackey continuity is needed to prove existence of an equilibrum in economies with infinitely many commodities.

there is a $j \in J(s_t)$ so that $A_j(s_t, s) = 1$ for all $s \in T$.

Assumption B can be replaced with the condition that for each node $s_t \in D$, there exists a portfolio of securities $z \in R^{J(s_t)}$ such that $\sum_j A_j(s_t, s) z_j > 0$ for all $s \in T$.

In this economy, assumptions A and B satisfy all conditions needed (see section 3 of [14]) for the existence of an equilibrium in open-ended incomplete markets economies. They are assumed to hold throughout this paper.

3.3 Equilibrium with a Transversality Condition

With the assumption that $z^i(s_{-1}) = 0$, and that preferences are strictly monotone, the trader's budget constraint at node $s_t \in \mathbb{D}$ is:

$$p(s_t)\left(x^i(s_t) - \omega^i(s_t)\right) = A(s_t)z^i(s_{t-1}) - q(s_t)z^i(s_t) \text{ for all } s_t \in \mathbb{D} \qquad (4)$$

In infinite horizon economies, a trader can borrow and roll over his debt *ad infinitum*. So we need a transversality condition to ensure that there is a bound on the rate at which the trader accumulates debt.

$$\lim_{T\to\infty} \sum_{s_T \in \mathbb{D}_T(s_t)} \pi^i(s_T)q(s_T)z^i(s_T) = 0 \text{ for all } s_t \in \mathbb{D} \qquad (5)$$

So the budget set for trader i is:

$$\mathcal{B}^{TC}_\infty(p, q, \pi^i, \omega^i, A) = \left\{x^i \in l^+_\infty(\mathbb{D} \times \mathbb{L}) : \exists z^i \in Z \text{ satisfying (4) and (5)}\right\}$$

Definition 1 *An equilibrium with a transversality condition of the economy* $\mathcal{E}_\infty(\mathbb{D}, \succeq, \omega, A)$ *is a pair* $(x, z), (p, q, (\pi^i)_{i\in\mathbb{I}}) \in l^+_\infty(\mathbb{D} \times \mathbb{L} \times I) \times Z^I \times \mathbb{R}^{\mathbb{D}\times\mathbb{L}} \times Q \times l^+_1(\mathbb{D} \times \mathbb{I})$ *such that:*

1. $\left(x^i, z^i\right)$ *is* $\succeq_i$*maximal in* $\mathcal{B}^{TC}_\infty(p, q, \pi^i, \omega^i, A)$

2. for each $i \in \mathbb{I}$:

(a) $\pi^i(s_t) > 0$,for all $s_t \in \mathbb{D}$ and $P^i \in l_1^+(\mathbb{D} \times \mathbb{L})$ where $P^i = (P^i(s_t), s_t \in \mathbb{D}) = \left(\pi^i(s_t)p(s_t), s_t \in \mathbb{D}\right)$

(b) x^i is $\succeq_i$maximal in $B_\infty(P^i, \omega^i) = \left\{x^i \in l_\infty^+(\mathbb{D} \times \mathbb{L}) : P^i(x^i - \omega^i) \leq 0\right\}$

(c) $\pi^i(s_t)q_j(s_t) = \sum_{s_{t+1}=(s_t,s)} \pi^i(s_{t+1})A_j(s_{t+1})$ for all $j \in j(s_t), s_t \in \mathbb{D}$

3. $\sum_{i\in\mathbb{I}} \left(x^i - \omega^i\right) = 0$

4. $\sum_{i\in\mathbb{I}} z^i = 0$

Theorem 2 *Each economy $\mathcal{E}_\infty(\mathbb{D}, \succeq, \omega, A)$ satisfying the above assumptions has an equilibrium with a transversality condition.*

Proof. Theorem 5.1 of [14]. ■

The assumption that assets must be short-lived ensures that an equilibrium exists. Is it however only a simplifying assumption as the results in this paper rest on analyzing the no arbitrage equation which must hold in equilibrium regardless of the particular asset structure.

4 Equilibrium Allocation and Beliefs

4.1 Effectively Identical Beliefs and Market Completeness

Note that $t(s) = t_0$ when $s = (s_{t_0}, ..)$. The Lagrangian associated with trader i's maximization problem is:

$$\begin{aligned} L^i \quad = \quad & \sum_{t\in\mathbb{T}} \sum_{s_t\in T^t} \rho^i(C(s_t))\delta_i^{t(s)} v^i(x^i(s_t))) \\ & - \sum_{t\in\mathbb{T}} \sum_{s_t\in T^t} \pi^i(s_t) \left[p(s_t)\left(x^i(s_t) - \omega^i(s_t)\right) - \left(A(s_t)z^i(s_{t-1}) - q(s_t)z^i(s_t)\right)\right] \end{aligned}$$

Necessary FOCs are:

$$\begin{aligned}\frac{\partial L^i}{\partial x_l^i(s_t)} &= 0 : \rho^i(C(s_t))\delta_i^t v_l^i(x^i(s_t)) = \pi^i(s_t)p_l(s_t) \text{ for all } s_t \in T^t, l \in \mathbb{L}, t \in \mathbb{T} \\ \frac{\partial L^i}{\partial z_j^i(s_t)} &= 0 : \pi^i(s_t)q_j(s_t) = \sum_{s_{t+1}\in\{(s_t,s):s\in T\}} \pi^i(s_{t+1})A_j(s_{t+1}) \text{ for all } s_t \in T^t, j \in J, t \in \mathbb{T}\end{aligned}$$

Recall that $p_1(s_t) = 1$ so $\pi^i(s_t) = \rho^i(C(s_t))\delta_i^t v_1^i(x^i(s_t))$. Note that the second set of FOCs is equivalent to the condition that there is no arbitrage (see page 861 of [14]). Also, note that the transversality condition must be satisfied by these multipliers:

$$\lim_{T\to\infty} \sum_{s_T\in\mathbb{D}_T(s_t)} \pi^i(s_T)q(s_T)z^i(s_T) = 0 \text{ for all } s_t \in \mathbb{D}$$

Or, in a more usual form:

$$\lim_{T\to\infty} \mathbb{E}_T^{\rho^i}\left(\delta_i^T v_1^i(x_T^i(.))q_T(.)z_T^i(.)\right)(s) = 0 \text{ for all } s \in \mathbb{S}$$

Where $x_t(s) = x(s_t)$ with $s = (s_t, ..)$.

The set of beliefs that a trader adopts that yield the same equilibrium outcome is the set of *effectively identical beliefs* for this trader, defined below.

Definition 3 *Suppose that* $(x,z),(p,q,(\pi^i)_{i\in\mathbb{I}})$ *is an equilibrium with a transversality condition of an economy* $\mathcal{E}_\infty(\mathbb{D},\succeq,\omega,A)$ *where trader i's preferences are represented by the expected utility* $\mathbb{E}^{\rho^i}\left[\sum_{t\in\mathbb{T}}\delta_i^t v^i(x_t^i)\right]$. *We say that trader i's beliefs* ρ^i *are effectively identical to* λ^i *(a probability measure on* $(T^\infty,\mathcal{F})$*) if there exists an equilibrium with a transversality condition* $(x,z),(p,q,(\psi^i)_{i\in\mathbb{I}})$ *of the economy* $\mathcal{E}_\infty(\mathbb{D},\succeq',\omega,A)$ *where trader i's preferences are represented by the expected utility* $\mathbb{E}^{\lambda^i}\left[\sum_{t\in\mathbb{T}}\delta_i^t v^i(x_t^i)\right]$ *and preferences for trader* $j \neq i$ *remain* $\succeq_j$.

Evidently, as beliefs change, so does the way traders value the future. Hence, the definition imposes that equilibrium allocations and prices are identical for different (but effectively identical) beliefs. The resulting state price processes are different precisely because the probability distributions ρ^i and λ^i are different.

Equilibrium security prices can reveal some information about a trader's beliefs. The price of a security in node s_t represents trader i's marginal utility of consuming the stream of this security's payoff across successor nodes. Along with a trader's actual consumption over these nodes, one can extract some information about this trader's beliefs over these successor nodes. In a complete markets economy, security prices reveal these beliefs perfectly. Equilibrium security prices and consumption for a given node s_t can be summarized in the no-arbitrage equation:

$$q_t = \rho^i(t+1|s_t)V(s_t) \tag{6}$$

where $V(s_t)$ is a matrix determined by the equilibrium consumption of trader i in successor nodes of s_t. This trader's conditional beliefs can then be extracted from this equation. These conditional beliefs, over all nodes, can be then put together to construct this trader's beliefs over the whole $\sigma-$field. This result is summarized in the proposition below.

Definition 4 *An economy $\mathcal{E}_\infty(\mathbb{D}, \succeq, \omega, A)$ has complete markets if $j(s_t) = b(s_t)$ for all $s_t \in \mathbb{D}$ and the $S \times j(s_t)$ matrix $A_{t+1}(s_t)$ has full rank for all $s_t \in \mathbb{D}$.*

Proposition 5 *Suppose that $(x, z), (p, q, (\pi^i)_{i\in\mathbb{I}})$ is an equilibrium with a transversality condition of a complete markets economy $\mathcal{E}_\infty(\mathbb{D}, \succeq, \omega, A)$ then the set of effectively identical beliefs for each trader is a singleton.*

In contrast, equation (6) doesn't determine trader i's conditional beliefs uniquely when markets are incomplete, because there are fewer security prices.

This is shown in the next proposition.

Assumption 1 (Markets are Incomplete at Some Node) There exists a finite path $\tilde{s}_{\tilde{t}} \in T^{\tilde{t}}$ such that $\mathrm{Rank}\big[A_{\tilde{t}+1}(\tilde{s}_{\tilde{t}})\big] < S - 1$.

This assumption is stronger than the usual one for market incompleteness. The additional degree of freedom is used in the proof of the next proposition to ensure that candidate solutions to equation (6) are probability distributions.

Proposition 6 *Under assumption 1, suppose that* $(x, z), (p, q, (\pi^i)_{i \in \mathbb{I}})$ *is an equilibrium with a transversality condition of an incomplete markets economy* $\mathcal{E}_\infty(\mathbb{D}, \succeq, \omega, A)$ *then the set of effectively identical beliefs for each trader is not a singleton.*

The above proposition has some straightforward implications in terms of belief selection in incomplete markets. Let ρ be the true probability distribution on $(T^\infty, \mathcal{F})$. We say that trader i has rational expectations (or correct beliefs) if $\rho^i = \rho$. Blume and Easley [9] use the following definition of survival.

Definition 7 *Trader* i *vanishes on a path* $s \in T^\infty$ *iff* $\lim_t x^i(s_t) = 0$*. She survives on path* $s \in T^\infty$ *iff* $\limsup_t x^i(s_t) > 0$.

Corollary 8 *Suppose that* $(x, z), (p, q, (\pi^i)_{i \in \mathbb{I}})$ *is any equilibrium with a transversality condition of an incomplete markets economy* $\mathcal{E}_\infty(\mathbb{D}, \succeq, \omega, A)$ *then each trader with rational expectations has effectively identical beliefs which are not correct.*

Corollary 9 *Suppose that* $(x, z), (p, q, (\pi^i)_{i \in \mathbb{I}})$ *is any equilibrium with a transversality condition of an incomplete markets economy* $\mathcal{E}_\infty(\mathbb{D}, \succeq, \omega, A)$*, all traders that survive* $\rho-$*almost surely have effectively identical beliefs which are not correct.*

Suppose we can observe all aspects of the economy except traders' beliefs. Then, given an equilibrium of that economy, we could not conclude that a trader who survives has correct beliefs. Of course, this definition of belief correctness is very strong. A trader whose conditional beliefs are identical to the truth in all nodes except one node, has incorrect beliefs. In the Pareto optimal economy discussed in Blume and Easley [9], this trader may survive (if we control for other factors).

4.2 Homogeneity of Beliefs and Market Completeness

Blume and Easley [9] show that a necessary condition for survival is that the truth is absolutely continuous with the beliefs of traders who survive. This formalizes the market selection hypothesis, that traders with incorrect beliefs are driven out of the market. Here, belief correctness refers to the concept of equivalence of a trader's beliefs with the truth. In this section, we show that survival in incomplete markets is consistent with beliefs not equivalent to the truth.

We require that all traders' conditional probabilities should be uniformly bounded away from zero by some $\varepsilon_0 > 0$: There must exist an $\varepsilon_0 > 0$ such that the ε_0-ball $B_{\varepsilon_0}(\rho(.|s_t)) \subset \mathbb{R}^S_{++}$ for all $s_t \in \mathbb{D}^+$.[7] This ensures that effectively identical beliefs can be chosen sufficiently far away from original beliefs. This allows "sufficient room for disagreement" from a trader's original beliefs.

The first step is to construct effectively identical beliefs that are not equivalent to a trader's original beliefs. We do this by constructing conditional beliefs uniformly bounded away from original beliefs, we then use Blackwell and Dubin's theorem to show that these new beliefs cannot be equivalent to original beliefs.

[7] We use the sup norm ($\|x\|_S = \sup_{i\in S}|x_i|$).

Definition 10 *Agent $i \in \mathbb{I}$ and $j \in \mathbb{I}$'s beliefs become homogeneous on a path $s \in S^\infty$ if:*

$$\sup_{B \in \mathcal{F}} |P^i_{s_t}(B) - P^j_{s_t}(B)| \to 0 \text{ as } t \to \infty$$

Definition 11 *Agent $i \in \mathbb{I}$ and $j \in \mathbb{I}$'s beliefs eventually become homogeneous if there is a set $A \in \mathcal{F}$ such that : $P^k(A) = 1$ for $k = i, j$ and for all $s \in A$, traders' beliefs are eventually homogeneous on s.*

Definition 12 *Agent $i \in \mathbb{I}$ and $j \in \mathbb{I}$'s beliefs are equivalent if:*

$$\rho^i(B) = 0 \Leftrightarrow \rho^j(B) = 0 \text{ for all } B \in \mathcal{F}$$

Proposition 13 *If two probability measures are equivalent then the posterior probabilities eventually become homogeneous.*

Proof. Blackwell and Dubins (1962). ■

Evidently, we must strengthen our notion of market incompleteness to ensure that we can choose effectively identical conditional beliefs sufficiently far away from original beliefs, infinitely often.

Assumption 2 (Markets are Sufficiently Incomplete Infinitely Often) For each $i \in \mathbb{I}$, there exists a set $A_i \in \mathcal{F}$ of positive measure ρ^i such that $\text{Rank}[A_{t+1}(s_t)] < S - 1$ i.o. on each path $s \in A_i$.

A sufficient condition for assumption 2 is that markets are incomplete at every node in the tree with $\text{Rank}(A_j(s_t, t+1)) < S - 1$.

Proposition 14 *Under assumption 2, suppose that $(x, z), (p, q, (\pi^i)_{i \in \mathbb{I}})$ is an equilibrium with a transversality condition of an economy $\mathcal{E}_\infty(\mathbb{D}, \succeq, \omega, A)$ where trader i's preferences are represented by the expected utility $\mathbb{E}^{\rho^i}\left[\sum_{t \in \mathbb{T}} \delta_i^t v^i(x_t^i)\right]$*

then the set of effectively identical beliefs for trader $i \in \mathbb{I}$ contains beliefs not equivalent to ρ^i.

The main result of this paper is an implication of the following corollary.

Corollary 15 *Under assumption 2, suppose that $(x, z), (p, q, (\pi^i)_{i \in \mathbb{I}})$ is an equilibrium with a transversality condition of an economy $\mathcal{E}_\infty(\mathbb{D}, \succeq, \omega, A)$ where trader i's preferences are represented by the expected utility $\mathbb{E}^{\rho^i}\left[\sum_{t \in \mathbb{T}} \delta_i^t v^i(x_t^i)\right]$ then the set of effectively identical beliefs for trader $i \in \mathbb{I}$ contains beliefs not equivalent to the true probability distribution ρ.*

Proof. If trader i's beliefs are not equivalent to ρ, then we're done. If they are, use the previous proposition. ■

If we can observe all aspects of the economy except for traders' beliefs, then given an equilibrium, a trader who survives $\rho-$a.s. has beliefs consistent with this survival that are not equivalent to ρ. This is in contrast to the Pareto optimal result of Blume and Easley [9].[8] Note that our result doesn't rely on assumptions about discount factors, or even the precise definition of survival. This is because it is the no-arbitrage equation along with the asset structure that determines a trader's set of effectively identical beliefs, in particular a surviving trader's beliefs.

We also obtain the result that two traders who survive may strongly disagree about the truth. This is a direct implication of the following

Corollary 16 *Under assumption 2, suppose that $(x, z), (p, q, (\pi^i)_{i \in \mathbb{I}})$ is an equilibrium with a transversality condition of an economy $\mathcal{E}_\infty(\mathbb{D}, \succeq, \omega, A)$ where trader i's preferences are represented by the expected utility $\mathbb{E}^{\rho^i}\left[\sum_{t \in \mathbb{T}} \delta_i^t v^i(x_t^i)\right]$ then each trader has effectively identical beliefs that are not equivalent to another trader's beliefs.*

[8] See theorem (17) in section (5).

5 Optimality vs. Survival

It is well known that generically outcomes in incomplete markets are not Pareto efficient. Suppose however that the endowment process ω is chosen such that the equilibrium outcome x is Pareto efficient. Also suppose that traders have identical discount factors. This way, traders' impatience does not affect their ability to survive. If we assume that the true probability distribution ρ is absolutely continuous with some trader's beliefs then theorem 3 of [9] applies:

Theorem 17 *If trader i survives $\rho - a.s$ then ρ is absolutely continuous with respect to ρ^i.*

This implies that in the presence of a trader whose beliefs don't strongly disagree with the truth, another trader can only survive if his own beliefs don't strongly disagree with the truth. But corollary (15) implies that trader i has an effectively identical belief ρ^i which is not equivalent to the truth. In an equilibrium where trader i's beliefs are ρ^i, trader i will still consume x^i but the overall outcome is not Pareto efficient. This is because his state-price process π^i differs from those of other traders. All the same, he still survives $\rho - a.s.$

We can restate this result as follows:

Proposition 18 *Under assumption 2, suppose that $(x, z), (p, q, (\pi^i)_{i \in \mathbb{I}})$ is an equilibrium with a transversality condition of an economy $\mathcal{E}_\infty(\mathbb{D}, \succeq, \omega, A)$ where trader i's preferences are represented by the expected utility $\mathbb{E}^{\rho^i}\left[\sum_{t \in \mathbb{T}} \delta^t v^i(x^i_t)\right]$ (identical discount factors) and where some trader j has equivalent beliefs to the truth ρ. Also suppose that trader i survives $\rho - a.s.$ Let λ^i be effectively identical to ρ^i, chosen such that λ^i is not equivalent to ρ.*

Then the resulting equilibrium in which trader i's beliefs are λ^i is not Pareto efficient.

Note that this equilibrium always exists. Under the assumptions of this proposition, any equilibrium where a trader who strongly disagrees with the truth survives according to the truth must be Pareto inefficient. While the result is hardly surprising (outcomes in incomplete markets are generically inefficient, regardless of traders' beliefs), this section shows that there are no contradictions between our results and those of [9].

6 Comments

Blume and Easley [9] and Sandroni [15] motivate their work partly on the premise that if traders with incorrect beliefs are driven out of the market, then prices will reflect correct beliefs only. As Blume and Easley [9] point out, those traders with incorrect beliefs are only allowed to disagree on a common probability space. If one takes the point of view that traders have subjective state spaces then the complete markets hypothesis becomes tenuous. Given a complete markets economy defined on a given state space, traders may disagree by adopting different probability distributions defined on its $\sigma-$field. But these disagreements may not be too strong: belief differences on the price process or on an extraneous process are not allowed. Of course, these kinds of disagreements can be accommodated by expanding the state space and allowing traders to disagree on this larger space. But then the original market structure is incomplete and the complete markets analysis does not apply. In light of our analysis, the incomplete markets assumption implies that prices reflect beliefs that may be very different from the truth.

A different motivation for this paper lies in the interpretation of rationality within the Bayesian framework. Bayesianism is agnostic about the choice of prior beliefs and this is unsatisfactory as "rationality" in common parlance refers to reasonable beliefs about the likelihood of events.[9] In the context of financial markets, we may reasonably expect those who make consistently wrong judgments in their portfolio choice to have their wealth be driven to naught. Lacking a theory of (prior) belief formation, natural selection in financial markets may provide a mechanism for choosing rational beliefs, as only those who survive have correct beliefs. But strong disagreements among traders may render markets incomplete and may make natural selection inefficient in separating correct from incorrect beliefs.

7 Conclusion

In this paper we model an infinite horizon economy, with a view to testing the market selection hypothesis under market incompleteness. We know from the literature (Sandroni [15], Blume and Easley [9]) that markets with a Pareto optimal outcome or, more narrowly, complete markets select for correct beliefs. All surviving traders have correct beliefs (i.e. beliefs that can be represented by probability distributions that merge with the truth). Both wealth and consumption of traders whose beliefs are incorrect converge to zero with probability one. Hence in the long run heterogeneity of beliefs is not persistent and market outcomes reflect the true probability distribution over returns.

The motivation for our study lies in two counterexamples provided by Blume and Easley [9] that point to the fact that the same need not hold under market incompleteness. In this paper we show that incomplete markets do not select for correct beliefs. In particular we prove that when markets are incomplete the set of beliefs that is consistent with a trader's survival admits beliefs which are not equivalent to the truth.

We build our main result on the characterisation of the set of *effectively*

[9] See Gilboa et al. [13] for a discussion.

identical beliefs. Given an economy and its corresponding equilibrium, this is the set of beliefs for a trader that are consistent with the same equilibrium allocation and prices. If a trader had to adopt different beliefs belonging to this set, the equilibrium outcome would remain unchanged. We show that, while in complete market economies the set of effectively identical beliefs admits only one element, under market incompleteness the set is not a singleton. Moreover, it always admits probability distributions that are not equivalent to the true probability distribution. This result holds for all traders and in particular for surviving traders. Hence one can always find beliefs that differ significantly from the true probability distribution and that still allow a trader to survive and have an impact on market outcomes in the long run.

An immediate corollary of our result is that heterogeneity of beliefs is persistent: surviving traders need not share the same beliefs in the long run. Under incomplete markets asset prices reflect a range of underlying probability distributions that generate them. These distributions offer conflicting evidence on the probability of some events.

8 Appendix

8.1 Preliminary

The following proposition is used in the proof of proposition (6).

Proposition 19 *Suppose that* $(x, z), (p, q, (\pi^i)_{i\in\mathbb{I}})$ *is an equilibrium with a transversality condition of an economy* $\mathcal{E}_\infty(\mathbb{D}, \succeq, \omega, A)$ *where trader i's preferences are represented by the expected utility* $\mathbb{E}^{\rho^i}\left[\sum_{t\in\mathbb{T}} \delta_i^t v^i(x_t^i)\right]$*. Let* $\left(\lambda^i\right)_{i\in\mathbb{I}}$ *be probability distributions on* $(\mathbb{S}, \mathcal{F})$ *such that:*

$$q_j(s_t) = \sum_{s_{t+1}\in\{(s_t,s):s\in T\}} \frac{\lambda^i(C(s_{t+1}))\delta_i v_1^i(x^i(s_{t+1}))}{\lambda^i(C(s_t))v_1^i(x^i(s_t))} A_j(s_{t+1}) \text{ for all } s_t \in T^t, j \in J, t \in \mathbb{T}$$

Then $\left(\lambda^i\right)_{i\in\mathbb{I}}$ *are effectively identical to* $\left(\rho^i\right)_{i\in\mathbb{I}}$*.*

Proof. Set:

$$\psi^i(s_t) = \delta_i^t v_1^i(x^i(s_t))\lambda^i(C(s_t)) \text{ for all } s_t \in T^t, t \in \mathbb{T}$$

So the no-arbitrage condition is satisfied:

$$\psi^i(s_t)q_j(s_t) = \sum_{s_{t+1}\in\{(s_t,s):s\in T\}} \psi^i(s_{t+1})A_j(s_{t+1}) \text{ for all } s_t \in T^t, j \in J, t \in \mathbb{T}$$

Note that the other FOCs of trader i's optimization problem are satisfied. Indeed, we know that:

$$\rho^i(C(s_t))\delta_i^t v_l^i(x^i(s_t)) = \pi^i(s_t)p_l(s_t) \text{ for all } s_t \in T^t, l \in \mathbb{L}, t \in \mathbb{T}$$

So that:

$$\frac{\rho^i(C(s_t))\delta_i^t}{\pi^i(s_t)} v_l^i(x^i(s_t)) = p_l(s_t) \text{ for all } s_t \in T^t, l \in \mathbb{L}, t \in \mathbb{T}$$

So that (with $p_1(s_t) = 1$):

$$\frac{p_l(s_t)}{p_1(s_t)} = p_l(s_t) = \frac{v_l^i(x^i(s_t))}{v_1^i(x^i(s_t))} \text{ for all } s_t \in T^t, l \in \mathbb{L}, t \in \mathbb{T}$$

So, given that:

$$\delta_i^t v_1^i(x^i(s_t))\lambda^i(C(s_t)) = \psi^i(s_t) \text{ for all } s_t \in T^t, t \in \mathbb{T} \tag{7}$$

It follows that:

$$\delta_i^t \frac{v_l^i(x^i(s_t))}{p_l(s_t)}\lambda^i(C(s_t)) = \psi^i(s_t) \text{ for all } s_t \in T^t, l \in \mathbb{L}, t \in \mathbb{T}$$

Or:

$$\lambda^i(C(s_t))\delta_i^t v_l^i(x^i(s_t)) = \psi^i(s_t)p_l(s_t) \text{ for all } s_t \in T^t, l \in \mathbb{L}, t \in \mathbb{T}$$

So all FOCs are satisfied. Since $(x, z), (p, q, (\pi^i)_{i\in\mathbb{I}})$ is an equilibrium with transversality condition for the economy $\mathcal{E}_\infty(\mathbb{D}, \succeq, \omega, A)$, it follows from theorem 5.2 of [14] that $((x, z), (p, q))$ is an equilibrium with implicit debt constraint for the economy $\mathcal{E}_\infty(\mathbb{D}, \succeq, \omega, A)$. So $\left(qz^i\right) \in l_\infty(\mathbb{D})$ for all $i \in \mathbb{I}$. So $((x, z), (p, q))$ is an equilibrium with implicit debt constraint for the economy $\mathcal{E}_\infty(\mathbb{D}, \succeq', \omega, A)$. Since preferences in the economy $\mathcal{E}_\infty(\mathbb{D}, \succeq', \omega, A)$ satisfy assumptions $A1 - A6$ in [14], theorem 5.2 of [14] implies the existence of present value vectors ν^i, $i \in \mathbb{I}$ so that $(x, z), (p, q, (\nu^i)_{i\in\mathbb{I}})$ is an equilibrium with transversality condition for the economy $\mathcal{E}_\infty(\mathbb{D}, \succeq', \omega, A)$. Incidentally, it follows that $\nu^i = \psi^i$ for all $i \in \mathbb{I}$, since $(\nu^i)_{i\in\mathbb{I}}$ satisfies equation (7). ■

8.2 Proof of Proposition (5)

Proof. Suppose not. Then there exists an equilibrium with a transversality condition $(x,z),(p,q,(\psi^i)_{i\in\mathbb{I}})$ where trader i's preferences are represented by the expected utility $\mathbb{E}^{\lambda^i}\left[\sum_{t\in\mathbb{T}}\delta_i^t v^i(x_t^i)\right]$. Note that $(\psi^i)_{i\in\mathbb{I}}$ must satisfy:

$$q_j(s_t) = \sum_{s_{t+1}\in\{(s_t,s):s\in T\}} \frac{\psi^i(s_{t+1})}{\psi^i(s_t)} A_j(s_{t+1}) \text{ for all } s_t\in T^t, j\in J, t\in\mathbb{T}$$

Set $\psi_{t+1}^i(s_t) = \left(\psi^i(s_t,1),..,\psi^i(s_t,S)\right)$. So, the above equation in matrix form is:

$$q(s_t) = \frac{\psi_{t+1}^i(s_t)}{\psi^i(s_t)} A_{t+1}(s_t) \text{ for all } s_t\in T^t, t\in\mathbb{T}$$

Where $A_{t+1}(s_t)$ is an $S\times j(s_t)$ matrix and $q(s_t)$ is a $1\times j(s_t)$ vector. Since markets are complete, A is square and has full rank. So the above equation has a unique solution, which we know is $\frac{\pi_{t+1}^i(s_t)}{\pi^i(s_t)}$. Hence $\frac{\psi_{t+1}^i(s_t)}{\psi^i(s_t)} = \frac{\pi_{t+1}^i(s_t)}{\pi^i(s_t)}$ for all $s_t\in T^t, t\in\mathbb{T}$. Finally, in period 0, $\psi(s_0)=\pi(s_0)$ by construction. So $\psi^i=\pi^i$. So equation (7) implies that $\lambda^i(C(s_t)) = \rho^i(C(s_t))$ for $s=(s_t,..)\in\mathbb{S}$. So λ^i and ρ^i agree on sets in $\cup_{t\in\mathbb{N}}\mathcal{F}_t$. This set is closed under finite intersections and hence is a $\pi-$system. The $\pi-\lambda$ theorem and it's implication (theorem 3.3 in [6]) in turn implies that $\lambda^i=\rho^i$, a contradiction. ■

8.3 Proof of Proposition (6)

Proof. Choose a process $\lambda^i(C(s_t)) \in [0,1]$ for all $s_t \in \mathbb{D}$ so that $\lambda^i(C(s_0)) = 1$ and:

$$\begin{aligned}
\sum_{s_{t+1} \in \{(s_t,s):s\in T\}} \left[\delta_i v_1^i(x^i(s_{t+1}))A_j(s_{t+1})\right] \lambda^i(C(s_{t+1})) &= v_1^i(x^i(s_t))q_j(s_t)\lambda^i(C(s_t)) \\
\sum_{s_{t+1} \in \{(s_t,s):s\in T\}} \lambda^i(C(s_{t+1})) &= \lambda^i(C(s_t))
\end{aligned}$$

Then, by Kolmogorov's Existence Theorem [theorem (20)], λ^i is a probability distribution on $(T^\infty, \mathcal{F})$, proposition (19) applies and $\left(\lambda^i\right)_{i\in\mathbb{I}}$ are effectively identical to $\left(\rho^i\right)_{i\in\mathbb{I}}$. We simplify this system by rewriting it.

$$\begin{aligned}
\sum_{s_{t+1} \in \{(s_t,s):s\in T\}} \left[\delta_i v_1^i(x^i(s_{t+1}))A_j(s_{t+1})\right] \lambda^i(s_{t+1}|s_t) &= v_1^i(x^i(s_t))q_j(s_t) \quad (8)\\
\sum_{s_{t+1} \in \{(s_t,s):s\in T\}} \lambda^i(s_{t+1}|s_t) &= 1
\end{aligned}$$

Given a process $\lambda^i(s_{t+1}|s_t)$, one can reconstruct a probability distribution on $(T^\infty, \mathcal{F})$ by setting, recursively:

$$\begin{aligned}
\lambda^i(C(s_1)) &= \lambda^i(s_1|s_0)\lambda^i(C(s_0)) = \lambda^i(s_1|s_0) \text{ for all } s_1 = (s_0, s) \\
\lambda^i(C(s_{t+1})) &= \lambda^i(s_{t+1}|s_t)\lambda^i(C(s_t)) \text{ for all } s_{t+1} = (s_t, s) \text{ for } t \in \mathbb{T} - \{0\}
\end{aligned}$$

Set $\lambda^i(.|s_t) = \rho^i(.|s_t)$ for all $s_t \neq \tilde{s}_{\tilde{t}}$. $\lambda^i(.|\tilde{s}_{\tilde{t}})$ is chosen such that $\lambda^i(.|\tilde{s}_{\tilde{t}}) \neq \rho^i(.|\tilde{s}_{\tilde{t}})$ and such that system of equations (8) is satisfied (this is possible because markets are incomplete, see below). Then the resulting probability distribution λ^i is different from ρ^i but effectively identical to ρ^i, by proposition (19) in section (8.1).

How to choose an appropriate $\lambda^i(.|\tilde{s}_{\tilde{t}}) \neq \rho^i(.|\tilde{s}_{\tilde{t}})$: Note that the set of equations in (8) can be rewritten as:

$$M(s_t)\lambda^i(.|s_t) = v(s_t)$$

Where:

$$M(s_t) = \begin{bmatrix} \delta_i v_1^i(x^i(s_{t+1}^1))A_1(s_{t+1}^1) & ... & \delta_i v_1^i(x^i(s_{t+1}^S))A_1(s_{t+1}^S) \\ \vdots & \ddots & \vdots \\ \delta_i v_1^i(x^i(s_{t+1}^1))A_J(s_{t+1}^1) & ... & \delta_i v_1^i(x^i(s_{t+1}^S))A_J(s_{t+1}^S) \end{bmatrix}$$

$$v(s_t) = \begin{bmatrix} v_1^i(x^i(s_t))q_1(s_t) \\ \vdots \\ v_1^i(x^i(s_t))q_J(s_t) \end{bmatrix}$$

Note that $(M(\tilde{s}_{\tilde{t}}))$ has full rank equal to the rank of $A_{\tilde{t}+1}(\tilde{s}_{\tilde{t}},) < S-1$. Since we know that $\rho^i(.|\tilde{s}_{\tilde{t}})$ solves the system of equations in (8), we know the solution set $\Lambda(\tilde{s}_{\tilde{t}})$ is linear and of dimension at least 1. We know that $\rho^i(.|\tilde{s}_{\tilde{t}}) \in \mathbb{R}^S_{++}$ and is interior to the unit simplex, by construction. Using the sup norm ($\|x\|_S = \sup_{i\in S}|x_i|$), choose an $\varepsilon > 0$ sufficiently small such that $B_\varepsilon(\rho^i(.|\tilde{s}_{\tilde{t}})) \subset \mathbb{R}^S_{++}$, and choose an element $\bar{\lambda}^i(.|\tilde{s}_{\tilde{t}}) \in B_\varepsilon(\rho^i(.|\tilde{s}_{\tilde{t}})) \cap \Lambda(\tilde{s}_{\tilde{t}})$ such that $\bar{\lambda}^i(.|\tilde{s}_{\tilde{t}}) \neq \rho^i(.|\tilde{s}_{\tilde{t}})$. ■

8.4 Proof of Proposition (14)

Proof. We use the construction in the proof of proposition (6) by choosing $\varepsilon = \varepsilon_0$ at the end of the proof. On each path $s \in A_i$, build a probability distribution λ^i by choosing $\lambda^i(.|s_t) \in \left[B_{\varepsilon_0}(\rho^i(.|s_t)) \cap \Lambda(s_t)\right] - B_{\varepsilon_0/2}(\rho^i(.|s_t))$ for all $s_t, t \in \mathbb{T}$ such that $\text{Rank}(A_j(s_t, t+1)) < S-1$ and such that $s = (s_t, ..)$. If the rank condition is not satisfied on these paths, choose $\lambda^i(.|s_t) = \rho^i(.|s_t)$. For paths $s \notin A_i$, choose $\lambda^i(.|s_t) = \rho^i(.|s_t)$.

For each path $s \in A_i$, we show that:

$$\lim_{t\to+\infty} \sup_{B\in\mathcal{G}} |\lambda^i_{s_t}(B) - \rho^i_{s_t}(B)| \geq \frac{\varepsilon_0}{2} \tag{9}$$

Where $\mathcal{G} = \{C(s_t) : s = (s_t, ..) \text{ for all } t \in \mathbb{T}\}$. Then we show that:

$$\lim_{t\to+\infty} \sup_{B\in\mathcal{G}} |\lambda^i_{s_t}(B) - \rho^i_{s_t}(B)| \leq \lim_{t\to+\infty} \sup_{B\in\mathcal{F}} |\lambda^i_{s_t}(B) - \rho^i_{s_t}(B)| \text{ when } \mathcal{G} \subset \mathcal{F} \tag{10}$$

This in turn implies that $\lim_{t\to+\infty} \sup_{B\in\mathcal{F}} |\lambda^i_{s_t}(B) - \rho^i_{s_t}(B)| > 0$ on a set of paths that trader i assigns positive measure. Blackwell and Dubins' result implies in turn that $\lambda^i$and ρ^i are not equivalent.

We now show inequality (9). On a path $s \in A_i$, let $a_t = \sup_{B\in\mathcal{G}} |\lambda^i_{s_t}(B) - \rho^i_{s_t}(B)|$ and $a = \lim_{t\to+\infty} a_t$. Suppose that $a < \frac{\varepsilon_0}{2}$. Choose $\delta > 0$ such that $B_\delta(a) \cap \{\frac{\varepsilon_0}{2}\} = \varnothing$. There is a $T_\delta \in \mathbb{T}$ such that $t \geq T_\delta \Rightarrow |a_t - a| < \delta$. Since $a_t < \frac{\varepsilon_0}{2}$ for $t \geq T_\delta$, it follows that $|\lambda^i_{s_t}(B) - \rho^i_{s_t}(B)| < \frac{\varepsilon_0}{2}$ for $t \geq T_\delta$. But this contradicts the existence of a $B \in \mathcal{G}$ such that $|\lambda^i_{s_t}(B) - \rho^i_{s_t}(B)| \geq \frac{\varepsilon_0}{2}$ i.o. on path $s \in A_i$. Take $B = C(s_{t+1})$ where $s_{t+1} = (s_t, s)$ and where s is chosen such that $\left|\lambda^i(s|s_t) - \rho^i(s|s_t)\right| \geq \frac{\varepsilon_0}{2}$. This s must exist by construction of $\lambda^i(.|s_t)$.

Inequality (10) is obvious: let $a_t = \sup_{B\in\mathcal{G}} |\lambda^i_{s_t}(B) - \rho^i_{s_t}(B)|$ and $a = \lim_{t\to+\infty} a_t$ and $b_t = \sup_{B\in\mathcal{F}} |\lambda^i_{s_t}(B) - \rho^i_{s_t}(B)|$ and $b = \lim_{t\to+\infty} b_t$. Suppose that $a > b$. Let $\eta = a - b > 0$. Choose $\varepsilon = \frac{\eta}{4}$. There exists a $T_\varepsilon \in \mathbb{T}$ such that $t \geq T_\varepsilon \Rightarrow |a_t - a| < \varepsilon$ and $|b_t - b| < \varepsilon$. So if $t \geq T_\varepsilon$, $a_t > b_t$ so $a_t > \frac{a_t+b_t}{2} \geq \sup_{B\in\mathcal{G}} |\lambda^i_{s_t}(B) - \rho^i_{s_t}(B)|$ so a_t is not the sup, a contradiction. ■

8.5 Kolmogorov's Existence Theorem

Given a process $\rho(C(s_t))$ such that $0 \leq \rho(C(s_t)) \leq 1$ and $\rho(C(s_t)) = \sum_{s_{t+1}=(s_t,s)} \rho(C(s_{t+1}))$ and $\rho(C(s_0)) = 1$, we wish to construct a probability distribution $\tilde{\rho}$ on $(\mathbb{S}, \mathcal{F})$

such that $\rho(C(s_t)) = \rho^{\tilde{}}(C(s_t))$.

For each $k-$tuple of distinct elements of $\mathbb{T}$, define the following probability measure $\rho_{t_1..t_k}(.)$ on T^k such that:

$$\rho_{t_1..t_k}(H_{t_1} \times .. \times H_{t_k}) = \sum_{s_{t_k} \in \cap_{i \in \{1,..,k\}} \left(T^{t_i-1} \times H_{t_i} \times T^{t_k-t_i}\right)} \rho(C(s_{t_k}))$$

Evidently, the system $\rho_{t_1..t_k}$ has the following property:

$$\rho_{t_1..t_k}(H_{t_1} \times .. \times H_{t_k}) = \rho_{t_{\pi 1}..t_{\pi k}}(H_{t_{\pi 1}} \times .. \times H_{t_{\pi k}})$$

Where π is a permutation map of $\{1,..,k\}$. Also:

$$\begin{aligned}
\rho_{t_1..t_{k-1}}\left(H_{t_1} \times .. \times H_{t_{k-1}}\right) &= \sum_{s_{t_{k-1}} \in \cap_{i \in \{1,..,k-1\}} \left(T^{t_i-1} \times H_{t_i} \times T^{t_{k-1}-t_i}\right)} \rho\left(C(s_{t_{k-1}})\right) \\
&= \sum_{s_{t_k} \in \cap_{i \in \{1,..,k-1\}} \left(T^{t_i-1} \times H_{t_i} \times T^{t_k-t_i}\right)} \rho(C(s_{t_k})) \\
&= \rho_{t_1..t_k}\left(H_{t_1} \times .. \times H_{t_{k-1}} \times T\right)
\end{aligned}$$

For each $t \in \mathbb{T}$, define the projection mapping $Z_t : T^\infty \to T$ by $Z_t(s^1,..,s^t,..) = s^t$.

Theorem 20 *If $\rho_{t_1..t_k}$ satisfies the above properties then there exists a probability measure $\rho^{\tilde{}}$ on $(T^\infty, \mathcal{F})$ such that the coordinate variable process $[Z_t : t \in \mathbb{T}]$ on $\left(T^\infty, \mathcal{F}, \rho^{\tilde{}}\right)$ has $\rho_{t_1..t_k}$ as its finite dimensional distribution.*

Proof. Theorem 36.1 p. 486 of [6]. ■

In particular:

$$\begin{aligned} P\left[(Z_1,..,Z_t) \in \left(s^1,..,s^t\right)\right] &= \rho_{1..t}\left(s^1,..,s^t\right) \\ &= \rho(C(s_t)) \\ &= \tilde{\rho}\,(C(s_t)) \end{aligned}$$

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