Intro Logic: Week 5

$\mathcal{L}_2$-Structures

1. An $\mathcal{L}_2$-structure $\mathcal{A}$ specifies a *domain of discourse* $D_{\mathcal{A}}$, which is a non-empty set of objects.

2. An $\mathcal{L}_2$-structure assigns elements of $D_{\mathcal{A}}$ to the constants as their semantic values.

3. Sentence letters (i.e. arity 0 predicate letters) are treated as in $\mathcal{L}_1$: they receive a truth value (either T or F) as their semantic values in an $\mathcal{L}_2$-structure.

4. Unary (arity 1) predicate letters have sets as their semantic values.

   - E.g. suppose P^1 formalises '... is green'. Then, in the relevant $\mathcal{L}_2$-structure, the semantic value of P^1 is the set of all green things.

5. Binary (arity 2) predicate letters have binary relations (i.e. sets of ordered pairs) as their semantic values.

   - E.g. suppose P^2 formalises '... loves ...'. Then, in the relevant $\mathcal{L}_2$-structure, the semantic value of P^1 is the set of all ordered pairs such that the first element in the pair loves the second element (e.g. $\{\langle \text{Romeo}, \text{Juliet}\rangle, \ldots\}$).

6. Generalising, the semantic values of arity n predicate letters are n-place relations.

In sum: An $\mathcal{L}_2$-structure $\mathcal{A}$ specifies a non-empty set $D_{\mathcal{A}}$ as its *domain of discourse*; assigns elements of $D_{\mathcal{A}}$ to constants; assigns a truth-value (T or F) to every sentence letter; and assigns an n-place relation to every predicate letter of arity $n > 0$.

Definition 1. **$\mathcal{L}_2$-structure:** *An $\mathcal{L}_2$-structure $\mathcal{A}$ is an ordered pair $\langle D_{\mathcal{A}}, I\rangle$, where $D_{\mathcal{A}}$ is some non-empty set, and I is a function from the set of all constants, sentence letters, and predicate letters such that the value of every constant is an element of $D_{\mathcal{A}}$, the value of every sentence letter is a truth-value T or F, and the value of every arity n predicate letter is an n-ary relation.*

Variable Assignments

So far, we haven't specified the value of variables in a given $\mathcal{L}_2$-structure. We do this by introducing a *variable assignment* over an $\mathcal{L}_2$-structure.

Definition 2. Variable assignment over an $\mathcal{L}_2$-structure: *A variable assignment α over an $\mathcal{L}_2$-structure $\mathcal{A}$ assigns an element of the domain $D_\mathcal{A}$ to each variable.*

- We write $|\cdot|^\alpha_\mathcal{A}$ for the semantic value of the expression $\cdot$ in the $\mathcal{L}_2$-structure $\mathcal{A}$ under the variable assignment α over $\mathcal{A}$.
- **To sum up everything so far:** For any $\mathcal{L}_2$-structure $\mathcal{A}$ and any variable assignment α over $\mathcal{A}$, the semantic values of the respective $\mathcal{L}_2$-expressions are as follows:

  (i) For any constant t, $|t|^\alpha_\mathcal{A}$ is the object in $D_\mathcal{A}$ assigned to t by $\mathcal{A}$.

  (ii) For any variable ν, $|\nu|^\alpha_\mathcal{A}$ is the object in $D_\mathcal{A}$ assigned to ν by α.

  (iii) For any sentence letter Φ, $|\Phi|^\alpha_\mathcal{A}$ is the truth-value (T or F) assigned to Φ by $\mathcal{A}$.

  (iv) For any arity $n > 0$ predicate letter Φ^n, $|\Phi^n|^\alpha_\mathcal{A}$ is the n-ary relation assigned to Φ^n by $\mathcal{A}$.

Semantics for $\mathcal{L}_2$-Sentences

How do we establish the truth values for sentences of $\mathcal{L}_2$?

1. Truth value of sentence letters are just specified in any $\mathcal{L}_2$-structure $\mathcal{A}$, as in $\mathcal{L}_1$.
2. Composite (i.e. not sentence letter) atomic formulae (recall: of the form $Z^n t_1 \dots t_n$):
   - $P^1 a$ is true in the $\mathcal{L}_2$-structure $\mathcal{A}$ iff the object in $D_\mathcal{A}$ assigned to a is in the extension (semantic value) of P^1 (i.e. the set associated to P^1 in $\mathcal{A}$).
   - $P^2 xa$ is true in the (variable assignment α over the) $\mathcal{L}_2$-structure $\mathcal{A}$ iff the pair $\langle |x|^\alpha_\mathcal{A}, |a|^\alpha_\mathcal{A} \rangle$ is in the extension (semantic value) of P^2 (i.e. the binary relation (set of pairs) associated to P^2 in $\mathcal{A}$).

3. Connectives are treated just like in $\mathcal{L}_1$ (see below).

4. Semantic value of quantified formulae (those containing $\forall\nu$ or $\exists\nu$, for some variable ν). Here is the idea:

   - $\exists x P^1 x$ is true iff there exists at least one element of $D_\mathcal{A}$ which is in the extension (semantic value) of P^1—i.e. iff there exists some variable assignment α over $\mathcal{A}$ such that $|x|^\alpha_\mathcal{A} \in |P^1|^\alpha_\mathcal{A}$.
   - $\forall x P^1 x$ is true iff all elements of $D_\mathcal{A}$ are in the extension (semantic value) of P^1—i.e. iff for all variable assignments α over $\mathcal{A}$, $|x|^\alpha_\mathcal{A} \in |P^1|^\alpha_\mathcal{A}$.

Let's make all this precise and maximally general:

Definition 3. Satisfaction: *Assume $\mathcal{A}$ is an $\mathcal{L}_2$-structure, α is a variable assignment over $\mathcal{A}$, ϕ and ψ are formulae of $\mathcal{L}_2$, and ν is a variable. For a formula ϕ either $|\phi|^\alpha_\mathcal{A} = \mathrm{T}$ or $|\phi|^\alpha_\mathcal{A} = \mathrm{F}$. Formulae other than sentence letters receive the following semantic values:*

(i) $|\Phi^n t_1 \dots t_n|^\alpha_\mathcal{A} = \mathrm{T}$ *iff* $\langle |t_1|^\alpha_\mathcal{A}, \dots, |t_n|^\alpha_\mathcal{A}\rangle \in |\Phi^n|^\alpha_\mathcal{A}$, *where Φ^n is an arity $n \geq 1$ predicate letter, and each of $t_1 \dots t_n$ is either a variable or a constant.*

(ii) $|\neg\phi|^\alpha_\mathcal{A} = \mathrm{T}$ *iff* $|\phi|^\alpha_\mathcal{A} = \mathrm{F}$.

(iii) $|\phi \wedge \psi|^\alpha_\mathcal{A} = \mathrm{T}$ *iff* $|\phi|^\alpha_\mathcal{A} = \mathrm{T}$ *and* $|\psi|^\alpha_\mathcal{A} = \mathrm{T}$.

(iv) $|\phi \vee \psi|^\alpha_\mathcal{A} = \mathrm{T}$ *iff* $|\phi|^\alpha_\mathcal{A} = \mathrm{T}$ *or* $|\psi|^\alpha_\mathcal{A} = \mathrm{T}$.

(v) $|\phi \rightarrow \psi|^\alpha_\mathcal{A} = \mathrm{T}$ *iff* $|\phi|^\alpha_\mathcal{A} = \mathrm{F}$ *or* $|\psi|^\alpha_\mathcal{A} = \mathrm{T}$.

(vi) $|\phi \leftrightarrow \psi|^\alpha_\mathcal{A} = \mathrm{T}$ *iff* $|\phi|^\alpha_\mathcal{A} = |\psi|^\alpha_\mathcal{A}$.

(vii) $|\forall\nu\phi|^\alpha_\mathcal{A} = \mathrm{T}$ *iff* $|\phi|^\beta_\mathcal{A} = \mathrm{T}$ *for all variable assignments β over $\mathcal{A}$ differing from α in ν at most.*

(viii) $|\exists\nu\phi|^\alpha_\mathcal{A} = \mathrm{T}$ *iff* $|\phi|^\beta_\mathcal{A} = \mathrm{T}$ *for at least one variable assignment β over $\mathcal{A}$ differing from α in ν at most.*

Exercise: Consider the following $\mathcal{L}_2$-structure (call it $\mathcal{E}$):

$$
\begin{aligned}
D_{\mathcal{A}} &= \{x : x \text{ is a city in Europe in the actual world}\} \\
\left|Q^1\right|_{\mathcal{E}}^{\alpha} &= \{\text{Florence}, \text{Stockholm}, \text{Barcelona}\} \\
\left|R^2\right|_{\mathcal{E}}^{\alpha} &= \{\langle d, e\rangle : d \text{ is smaller than } e\} \\
|a|_{\mathcal{E}}^{\alpha} &= \text{Florence} \\
|b|_{\mathcal{E}}^{\alpha} &= \text{London}
\end{aligned}
$$

Here, α is any variable assignment over $\mathcal{E}$.

Show that:

1. The $\mathcal{L}_2$-sentence R^2ab is true in $\mathcal{E}$.
2. The $\mathcal{L}_2$-sentence $\forall x\,(Q^1x \rightarrow R^2xb)$ is true in $\mathcal{E}$.

Validity, Logical Truth, and Contradiction

As in the case of $\mathcal{L}_1$, we have the following:

1. A sentence ϕ of $\mathcal{L}_2$ is *logically true* iff ϕ is true in all $\mathcal{L}_2$-structures.
2. A sentence ϕ of $\mathcal{L}_2$ is a *contradiction* iff ϕ is not true in any $\mathcal{L}_2$-structure.
3. Sentences ϕ and ψ of $\mathcal{L}_2$ are *logically equivalent* if they are true in exactly the same $\mathcal{L}_2$-structures.
4. A set Γ of $\mathcal{L}_2$-sentences is *semantically consistent* iff there exists an $\mathcal{L}_2$-structure $\mathcal{A}$ in which all sentences in Γ are true.
5. Let Γ be a set of sentences of $\mathcal{L}_2$ and ϕ a sentence of $\mathcal{L}_2$. The argument with all sentences in Γ as premisses and ϕ as conclusion is *valid* iff there is no $\mathcal{L}_2$-structure in which all sentences in Γ are true and ϕ is false. In this case, we write $\Gamma \vDash \phi$; otherwise, write $\Gamma \nvDash \phi$.

Counterexamples

- An $\mathcal{L}_2$-structure $\mathcal{A}$ is a counterexample to the validity of an argument iff all premisses of the argument are true in $\mathcal{A}$ but the conclusion is false in $\mathcal{A}$.
- An $\mathcal{L}_2$-structure $\mathcal{A}$ is a counterexample to the logical truth of an $\mathcal{L}_2$-sentence iff that sentence is false in $\mathcal{A}$.

Exercise: Show that:

1. The $\mathcal{L}_2$-sentence $Q^1b \to \forall x Q^1x$ is not logically true (i.e. $\nvDash Q^1b \to \forall x Q^1x$).
2. $\forall x \exists y Rxy \nvDash \exists y \forall x Rxy$.

Work for Week 5

1. Halbach week 5, whole sheet
2. Peter Fritz week 5, exercise 5.7

Links to both sets of exercises are available at logicmanual.philosophy.ox.ac.uk/

Solutions due at noon on Thursday week 5