Intro Logic: Week 4

Propositional Logic and Formalisation

- **Task:** Formalise the following argument in $\mathcal{L}_1$:

  Zeno is a tortoise. All tortoises are toothless. Therefore, Zeno is toothless.

- This argument is valid in English, but *not* valid in $\mathcal{L}_1$. So this is a problem with $\mathcal{L}_1$!

Subjects and Predicates

- When formalising into $\mathcal{L}_1$, we can't break down sentences such as 'Zeno is a tortoise', or 'All tortoises are toothless'—the best we can do is to formalise with sentence letters.
- We'd like to be able to split such sentences up further, by identifying the *subject* and the *predicate*.
  - Consider the sentence 'Zeno is a tortoise'.
  - The subject is the thing (proper noun) being described—here, Zeno.
  - The predicate is the description applied to the subject—here, '...is a tortoise'.
- Identifying subjects and predicates is the starting point for the development of *predicate logic*, also called $\mathcal{L}_2$.
- To formalise subject-predicate sentences, use $a, b, c, \ldots$ for proper nouns (e.g. 'Zeno'), and *predicate letters*, of the form $P^1, Q^1, R^1, P^1_1, \ldots$, for descriptions.
  - The upper index in a predicate letter denotes the number of 'slots' (into which subjects can fit).
  - Since '...is a tortoise' has one 'slot', we would formalise it using a one-place predicate letter: e.g. P^1. Then, we would formalise 'Zeno is a tortoise' as P^1a.
  - **Question:** How would you formalise 'Tom loves Mary' in this way? 'Tom loves Mary and Ebenezer is a scrooge'? 'Caesar came, he saw, he conquered'?

Quantifiers

- How would one formalise a sentence like ‘Everything is a substance’?
  - ‘Everything’ isn’t a proper noun—so not a subject! Therefore, shouldn’t formalise with $a, b, c, \ldots$.
  - Could reformulate the sentence as: ‘For all x: x is a substance’.
  - We can introduce more notation for ‘for all’: $\forall$.
  - Then, the above sentence reads $\forall x P^1 x$.
- What about e.g. ‘At least one thing is a substance’?
  - Can also introduce notation for ‘there is at least one’: $\exists$.
  - Then, the above sentence reads $\exists x P^1 x$.
- It’s crucial to distinguish *constants*—$a, b, c, \ldots$—from *variables*—$x, y, z, \ldots$. Constants represent proper nouns, e.g. ‘Tom’. Variables are *placeholders* for proper nouns.
- **Task:** How to formalise (i) ‘Every visitor is a classicist’?; (ii) ‘There’s at least one good person’?; (iii) ‘Tom owns at least one bicycle’?

Formulae of Predicate Logic

The above informal discussion in hand, we now introduce more formally the syntax of *predicate logic*, also known as $\mathcal{L}_2$.

Definition 1. Predicate letter: *All expressions of the form P^k_n, Q^k_n or R^k_n are* predicate letters*, where k is either missing (no symbol) or a numeral ‘1’, ‘2’, ‘3’, ... and similarly, n is either missing (no symbol) or a numeral ‘1’, ‘2’, ‘3’, ...*

Definition 2. Arity: *The value of the upper index of a predicate letter is called its* arity. *If a predicate letter does not have an upper index, its arity is 0.*

- Predicate letters of arity 0 can be understood as being equivalent to the sentence letters of $\mathcal{L}_1$.

$\mathcal{L}_2$ contains constants, which will be used to translate natural language proper names and some similar expressions.

Definition 3. Constants: $a, b, c, a_1, b_1, c_1, \ldots$ *are* constants.

Moreover, $\mathcal{L}_2$ contains infinitely many variables:

Definition 4. Variables: $x, y, z, x_1, y_1, z_1, \ldots$ *are* variables.

Now the notion of an atomic $\mathcal{L}_2$-formula can be defined:

Definition 5. Atomic formula of $\mathcal{L}_2$: *If Z is a predicate letter of arity n and each of $t_1, \ldots, t_n$ is a variable or a constant, then $Zt_1 \ldots t_n$ is an* atomic formula of $\mathcal{L}_2$.

- Here, Z serves as a metavariable for predicate letters, e.g. P, R^2_4, Q^1, etc.
- According to this definition, Q^1x, P^2cy, P^3_5xcy, R^2xx are examples of atomic formulae.
  - Note that the number of variables/constants in each atomic formula of $\mathcal{L}_2$ matches the upper index of the predicate letter!
- The above definition allows for the case in which $n = 0$. This means that all sentence letters $P, Q, R, P_1, \ldots$ are atomic formulae.

Definition 6. Quantifiers: *A* quantifier *is an expression $\forall\nu$ or $\exists\nu$, where ν is a variable.*

- **Question:** Which of (i) $\forall x_{34}$, (ii) $\forall a$, (iii) $\exists y$ is a quantifier?

Definition 7. Formulae of $\mathcal{L}_2$:

1. *All atomic formulae of $\mathcal{L}_2$ are formulae of $\mathcal{L}_2$.*
2. *If ϕ and ψ are formulae of $\mathcal{L}_2$, then $\neg\phi$ $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, and $(\phi \leftrightarrow \psi)$ are formulae of $\mathcal{L}_2$.*
3. *If ν is a variable and ϕ is a formula, then $\forall\nu\phi$ and $\exists\nu\psi$ are formula of $\mathcal{L}_2$.*
4. *Nothing else is a formula of $\mathcal{L}_2$.*

Free and Bound Variables

- In the $\mathcal{L}_2$-formula $\forall x(P^1x \to Q^1x)$, the last two occurrences of x refer back to the quantifier $\forall x$.
- In the $\mathcal{L}_2$-formula $P^1x \to Q^1x$, by contrast, there is no quantifier to which these x refer back.
- We say that occurrences of x in the former case are *bound*; occurrences of x in the latter case are *free*.

Definition 8. Free and bound variables:

(i) All occurrences of variables in atomic formulae are free.

(ii) The occurrences of a variable that are free in ϕ and ψ are also free in $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \to \psi)$, and $(\phi \leftrightarrow \psi)$.

(iii) In a formula $\forall\nu\phi$ or $\exists\nu\phi$, no occurrence of the variable ν is free; all occurrences of variables other than ν that are free in ϕ are also free in $\forall\nu\phi$ and $\exists\nu\phi$.

An occurrence of a variable is bound *in a formula iff it is not free.*

Occurrences of variables are free as long as they are not 'caught' by a quantifier.

Sentences of Predicate Logic

Having introduced the above apparatus, we can now distinguish between *formulae* (introduced above) and *sentences* of $\mathcal{L}_2$:

Definition 9. Sentence of $\mathcal{L}_2$: *A formula of $\mathcal{L}_2$ is a* sentence *of $\mathcal{L}_2$ iff no variable occurs freely in the formula.*

Given a formula or a sentence of $\mathcal{L}_2$, we may:

(a) Apply the bracketing conventions (1)-(3) that we're used to from $\mathcal{L}_1$.

(b) Omit the upper index on predicate letters (the arity indices).

- **Task:** Apply this to $\forall x \forall y (P^1 x \leftrightarrow P^2 xy)$.

Exercise: Which of the following are $\mathcal{L}_2$-formulae? Which are $\mathcal{L}_2$-sentences?

1. $\forall x (P^1_1 x \to Q^1 y)$
2. $\exists x \neg(\neg\neg\exists y P^1 y \wedge R^2 xa)$
3. $R^4 ab$
4. P
5. $\forall x \exists y \exists z (R^3_{25} xyz)$
6. $\forall x \exists y \exists a R^3 xya$

Formalisation into Predicate Logic

As with $\mathcal{L}_1$, best to proceed by way of examples. So consider:

1. Everything is material.
2. All frogs are amphibians.
   - Warning! $\forall x (Px \to Qx)$ (correct), *vs.* $\forall x (Px \wedge Qx)$ (incorrect).
3. No frog is poisonous.
4. Every student has a computer.
5. If it's raining, then Bill reads a book or a newspaper.
   - $P \to \exists x (P^2 ax \wedge (Qx \vee Rx))$.
6. There is a country between Spain and France.
   - $\exists x (Px \wedge Qxbc)$.

Work for Week 4

1. Halbach week 4, whole sheet
2. Peter Fritz week 4, exercise 4.3

Links to both sets of exercises are available at logicmanual.philosophy.ox.ac.uk/

Solutions due at noon on Thursday week 4