Intro Logic: Week 8

Qualitative and Numerical Identity

1. **Qualitative identity:**

> There is a fountain pen in my teaching room and another fountain pen in my study at home. They are the same model, the same colour, and both are still in pristine condition. Thus, I have two identical fountain pens.

There are two pens, but we might describe them as *qualitatively identical*, because they are similar in all relevant respects.

2. **Numerical identity:**

> A fountain pen expert sees my pen at home after having seen the pen in my teaching room the day before. He may wonder whether I have taken the pen home and asks: 'Is this the same pen as the pen in your teaching room?'

The fountain pen expect knows all the ways the pen at my home and the pen in my teaching room are similar, so he is not asking whether they are the same colour or brand. Rather, he wants to know if they are the *very same* pen—he wants to ascertain whether they are *numerically identical*.

- *Philosophical aside:* Leibniz's *principle of the identity of indiscernibles* says that if two things that are qualitatively identical in all respects, then they are numerically identical. This is very controversial! (Compare the less controversial *indiscernibility of indenticals*.)
  - Identity of indiscernibles: $\forall x \forall y(\forall X(Xx \leftrightarrow Xy) \rightarrow x = y)$
  - Indiscernibility of identicals: $\forall x \forall y(x = y \rightarrow \forall X(Xx \leftrightarrow Xy))$
- Qualitative identity may be formalised by a binary predicate letter of $\mathcal{L}_2$. By contrast, numerical identity is given a special status. In the following we focus on numerical identity.

The Syntax of $\mathcal{L}_=$

All atomic formulae of $\mathcal{L}_2$ are atomic formulae of $\mathcal{L}_=$. But $\mathcal{L}_=$ also contains a new kind of atomic formula.

Definition 1. (Atomic formulae of $\mathcal{L}_=$) *All atomic formulae of $\mathcal{L}_2$ are atomic formulae of $\mathcal{L}_=$. Furthermore, if s and t are variables or constants, then $s = t$ is an atomic formula of $\mathcal{L}_=$.*

Definition 2. (Formulae of $\mathcal{L}_=$):

(i) All atomic formulae of $\mathcal{L}_=$ are formulae of $\mathcal{L}_=$.

(ii) If ϕ and ψ are formulae of $\mathcal{L}_=$, then $\neg\phi$ $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \to \psi)$, and $(\phi \leftrightarrow \psi)$ are formulae of $\mathcal{L}_=$.

(iii) If ν is a variable and ϕ is a formula, then $\forall\nu\phi$ and $\exists\nu\psi$ are formula of $\mathcal{L}_=$.

(iv) Nothing else is a formula of $\mathcal{L}_=$.

The Semantics of $\mathcal{L}_=$

- For the semantics of $\mathcal{L}_=$, we just add one extra clause to the semantics of $\mathcal{L}_2$:
  - $|s = t|^{\alpha}_{\mathcal{A}} = \mathrm{T}$ iff $|s|^{\alpha}_{\mathcal{A}} = |t|^{\alpha}_{\mathcal{A}}$.
- The definitions of validity of arguments, semantic consistency, logically true sentences, etc., carry over from $\mathcal{L}_2$.
- **Exercise:** Show: $\exists x Px \wedge \exists y Py \not\models \exists x \exists y \neg x = y$.
- In order to obtain a proof system that is sound and complete with respect to the semantics of $\mathcal{L}_=$, the system of Natural Deduction needs to be expanded to include introduction and elimination rules for $=$ (see final page).
- **Exercises:** Show: (i) $\vdash \forall x x = x$; (ii) $\vdash \forall x \forall y (x = y \to y = x)$; (iii) $\vdash \forall x \forall y \forall z (x = y \wedge y = z \to x = z)$.

Formalising Numerical Identity

- *Prima facie*, numerical identity ('is', as in e.g. 'Ratzinger is Benedict XVI') seems to be merely another binary predicate in English. So why should it deserve special treatment?
- Consider the argument:

  The morning star is the evening star. The morning star is a planet. Therefore, the evening star is a planet.

  (i) This argument is valid in English.

  (ii) The formalisation of the argument in $\mathcal{L}_=$ ($a = b, Pa \therefore Pb$) is valid.

  (iii) The formalisation of the argument in $\mathcal{L}_2$ ($Rab, Pa \therefore Pb$) is *not* valid.

  (iv) **Exercise:** Show the claims in (ii) and (iii).

Uses of Identity

- In $\mathcal{L}_2$, we were only able to express (using $\exists$) that there is *at least one* thing of a certain type (e.g. $\exists x Px$ means 'there is at least one thing that is P'). In $\mathcal{L}_=$, we can do much more. For example:

  (a) 'There are at least two things that are P': $\exists x \exists y (Px \wedge Py \wedge \neg x = y)$.

  (b) 'There are at least three things that are P': $\exists x \exists y \exists z (Px \wedge Py \wedge Pz \wedge \neg x = y \wedge \neg x = z \wedge \neg y = z)$.

  (c) 'There is at most one P': $\forall x \forall y (Px \wedge Py \rightarrow x = y)$.

  (d) 'There are at most two Ps': $\forall x \forall y \forall z (Px \wedge Py \wedge Pz \rightarrow x = y \vee y = z \vee x = z)$.

  (e) 'There is exactly one P': $\exists x (Px \wedge \forall y (Py \rightarrow x = y))$.

  (f) 'There are exactly two Ps': $\exists x \exists y (Px \wedge Py \wedge \neg x = y) \wedge \forall x \forall y \forall z (Px \wedge Py \wedge Pz \rightarrow x = y \vee y = z \vee x = z)$.

Definite Descriptions

- Consider the argument in English:

  The car owned by Tim is red. Therefore, there is a red car.

  This argument is valid in English, However, the best formalisation in $\mathcal{L}_2$ that we seem to be able to come up with is $Pa \quad \therefore \quad \exists x(Px \wedge Qx)$, with the dictionary

  $$\begin{aligned} a: &\quad \text{the car owned by Tim} \\ P: &\quad \text{... is red} \\ Q: &\quad \text{... is a car.} \end{aligned}$$

  This argument is not valid—the problem is that by formalising a *definite description* (here: 'the car owned by Tim') as a constant, we lose all information contained in the definite description.

  It would be better to formalise the premise in $\mathcal{L}_=$, as $\exists x((Qx \wedge Rbx) \wedge \forall y(Qy \wedge Rby \to x = y) \wedge Px)$, with the dictionary

  $$\begin{aligned} b: &\quad \text{Tim} \\ R: &\quad \text{... owns ....} \end{aligned}$$

  Exercise: Show that the argument is valid, via soundness and a proof of

  $$\exists x((Qx \wedge Rbx) \wedge \forall y(Qy \wedge Rby \to x = y) \wedge Px) \vdash \exists x(Px \wedge Qx)$$

- Identifying definite descriptions:

  (i) Jane is a classicist $\quad \longrightarrow \quad Pa$

  (ii) Jane is *the* classicist $\quad \longrightarrow \quad \exists x(Px \wedge \forall y(Py \to y = x) \wedge a = x)$.

  In (i), 'is' expresses predication. In (ii), 'is' expresses an identity statement.

- Consider the argument:

  It is not the case that the king of France is bald. Therefore, something is not bald.

  - Formalised in $\mathcal{L}_2$: $\neg Pa \vdash \exists x \neg Px$.
  - Formalised in $\mathcal{L}_=$: $\neg \exists x(Rxa \land \forall y(Rya \to y = x) \land Px) \nvdash \exists x \neg Px$.
  - The English argument is not valid—so the formalisation of the argument in $\mathcal{L}_2$ goes wrong.

Validity of natural language arguments (and logical truth, etc.) in predicate logic with identity is defined analogously to validity of natural language arguments in propositional logic and predicate logic.

Work for Week 8

1. Halbach week 8, whole sheet.
2. Peter Fritz week 8, exercise 8.7.

Links to both sets of exercises are available at logicmanual.philosophy.ox.ac.uk/

Solutions due at noon on Thursday week 8.