Intro Logic: Week 7

Syntactic Consistency

Recall from last week:

- 'If $\Gamma \vdash \phi$ then $\Gamma \vDash \phi$' $\longrightarrow$ Soundness
- 'If $\Gamma \vDash \phi$ then $\Gamma \vdash \phi$' $\longrightarrow$ Completeness

Contraposing, we have the following equivalent forms of soundness and completeness:

- 'If $\Gamma \nvDash \phi$ then $\Gamma \nvdash \phi$' $\longrightarrow$ Soundness
- 'If $\Gamma \nvdash \phi$ then $\Gamma \nvDash \phi$' $\longrightarrow$ Completeness

Definition 1. Syntactic Consistency: *A set Γ of $\mathcal{L}_2$-sentences is syntactically consistent iff there is a sentence ϕ such that $\Gamma \nvdash \phi$.*

Theorem 1. *A set Γ of $\mathcal{L}_2$-sentences is semantically consistent if and only if Γ is syntactically consistent.*

> **Proof:** (Left to right: 'If Γ is semantically consistent then Γ is syntactically consistent.') Assume that Γ is semantically consistent. Then there is an $\mathcal{L}_2$-structure $\mathcal{A}$ in which all sentences in Γ are true. Choose a sentence ϕ which is false in that structure. Then $\Gamma \nvDash \phi$, and by soundness $\Gamma \nvdash \phi$, so Γ is syntactically consistent.
>
> (Right to left: 'If Γ is syntactically consistent then Γ is semantically consistent.') Prove the contrapositive: 'If Γ is not semantically consistent, then Γ is not syntactically consistent'. Assume that Γ is not semantically consistent. Then there is no $\mathcal{L}_2$-structure $\mathcal{A}$ in which all sentences in Γ are true. Consequently, any sentence ϕ will be true in all $\mathcal{L}_2$-structures in which all sentences of Γ are true (because there are no such structures). So $\Gamma \vDash \phi$ for all sentences ϕ of $\mathcal{L}_2$. Then, by completeness, $\Gamma \vdash \phi$ for all sentences ϕ of $\mathcal{L}_2$. So there is no sentence ϕ that is not provable from premisses in Γ, and therefore Γ is not syntactically consistent.

Ambiguity

$\mathcal{L}_2$ can capture various ambiguities in natural language. To take two examples, consider:

1. An electron is positively charged.
2. A mistake was made by every student.

Exercise: Give different formalisations of these sentences in $\mathcal{L}_2$, to bring out the ambiguity.

Extensionality

If constants, sentence letters, and predicate letters are replaced in an $\mathcal{L}_2$-sentence by other constants, sentence letters, and predicate letters (respectively) that have the same extension in a given $\mathcal{L}_2$-structure, then the truth value of the sentence in that $\mathcal{L}_2$-structure does not change. This property of $\mathcal{L}_2$ is called extensionality.

For example, consider the $\mathcal{L}_2$-structure $\mathcal{A}$ such that

$$\begin{aligned} D_{\mathcal{A}} &= \{1\} \\ |P|_{\mathcal{A}} &= \{1\} \\ |Q|_{\mathcal{A}} &= \{1\} \\ |a|_{\mathcal{A}} &= 1 \\ |b|_{\mathcal{A}} &= 1 \end{aligned}$$

In this structure, $|Pa|_{\mathcal{A}} = \text{T}$. But since $|a|_{\mathcal{A}} = |b|_{\mathcal{A}} = 1$, we also have $|Pb|_{\mathcal{A}} = \text{T}$. Similarly, since $|P|_{\mathcal{A}} = |Q|_{\mathcal{A}}$, we also have $|Qa|_{\mathcal{A}} = \text{T}$, etc.

In many cases, natural language like English also appear to be extensional. For example, consider ‘Everest is 8,848 metres high.’, versus ‘Sagarmatha is 8,848 metres high.’ Since

'Everest' and 'Sagarmatha' denote exactly the same thing (namely, Everest[1]), substitution of the latter word for the former should not affect the truth value of the resulting sentence.

That said, there are also failures of extensionality in English, e.g.:

(A) Tom believes that Hesperus rises in the morning.

(B) Tom believes that Phosphorus rises in the morning.

In fact, Hesperus *is* Phosphorus, but Tom doesn't know this. So (A) might be true without (B) being true. This failure of extensionality in English in turn leads to problems when formalising in $\mathcal{L}_2$.

Predicate Logic and Arguments in English

(i) A natural language sentence is *logically true in predicate logic* iff its formalisation in predicate logic is logically true.

(ii) A natural language sentence is a *contradiction in predicate logic* iff its formalisation in predicate logic is a contradiction.

(iii) A set of natural language sentences is *consistent in predicate logic* iff the set of their formalisations in predicate logic is semantically consistent.

(iv) A natural language argument is *valid in predicate logic* iff its formalisation in $\mathcal{L}_2$ is valid.

Now recall our recurring argument:

Zeno is a tortoise. All tortoises are toothless. Therefore, Zeno is toothless.

Exercise: Show that this argument is valid in predicate logic, using (a) a semantic argument; (b) a natural deduction proof and appeal to soundness.

[1]This is a case where our quotation mark conventions from week 2 are useful.

If a partial formalisation of a natural language sentence is logically true, then that natural language sentence is logically true in predicate logic. Similarly, if a partial formalisation of a natural language argument is valid, then that natural language argument is valid in predicate logic.

For example, consider

> Every student has a computer. Wilma doesn't have a computer. Therefore, Wilma isn't a student.

Partially formalised, this yields

$$\forall x(Px \to Qx), \neg Qa \vdash \neg Pa,$$

This is true (i.e. there exists such a proof—**task:** show this). It's not necessary here to go to the 'full' formalisation, where would render the first premise as e.g. $\forall x(Px \to \exists y(Rxy \wedge Qy))$, to show that the argument is valid in $\mathcal{L}_2$.

Exercise: Formalise the following argument. Is it valid in predicate logic?

> All tickets are winners or losers. Therefore, all tickets are winners.

Work for Week 7

1. Halbach week 7, whole sheet.
2. Peter Fritz week 7, exercise 7.6.

Links to both sets of exercises are available at logicmanual.philosophy.ox.ac.uk/

Solutions due at noon on Thursday week 7.