Intro Logic: Week 2

Quotation Marks

- A word without quotation marks denotes the referent of that word (e.g. "Grass is green.").
- A word with quotation marks, denotes the *word itself* (e.g. "'Cheese' is derived from the Old English word 'cyse'.").
  - "Grass is green" is legitimate, but "'Grass' is green" is dubious.
  - "'Cheese' is derived from the Old English word 'cyse'" is legitimate, but "Cheese is derived from the Old English word 'cyse'" is dubious.
- This is the *use/mention* distinction.

The Syntax of Propositional Logic

- The simplest formal language that we will study is propositional logic, $\mathcal{L}_1$.
- Let's introduce its syntax (i.e. its words, grammar, sentences—but not yet discussing the *meaning* of those words/sentences (recall syntax vs. semantics).)

Definition 1. (Sentence letter): *P, Q, R, P_1, Q_1, R_1, . . . are sentences letters.*

Definition 2. (Sentence of $\mathcal{L}_1$):

(i) All sentence letters are sentences of $\mathcal{L}_1$.

(ii) If ϕ and ψ are sentences of $\mathcal{L}_1$, then $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \to \psi)$ and $(\phi \leftrightarrow \psi)$ are sentences of $\mathcal{L}_1$.

(iii) Nothing else is a sentence of $\mathcal{L}_1$.

- Note: Here (and throughout), the Greek letters ϕ ('phi'), ψ ('psi') and χ ('chi') are *metavariables*—i.e. placeholders for sentences of $\mathcal{L}_1$.

- How should we read these new symbols?

  (a) '$\neg$' is called *negation*, and can be read in English as 'not'.

  (b) '$\wedge$' is called *conjunction*, and can be read in English as 'and'.

  (c) '$\vee$' is called *disjunction*, and can be read in English as 'or'.

  (d) '$\rightarrow$' is called the *conditional* (sometimes: *material implication*), and can be read in English as 'if ... then ...'.

  (e) '$\leftrightarrow$' is called the *biconditional*, and can be read in English as 'if and only if'.

- **Test:** Which of the following are sentences of $\mathcal{L}_1$?

  (i) $(((P_1 \rightarrow P_1) \rightarrow P_1) \vee Q)$

  (ii) $(((P_2 \wedge R)) \rightarrow Q_4)$

  (iii) $P \rightarrow \neg P \wedge R$

  (iv) $(X\neg \rightarrow P)$

  (v) $(\neg P \rightarrow P)$

  (vi) $(P \rightarrow \neg\neg\neg(R \vee \neg R))$

Bracketing Conventions

- Strictly, if a 'sentence' of $\mathcal{L}_1$ doesn't have the right bracket structure (as specified in Def. 2), then it's *not* a sentence of $\mathcal{L}_1$.

- But life's too short to be writing brackets all the time, so we introduce *bracketing conventions*.

Bracketing Convention 1. *The outer brackets may be omitted from a sentence that is not part of another sentence.*

– E.g. one may write $P \rightarrow (Q \vee R)$ for $(P \rightarrow (Q \vee R))$.

Bracketing Convention 2. *The inner set of brackets may be omitted from a sentence of the form* $((\phi \wedge \psi) \wedge \chi)$. *An analogous convention applies to* $\vee$.

- E.g. one may write $(P \wedge Q_2 \wedge P_2)$ for $((P \wedge Q_2) \wedge P_2)$.
- **Test:** Can one abbreviate this further?
- **Exercise** $(*)$**:** Abbreviate $((((P_2 \wedge P_3) \wedge Q) \wedge R) \rightarrow ((P_2 \vee \neg P_3) \vee Q))$ using the bracketing conventions introduced so far.

Bracketing Convention 3. *Assume that ϕ, ψ, and χ are sentences of $\mathcal{L}_1$; $\diamond$ represents either $\wedge$ or $\vee$; and $\circ$ represents either $\rightarrow$ or $\leftrightarrow$. Then, if $(\phi \circ (\psi \diamond \chi))$ or $((\phi \diamond \psi) \circ \chi)$ occurs as part of the sentence that is to be abbreviated, the inner set of brackets may be omitted.*

- E.g. after dropping the outer set of brackets from $((P \wedge Q) \rightarrow R)$ according to Convention 1, one may shorten the sentence further to $P \wedge Q \rightarrow R$.
- The idea is that $\wedge$ and $\vee$ 'bind' more strongly than $\rightarrow$ and $\leftrightarrow$. (Cf. $\times$ vs. $+$ in mathematics).
- **Exercise:** Finish off exercise $(*)$.

The Semantics of Propositional Logic

- Sentences of $\mathcal{L}_1$ can be understood to *represent* sentences of natural language—e.g. English.
  - E.g. we may take $(P \wedge Q)$ to represent the sentence 'Grass is green and it is raining', if we take P to represent 'grass is green' and Q to represent 'it is raining' (see week 3 for more).
- Recall that we're only interested in *declarative* natural language sentences—i.e. sentences which are either true or false.
- We now assign *truth values* to sentences of $\mathcal{L}_1$—since we're now going beyond the bare grammar of $\mathcal{L}_1$ (by discussing whether sentences of $\mathcal{L}_1$ are true or false), we're doing *semantics*.

Definition 3. **($\mathcal{L}_1$-structure):** *An $\mathcal{L}_1$-structure is an assignment of exactly one truth-value (*T *or* F*) to every sentence letter of $\mathcal{L}_1$.*

– I.e. An $\mathcal{L}_1$-structure is a function from the sentence letters into $\{\mathrm{T}, \mathrm{F}\}$, e.g.:

$$\begin{aligned} P &\longrightarrow \mathrm{T} \\ Q &\longrightarrow \mathrm{T} \\ R &\longrightarrow \mathrm{F} \\ P_1 &\longrightarrow \mathrm{T} \\ Q_1 &\longrightarrow \mathrm{F} \\ &\vdots \end{aligned}$$

Note: Sometimes, an $\mathcal{L}_1$-structure is called an $\mathcal{L}_1$-*interpretation*.

Definition 4. (Truth in an $\mathcal{L}_1$-structure): *Let $\mathcal{A}$ be some $\mathcal{L}_1$-structure. Then $|\ldots|_{\mathcal{A}}$ assigns either* T *or* F *to every sentence of $\mathcal{L}_1$ in the following way:*

(i) If ϕ is a sentence letter, then $|\phi|_{\mathcal{A}}$ is the truth-value assigned to ϕ by the $\mathcal{L}_1$-structure $\mathcal{A}$.

(ii) $|\neg\phi|_{\mathcal{A}} = \mathrm{T}$ *if and only if* $|\phi|_{\mathcal{A}} = \mathrm{F}$.

(iii) $|\phi \wedge \psi|_{\mathcal{A}} = \mathrm{T}$ *if and only if* $|\phi|_{\mathcal{A}} = \mathrm{T}$ *and* $|\psi|_{\mathcal{A}} = \mathrm{T}$.

(iv) $|\phi \vee \psi|_{\mathcal{A}} = \mathrm{T}$ *if and only if* $|\phi|_{\mathcal{A}} = \mathrm{T}$ *or* $|\psi|_{\mathcal{A}} = \mathrm{T}$.

(v) $|\phi \rightarrow \psi|_{\mathcal{A}} = \mathrm{T}$ *if and only if* $|\phi|_{\mathcal{A}} = \mathrm{F}$ *or* $|\psi|_{\mathcal{A}} = \mathrm{T}$.

(vi) $|\phi \leftrightarrow \psi|_{\mathcal{A}} = \mathrm{T}$ *if and only if* $|\phi|_{\mathcal{A}} = |\psi|_{\mathcal{A}}$.

- (iv) is more intuitively written: $|\phi \rightarrow \psi|_{\mathcal{A}} = \mathrm{F}$ if and only if $|\phi|_{\mathcal{A}} = \mathrm{T}$ and $|\psi|_{\mathcal{A}} = \mathrm{F}$.
- **Exercise:** Show this follows from (iv) as given in Def. 4.
- Plausible because the idea is that if a conditional is to hold, then if its antecedent (i.e. first) claim holds, its consequent (i.e. second) claim must also hold.

Truth Tables

1. Truth tables!

   – For (i)-(vi).

   – For some generic sentences.

Definition 5. (Logical truth): *A sentence ϕ of $\mathcal{L}_1$ is* logically true *(sometimes: a* tautology*) if and only if it is true in all $\mathcal{L}_1$-structures.*

Definition 6. (Contradiction): *A sentence ϕ of $\mathcal{L}_1$ is a* contradiction *if and only if it is not true in any $\mathcal{L}_1$ structure.*

Definition 7. (Logical equivalence): *A sentence ϕ and a sentence ψ of $\mathcal{L}_1$ are* logically equivalent *if and only if ϕ and ψ are true in exactly the same $\mathcal{L}_1$-structures.*

- These definitions in terms of truth tables.
- Note: just as ϕ, ψ, χ are used to stand in for sentences of $\mathcal{L}_1$, so *capital* Greek letters (e.g. Γ ('Gamma'), Δ ('Delta'), Θ ('Theta')) are used to stand in for *sets* of sentences of $\mathcal{L}_1$.

Definition 8. (Logical validity): *Let Γ be a set of sentences of $\mathcal{L}_1$ and ϕ be a sentence of $\mathcal{L}_1$. The argument with all sentences in Γ as premisses and ϕ as conclusion is* logically valid *if and only if there is no $\mathcal{L}_1$-structure in which all sentences in Γ are true and ϕ is false.*

- If Γ are the premisses and ϕ is the conclusion and the argument is *valid*, we write $\Gamma \vDash \phi$.
- If Γ are the premisses and ϕ is the conclusion and the argument is *not valid* (i.e. *invalid*), we write $\Gamma \nvDash \phi$.

Definition 9. (Counterexample): *An $\mathcal{L}_1$-structure is a* counterexample *to the argument with Γ as the set of premisses and ϕ as conclusion if and only if $|\psi|_{\mathcal{A}} = \mathrm{T}$ for all $\psi \in \Gamma$ and $|\phi|_{\mathcal{A}} = \mathrm{F}$.*

Definition 10. (Semantic consistency): *A set Γ of $\mathcal{L}_1$-sentences is* semantically consistent *if and only if there is an $\mathcal{L}_1$-structure $\mathcal{A}$ such that $|\phi|_{\mathcal{A}} = \mathrm{T}$ for all $\phi \in \Gamma$.*

- **Exercise:** Show (i) $\{P \to Q, P\} \vDash Q$ (*modus ponens*); (ii) $\{P \to Q, \neg Q\} \vDash \neg P$ (*modus tollens*).

- Working backwards to show a sentence is a tautology (*partial truth tables*): put an F in the main column and derive a contradiction.

  – **Exercise:** Do this for (i) $P \vee \neg P$; (ii) $(P \vee \neg Q) \leftrightarrow (Q \to P)$.

Work for Week 2

1. Halbach week 2, whole sheet

2. Peter Fritz week 2, exercises 2.4, 2.7 and 2.9

Links to both sets of exercises are available at logicmanual.philosophy.ox.ac.uk/

Solutions due at noon on Thursday week 2