Philosophy of Quantum Mechanics: Week 3

Entanglement

Composite systems

Consider two quantum mechanical systems, labelled 1 and 2.

- System 1 has Hilbert space $\mathcal{H}_1$; orthonormal basis $\{|\phi_i\rangle\}$.
- System 2 has Hilbert space $\mathcal{H}_2$; orthonormal basis $\{|\chi_j\rangle\}$.

There is a Hilbert space $\mathcal{H}_{12}$ for the joint system,

$$\mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

$\mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2$ is defined as the space spanned by all linear combinations of basis vectors of the form

$$|\psi_{ij}\rangle = |\phi_i\rangle \otimes |\chi_j\rangle \,.$$

A generic state $|\Psi\rangle_{12} \in \mathcal{H}_1 \otimes \mathcal{H}_2$ may therefore be written

$$|\Psi\rangle_{12} = \sum_{ij} \alpha_{ij} \, |\psi_i\rangle_1 \otimes |\chi_j\rangle_2 \,.$$

If $\mathcal{H}_1$ is n-dimensional and $\mathcal{H}_2$ is m-dimensional, then $\mathcal{H}_1 \otimes \mathcal{H}_2$ is mn-dimensional.

Tensor product of operators

To get a grip on the meaning of '$\otimes$':

- If A acts on $\mathcal{H}_1$ and B acts on $\mathcal{H}_2$ ...
- ... then $A \otimes B$ acts (linearly) on $\mathcal{H}_1 \otimes \mathcal{H}_2$:

$$\begin{aligned}(A \otimes B) |\Psi\rangle_{12} &= (A \otimes B) \left(\sum_{ij} \alpha_{ij} |\phi_i\rangle_1 \otimes |\chi_j\rangle_2 \right) \\ &= \sum_{ij} \alpha_{ij} A |\phi_i\rangle_1 \otimes B |\chi_j\rangle_2 .\end{aligned}$$

Note: It is always possible to write any linear operator O_{12} acting on $\mathcal{H}_1 \otimes \mathcal{H}_2$ in the form

$$O_{12} = \sum_{kl} c_{kl} A_k \otimes B_l.$$

Entanglement

Schrödinger: "Not *a*, but *the* characteristic feature of quantum mechanics."

- Product state: $|\Psi\rangle_{12} = |\phi\rangle_1 \otimes |\chi\rangle_2$.
- A system is *entangled* if its state cannot be written in the product form, e.g. if

$$|\Psi\rangle_{12} = \sum_i \alpha_i |\phi_i\rangle_1 \otimes |\chi_i\rangle_2 .$$

- NB: We need to be a bit careful, for suppose e.g.

$$|\Psi\rangle_{12} = \sum_{ij} c_{ij} |a_i\rangle_1 \otimes |b_j\rangle_2 .$$

Is this an entangled state? If $c_{ij} = \alpha_i \beta_j$, then *no*, for

$$
\begin{aligned}
|\Psi\rangle_{12} &= \sum_{ij} c_{ij} \, |a_i\rangle_1 \otimes |b_j\rangle_2 \\
&= \left(\sum_i \alpha_i \, |a_i\rangle_1 \right) \otimes \left(\sum_j \beta_j \, |b_j\rangle_2 \right) \\
&= |\psi\rangle_1 \otimes |\chi\rangle_2 \, .
\end{aligned}
$$

- If there's no way to write the state $|\Psi\rangle_{12}$ as a product of a state in $\mathcal{H}_1$ and a state in $\mathcal{H}_2$ then we have a *qualitatively new phenomenon*.
- *The individual systems* ***do not have states of their own****—only the global system does*.
- Some of the most famous (and relevant, for us) entangled states are the spin states of two-electron systems—the so-called *Bell states*:

$$
\begin{aligned}
|\phi^+\rangle_{12} &= \frac{1}{\sqrt{2}} \left(|\uparrow\rangle_1 \, |\uparrow\rangle_2 + |\downarrow\rangle_1 \, |\downarrow\rangle_2 \right) \\
|\phi^-\rangle_{12} &= \frac{1}{\sqrt{2}} \left(|\uparrow\rangle_1 \, |\uparrow\rangle_2 - |\downarrow\rangle_1 \, |\downarrow\rangle_2 \right) \\
|\psi^+\rangle_{12} &= \frac{1}{\sqrt{2}} \left(|\uparrow\rangle_1 \, |\downarrow\rangle_2 + |\downarrow\rangle_1 \, |\uparrow\rangle_2 \right) \\
|\psi^-\rangle_{12} &= \frac{1}{\sqrt{2}} \left(|\uparrow\rangle_1 \, |\downarrow\rangle_2 - |\downarrow\rangle_1 \, |\uparrow\rangle_2 \right)
\end{aligned}
$$

  - The former three are known as *triplet states*; the fourth is the so-called *singlet state*.
  - What is the spin state of system 1 or system 2 in any of the four above joint states? Answer: *undefined*!

ENTANGLEMENT SHOWS THAT THERE ARE GLOBAL PROPERTIES OF A JOINT SYSTEM WHICH ARE NOT REDUCIBLE TO PROPERTIES OF SUBSYSTEMS!

The Einstein-Podolsky-Rosen Argument

- In 1935, Einstein, Podolsky and Rosen (EPR) presented a dilemma: *either quantum mechanics is incomplete, or it is non-local*.
- This argument was driven by a (simple and reasonable) requirement of *separability* for physical systems.
- Einstein criterion for separability: The genuine properties possessed by a system should not be affected by what is done to another, distinct, spatially separated system, even if the two have interacted in the past.
- Quantum mechanics seems to fail to meet this criterion.

The dilemma

Consider the singlet two-electron spin state:

$$|\psi^-\rangle_{12} = \frac{1}{\sqrt{2}} \left(|{\uparrow_z}\rangle_1 \, |{\downarrow_z}\rangle_2 - |{\downarrow_z}\rangle_1 \, |{\uparrow_z}\rangle_2 \right)$$

Recalling that

$$\begin{aligned} |{\uparrow_z}\rangle &= \frac{1}{\sqrt{2}} \left(|{\uparrow_x}\rangle + |{\downarrow_x}\rangle \right) \\ |{\downarrow_z}\rangle &= \frac{1}{\sqrt{2}} \left(|{\uparrow_x}\rangle - |{\downarrow_x}\rangle \right) \end{aligned}$$

we see that $|\psi^-\rangle_{12}$ could also be written

$$|\psi^-\rangle_{12} = \frac{1}{\sqrt{2}} \left(|{\uparrow_x}\rangle_1 \, |{\downarrow_x}\rangle_2 - |{\downarrow_x}\rangle_1 \, |{\uparrow_x}\rangle_2 \right)$$

(In fact, the singlet is *spherically symmetric*, i.e. it takes the same form for all spin directions.)

- Now consider an experiment in which a pair of electrons is prepared in this joint state at one location at time t_0, before being widely separated.
- At time t_1 the systems 1 and 2 are far apart and no longer interacting and we consider performing measurements on one of them, say system 1.
- z-spin measurement: before measurement, neither system has a definite value of spin in the z-direction. At $t > t_1$ we obtain a definite value of spin for system 1. But not only has the property of system 1 changed, but so also have the properties of the far away system 2 ("instantaneous action at a distance"—Einstein: "spooky action").
- Similarly if we had performed an x-spin measurement on system 1 at time $t > t_1$. Our choice of which measurement to perform in the region of system 1 affects at a distance the properties of system 2.

So two possibilities:

1. **Either:** Quantum mechanics is non-local—the effect of measurement is to cause a collapse which has an instantaneous causal effect at a distance;
2. **Or:** Quantum mechanics is incomplete—the genuine properties of systems should not behave like this, which means that the quantum state cannot be the whole story. (This was EPR's preferred horn of the dilemma.)

The no-signalling theorem

In EPR's hands (and on the assumption of collapse), entanglement gives rise to a special kind of *action at a distance*. Following measurement on one half of a spacelike separated pair of systems in an entangled state, the non-classical global correlation properties of the joint system are transmuted instantaneously into definite locally possessed values for *both* halves. The far system jumps into having a definite value, where it didn't have one before. This holds independently of how far apart the systems may be separated in space.

- Can these global correlation properties be used to signal instantaneously?
- *No*: see the *no-signalling theorem.*

Hidden variables

The natural way to cash out the incompleteness horn of the dilemma is via *hidden variables*:

- Additional determinate elements about whose values we are ignorant (hence the term 'hidden') and which explain why we get the particular outcomes we do when we perform (quantum mechanical) experiments.
- Statistical features of quantum mechanics just arise as 'ignorance' probabilities regarding the true values as described the hidden variables.
- In general, incompatible properties like position and momentum *are* definite at the same time, it's just that we are *ignorant* of what their values are.
- Note that introducing hidden variables may help us with the measurement problem: the value of the hidden variable will determine which outcome of the experiment is definite, even if the *quantum state* happens to be left in a superposition of distinct measurement outcomes.

Bell's Theorem

- Bell (1964) made the discovery that if a hidden variable theory is to be consistent with the empirical predictions of quantum mechanics, it must be *nonlocal* ('Bell's theorem').
- Therefore, taking the incompleteness horn of the EPR dilemma does not lead us away from non-locality after all.

Setup

- Consider a **deterministic** theory which ascribes definite values to the results of all experiments we might perform on a system (deterministic hidden variable theory).
- Specifying the initial state leads to deterministic predictions about what the results of any measurement will be.

- Assume, furthermore, that the theory satisfies a **locality** condition: the values for the outcomes of experiments on a system depend *only* on the initial state and on the particular experiment performed on that system.
- The aim is to show that the statistical predictions of any such theory will satisfy a certain inequality (a *Bell inequality*) regarding the correlations in EPR-type experiments.
- Such an inequality will be violated by the predictions of quantum mechanics.
- Conclusion: The results predicted by quantum mechanics cannot be modelled by any *local* deterministic hidden variable theory.

The EPR-type experiment

- We will consider an experiment testing the correlations between various spin measurements on a pair of spin-half systems prepared in an initial entangled singlet state.
- Measurements of spin will be made in the direction $\mathbf{a}$ or $\mathbf{a}'$ on system 1; and in the direction $\mathbf{b}$ or $\mathbf{b}'$ on system 2.
- The hidden variable theory will assign definite values to all four spin quantities at the same time, i.e. $\mathbf{a}$ and $\mathbf{a}'$ for system 1 possess a definite value just before measurement, as do $\mathbf{b}$ and $\mathbf{b}'$. Denote these quantities by a, a', b, b', respectively. They represent the values that *would* be observed if one *were* to perform the measurement of that quantity on the given system.
- On each run of the experiment, all four spin quantities possess a definite value. For run n, denote these quantities by a_n, a'_n, b_n, b'_n.
- Although the initial *quantum* state of the joint system is the same for each run, the hidden variable takes different values and therefore a spread of outcomes is observed.

Average values and correlation coefficient are given by, respectively,

$$
\begin{aligned}
\bar{a} &= \frac{1}{N}\sum_{n=1}^{N} a_n, \\
c\left(\mathbf{a}, \mathbf{b}\right) &= \frac{1}{N}\sum_{n=1}^{N} a_n b_n.
\end{aligned}
$$

Now consider the following function:

$$
\begin{aligned}
\gamma_n &= a_n b_n + a_n b'_n + a'_n b_n - a'_n b'_n \\
&= a_n \left(b_n + b'_n\right) + a'_n \left(b_n - b'_n\right).
\end{aligned}
$$

Since each of a, a', b, b' can be equal to ± 1, γ_n must be equal to ± 2. Now calculate the average value of this quantity for a large number of runs of the experiment:

$$
\frac{1}{N}\sum_{n=1}^{N} \gamma_n = \frac{1}{N}\sum_{n=1}^{N} \left(a_n b_n + a_n b'_n + a'_n b_n - a'_n b'_n\right)
$$

Given that on each run $\gamma_n = \pm 2$, this average must lie between ± 2, so its modulus will be less than or equal to 2:

$$
\left|\frac{1}{N}\sum_{n=1}^{N} \left(a_n b_n + a_n b'_n + a'_n b_n - a'_n b'_n\right)\right| \leq 2
$$

In terms of correlation coefficients, this gives us the *Bell inequality* (in the Clauser-Horne-Shimony-Holt (CHSH) version):

$$
\left|c\left(\mathbf{a}, \mathbf{b}\right) + c\left(\mathbf{a}, \mathbf{b}'\right) + c\left(\mathbf{a}', \mathbf{b}\right) - c\left(\mathbf{a}', \mathbf{b}'\right)\right| \leq 2
$$

How does this compare with the predictions that *quantum mechanics* makes for the correlations between spin measurement outcomes that would be observed in the EPR-type experiment? When the quantum mechanical values for the correlation coefficients are calculated, we instead obtain:

$$|c(\mathbf{a}, \mathbf{b}) + c(\mathbf{a}, \mathbf{b}') + c(\mathbf{a}', \mathbf{b}) - c(\mathbf{a}', \mathbf{b}')| \leq 2\sqrt{2}$$

That is, the quantum mechanical correlations are *stronger* than any correlations that can be obtained in a local hidden variable model.

This can be illustrated in a special case—suppose for example that:

1. Measurements of $\mathbf{a}$ and $\mathbf{b}$ are along the same axis.
2. Measurements of $\mathbf{a}$ and $\mathbf{b}'$ or $\mathbf{b}$ and $\mathbf{a}'$ are at an angle θ.
3. Measurements of $\mathbf{a}'$ and $\mathbf{b}'$ are at an angle 2θ.

Quantum mechanics has its own predictions of correlation coefficients. If ϕ is the angle between directions $\mathbf{a}$ and $\mathbf{b}$, then

$$c(\mathbf{a}, \mathbf{b}) = \cos\phi$$

Using the above assumptions, we therefore have

$$\begin{aligned} |c(\mathbf{a}, \mathbf{b}) + c(\mathbf{a}, \mathbf{b}') + c(\mathbf{a}', \mathbf{b}) - c(\mathbf{a}', \mathbf{b}')| &= |\cos(0) + 2\cos(\theta) - \cos(2\theta)| \\ &= \left|2\left(1 + \cos\theta - \cos^2\theta\right)\right| \\ &> 2 \end{aligned}$$

Note: It's not that standard quantum mechanics predicts different *inequalities* to local hidden variable theories. It's that some of its specific predictions involve correlations that violate the Bell inequalities.

SO NO LOCAL HIDDEN VARIABLE MODEL CAN BE EMPIRICALLY EQUIVALENT TO QUANTUM MECHANICS!

Experimental evidence

- Which is right—quantum mechanics, or local hidden variable theories?
- We need to do the experiment and see whether the quantum predictions or the local hidden variable predictions for the correlations hold.
- The overwhelming evidence is that *quantum mechanics is correct* (see e.g. Aspect's experiment).

SO NO LOCAL HIDDEN VARIABLE MODEL IS EMPIRICALLY ADEQUATE!

Thus we conclude that any acceptable hidden variable theory must be non-local. But note that this doesn't imply anything about *quantum mechanics* being non-local, just that any hidden variable theory put in its place must be. We know that collapse in quantum mechanics gives rise to action at a distance, but what if we deny that collapse is a real physical process?

NB: Bell's theorem does not force us to accept "the bare formalism of quantum mechanics" *sans* interpretation, as some (physics texts) claim. We still need to solve the measurement problem!