

## Paradoxes—Reading group 6

### Chapter 6: Classes and truth

In this chapter, Sainsbury considers two paradoxes, both famous and fecund: *Russell's paradox*, and the *liar paradox*.

#### Russell's paradox

Consider the set

$$R := \{x : x \notin x\}.$$

This is the set of all objects which are not elements of themselves. Is  $x$  an element of  $x$ ? If it is, then it is not, and if it is not, then it is! Contradiction. This is *Russell's paradox*.<sup>1</sup>

Russell's response to his paradox was his *theory of types*, “according to which classes are arranged in a hierarchy, in such a way that every class is on a higher level than any of its members” (Sainsbury p. 126). Then, the expression ‘ $x \in x$ ’ is not meaningful, so  $R$  is ruled out as a legitimate set.

Nowadays, people work with *Zermelo-Fraenkel set theory* (ZFC), which is a version of set theory which avoids Russell's paradox.

#### The liar paradox

Consider the following sentence:

**Liar:** This sentence is false.

Is **Liar** true or false? If it is true, then it is false, and if it is false, then it is true! Contradiction.

There has been a rich array of responses to the liar paradox. One is to reject the *principle of bivalence*: to think that some sentences are *neither true nor false*—that is, to think that there are

<sup>1</sup>First presented by Russell in his *Principles of Mathematics* (1903).

truth *gaps*. If this applies to **Liar**, perhaps (the thought goes) the right response to the paradox is that this sentence is neither true nor false.

A second response to the liar paradox would be to endorse Priest's *logic of paradox*, according to which some sentence (presumably including **Liar**) can be *both* true and false—that is, to think that there are truth *gluts*.

### Grounding and truth

Some seek to argue that **Liar** is *semantically defective*. One line of thought is that **Liar** is not *grounded* in real world facts—and so is not the kind of thing which could be true or false (this is consistent with the rejection of bivalence, but goes further, because the claim now is that **Liar** is not even a *candidate* to be true/false). As Sainsbury notes, this response would also adjudicate that sentences such as

This sentence is true.

do not have a truth value, in spite of being non-paradoxical—for such sentences are just as ungrounded as **Liar**.

### Levels

Tarski (1956) had a different response to the liar paradox—superficially reminiscent of Russell's theory of types. This also maintains that **Liar** is semantically defective, but now for a different reason. Here's how Sainsbury puts it:

*Tarski's response is that the ordinary concept of truth, the one we use every day, is incoherent and must be rejected. According to Tarski, it needs to be replaced by a series of concepts of truth, hierarchically arranged, and each expressed in a language different from any natural language (i.e. from any language that has evolved naturally). (Sainsbury p. 134)*

How does this solve the paradox? Sainsbury explains:

*No paradoxical Liar sentence can be formulated in any of the languages in Tarski's hierarchy. How is this supposed to provide a "solution" to the paradox? The paradox arises in our language, so a proper defusing of it must say something about our language, and not merely offer a replacement.*

*What Tarski says about our language is that the Liar shows it to be incoherent. We must replace our actual, but incoherent, concept of truth by a family of new concepts, each fixed to a level in the hierarchy, in the way just described. Many people have sought something less radical, a response that preserves more of our ordinary thought and talk. (Sainsbury p. 135)*

### Self-reference

Another possible response is to maintain that **Liar** is paradoxical because it is self-referential. But, as Sainsbury explains (p. 137), it's hard to get this response off the ground:

*It is natural to think that something about the self-referential character of Liar paradoxical sentences is the main source of their paradoxical nature. There may be something in this thought, but as it stands it is both incorrect and inadequate.*

*It is incorrect because a sentence can refer to itself, as for example this very sentence does, without leading to any kind of semantic defect or paradox. So sentential self-reference cannot be the sole source of Liar paradoxes.*

*It is inadequate because one can construct Liar paradoxes without using any sentence which refers to itself. One example of this phenomenon involves liar cycles like the following.*

(A) (said by  $\alpha$  on Monday): Everything  $\beta$  will say on Tuesday is true.

(B) (said by  $\beta$  on Tuesday): Nothing  $\alpha$  said on Monday is true.

### Comparison of the paradoxes

It's interesting to think about whether Russell's paradox and the liar paradox are merely different aspects of one and the same paradox. Initially, following Ramsey (1925), the answer to this question given by many philosophers was negative:

*Ramsey urged that the paradoxes are different in kind, and his view has been predominant, at least until relatively recently. He based the distinction on their different subject matter: the logical paradoxes, under which heading he included Russell's paradox, arise from logical notions, like that of class; the semantic paradoxes, under which he included the Liar, arise from semantic notions, like that of truth. (Sainsbury p. 142)*

In reaction to this, Sainsbury points to five senses in which the paradoxes are at least very similar:

1. "The Class paradox resembles a paradox about properties, and the Property paradox in turn resembles the Liar." (p. 143)
2. "Both the Class paradox and the Liar involve self-reference, or some similar circularity." (p. 143)
3. "The principles appealed to in the derivation of the two paradoxes (CE: for every intelligible condition  $F$ , there is a class  $x$ , such that: for any object  $y$ ,  $y \in x$  if and only if  $y$  satisfies  $F$ ; and T:  $\sigma$  is true iff  $p$ ) are structurally similar, and appear to play similarly constitutive roles with respect to the intuitive notions of class and truth." (p. 143)
4. "Hierarchies have been used in response to both kinds of paradox, beginning with one of the earliest systematic treatments, in Russell (1908)." (p. 144)
5. "Russell's classification of the Class paradox and the Liar as of a common kind is based on the claim that they both alike derive from an infringement of the Vicious Circle Principle:

**VCP:** No totality can contain members fully specifiable only in terms of itself." (p. 144)