Introduction to symmetries
Problem set 1/3: Translations, rotations

Michaelmas 2017

1 Operators

a) Show explicitly that

$$\hat{U}(\vec{a})\ \vec{X}\ \hat{U}^\dagger(\vec{a}) \ = \ \vec{X} - \vec{a}\,,$$

where $\hat{U}(\vec{a}) = \exp(-i\vec{a}\cdot\vec{p})$. Remember that $\vec{X}$ and $\vec{p}$ are operators, whereas $\vec{a}$ is a vector in $\mathbb{R}^3$. You need to use the Baker-Campbell-Hausdorff formula:

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + ...$$

b) Keeping the i and j indexes generic (without choosing for example $i=1$ and $j=2$), check that

$$[\hat{L}_i, \hat{L}_j] = i\epsilon_{ijk}\hat{L}_k\,.$$

You'll need to use the definition $\hat{L}_i = \epsilon_{ijk}\hat{x}_j\hat{p}_k$, the canonical commutation relation $[\hat{x}_i, \hat{p}_j] = i\delta_{ij}$, and the property $\epsilon_{iab}\epsilon_{icd} = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}$.

2 Translations

a) Two classical non-relativistic particles of mass m_1 and m_2 interact in one dimension by a mutual potential energy function $V(x_1 - x_2)$. The force on m_1 is $F_1 = -\frac{\partial V}{\partial x_1}$ and similarly for the force on m_2. Show that $F_1 = -F_2$ and that the total momentum $P = m_1 v_1 + m_2 v_2$ is constant in time.

b) Suppose a third particle of mass m_3 is added. Suggest a form for the mutual potential energies (three of them) so that the total momentum is constant in time.

c) Two quantum mechanical particles in one dimension are described by the Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m_1}\frac{\partial^2}{\partial x_1^2} - \frac{\hbar^2}{2m_2}\frac{\partial^2}{\partial x_2^2} + V(x_1 - x_2)\,.$$

Show that the eigenvalues of the total momentum operator $\hat{P} = -i\hbar\frac{\partial}{\partial x_1} - i\hbar\frac{\partial}{\partial x_2}$ are constant in time.

3 Vectorial representation of SO(3)

Consider the operator $U(\hat{z},\alpha) = \exp\left(-i\frac{\alpha}{\hbar}L_z^{(1)}\right)$, where

$$L_z^{(1)} = \hbar \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

Using the definition of the exponential, compute the matrix $U(\hat{z},\alpha)$ and show that

$$U(\hat{z},\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

4 Spinorial representation of SO(3)

a) Show that the Pauli equation

$$\frac{\left[\vec{\sigma}\cdot(\vec{p}-e\vec{A}(\vec{x}))\right]^2}{2m}\psi = E\psi,$$

where $\{\sigma_i\}$ are the Pauli matrices, $\vec{A}$ is the magnetic potential and ψ is a bidimensional wave function, corresponds to

$$\left[\frac{(\vec{p}-e\vec{A})^2}{2m}\mathbb{1} - \frac{e\hbar}{2m}\vec{\sigma}\cdot\vec{B}\right]\psi = E\psi\,.$$

To obtain this in general, without making a particular choice for $\vec{A}$, you need to use $\sigma_i\sigma_j = \delta_{ij}\mathbb{1} + i\epsilon_{ijk}\sigma_k$ and recall that $(\vec{\nabla}\times\vec{A})_i = \epsilon_{ijk}\partial_j A_k$. Remember that $\vec{p}$ and $\vec{A}$ don't commute.

b) Prove that, by exponentiating a linear combination of Pauli matrices, one obtains the identity

$$\exp\left(-i\frac{\alpha}{2}\hat{n}\cdot\vec{\sigma}\right) = \cos(\alpha/2)\ \mathbb{1} - i\sin(\alpha/2)\ (\hat{n}\cdot\vec{\sigma})\,,$$

where $\hat{n}$ is a generic 3-dimensional unit vector.

5 Degeneracies and vectorial representations of SO(3)

Consider the states with wavefunctions

$$\begin{aligned} \psi_+ &= (x+iy)/\sqrt{2}\,, \\ \psi_- &= (x-iy)/\sqrt{2}\,, \\ \psi_0 &= z\,. \end{aligned}$$

They are degenerate because for all of them $\ell = 1$ ($L^2 = 2\hbar^2$). Now consider the rotation $R_z(\alpha)$ such that

$$\begin{aligned} x' &= x\ \cos\alpha - y\ \sin\alpha \\ y' &= x\ \sin\alpha + y\ \cos\alpha \\ z' &= z\,. \end{aligned}$$

We have that, for $m = +, -, 0$,

$$\begin{aligned} \psi'_m(\vec{r}) &= \psi_m(R_z^{-1}\vec{r}) \\ &= \psi_m(x\cos\alpha + y\sin\alpha, -x\sin\alpha + y\cos\alpha, z) \\ &= \sum_{n=+,-,0} D_{nm}(R_z)\psi_n\,. \end{aligned}$$

a) Calculate the elements of the matrix $D(R_z)$ written out as

$$D(R_z) = \begin{pmatrix} D_{++} & D_{+-} & D_{+0} \\ D_{-+} & D_{--} & D_{-0} \\ D_{0+} & D_{0-} & D_{00} \end{pmatrix}.$$

b) Taking the infinitesimal limit $\alpha \to \epsilon$, and writing as usual $D(R_z) = 1 - \frac{i\epsilon}{\hbar}\tilde{L}_z^{(1)}$, find the matrix $\tilde{L}_z^{(1)}$.

c) In the lectures, $L_z^{(1)}$ was found using the basis $\psi_1 = x,\ \psi_2 = y,\ \psi_3 = z$. Explain as fully as you can the relation between $L_z^{(1)}$ and $\tilde{L}_z^{(1)}$.
hint: if you want (**Optional !**) to find the transformation that brings $L_z^{(1)}$ into $\tilde{L}_z^{(1)}$, it's a good idea to use the Dirac "bra-ket" notation.