Introduction to symmetries
Problem set 3/3: Higgs mechanism

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1 Higgs sector of the SM Lagrangian

The Lagrangian for the Higgs sector of the Standard Model, including no interactions with matter or gauge fields, is

$$\mathcal{L} = \partial_\mu \phi^\dagger \, \partial^\mu \phi + \frac{\mu^2}{2} \phi^\dagger \phi - \frac{\lambda^2}{2} (\phi^\dagger \phi)^2 \, ,$$

where $\mu^2 > 0$, $\lambda^2 > 0$, and $\phi = (\phi_1, \phi_2)$ is an $SU(2)$ "isospinor", *i.e.* both ϕ_1 and ϕ_2 are complex fields, so $\phi^\dagger \phi = |\phi_1|^2 + |\phi_2|^2$ (this $SU(2)$ group is the $SU(2)_L$ gauge group of the SM, but the fact that it is a gauge group doesn't matter for this exercise – it matters of course for the SM !).

a) Show that the minimum of the potential is at $\phi^\dagger \phi = \mu^2/(2\lambda^2)$.

Since both ϕ_1 and ϕ_2 are complex fields, there are 4 real degrees of freedom involved (4 real fields). The way of parameterizing the fluctuations away from the minimum is to set

$$\phi = \frac{1}{\sqrt{2}} e^{-i\frac{\vec{\tau}\cdot\vec{\theta}(x)}{2f}} \begin{pmatrix} 0 \\ f + h(x) \end{pmatrix} , \quad f = \frac{\mu}{\lambda} \, ,$$

which has, consistently, 4 real degrees of freedom: $\vec{\theta}(x)$ and $h(x)$ (τ_i are the Pauli matrices).

b) Show that the $\theta_i(x)$ fields are all massless and find the mass of the h field.

2 Abelian Higgs model

The Lagrangian for the Abelian Higgs model is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (\mathcal{D}_\mu \phi)^* (\mathcal{D}_\mu \phi) + \frac{\mu^2}{2} |\phi|^2 - \frac{\lambda^2}{2} |\phi|^4 \, ,$$

where ϕ is a complex field ($|\phi|^2 = \phi^* \phi$), $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\mathcal{D}_\mu \phi = (\partial_\mu + ieA_\mu)\phi$ is the "covariant derivative", and μ and λ are real parameters.

a) Show that the potential $\mathcal{V} = -\frac{\mu^2}{2}|\phi|^2 + \frac{\lambda^2}{2}|\phi|^4$ has a minimum at $|\phi| = |\frac{\mu}{\sqrt{2}\lambda}|$.

b) Verify that this theory is invariant under the gauge transformation

$$\begin{aligned} \phi(x) &\rightarrow e^{-ie\chi(x)}\phi(x)\,, \\ A_\mu(x) &\rightarrow A_\mu(x) + \partial_\mu\chi(x)\,. \end{aligned}$$

c) Consider the following expansion of ϕ about the minimum of the potential:

$$\phi = \frac{1}{\sqrt{2}} e^{i\theta(x)/f}(f + \rho(x))\,, \quad f = \left|\frac{\mu}{\lambda}\right|\,.$$

Show that by a choice of gauge (*i.e.* of the function $\chi(x)$), $\theta(x)$ can be reduced to zero. If $A_\mu(x)$ is the vector potential in the original gauge where $\theta(x) \neq 0$, what is the vector potential $A'_\mu(x)$ in the gauge where $\theta(x) = 0$?

d) In the gauge where $\theta = 0$, we have $\phi(x) = \frac{1}{\sqrt{2}}(f + \rho(x))$. Insert this expression of ϕ in $\mathcal{L}$, and retain only the terms that are quadratic in ρ and A'_μ (keep A'_μ as it is, don't re-expand it terms of the fields in the gauge where $\theta \neq 0$). What particles (spin and masses) does this quadratic part describe?