$\mathbb{N}$ VERSUS THE OTHER INFINITE

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First, crucial differences are demonstrated between Predicativity given the totality of ω and Predicativity given the totality of other infinite mathematical entities, such as the set $\mathcal{P}(\omega)$ of reals, the set $\mathcal{P}(\mathcal{P}(\omega))$ of functions and the universe V of sets: Let us concentrate on how we can proceed to generate subsets (subclasses) of the given entities. On the one hand, the following characterizations (A)-(C') of tradtional ω-predicativity have been known to be all proof-theoretically equivalent (or equiconsistent) and hence considered as the right characterizations. On the other hand, (A)-(C') for the other kinds of predicativity, we lost such robustness and could not conclude which one is the right.

- (A) autonomous progression of elementary comprehension: since the quantifiers over the given entity make sense, elementary comprehension must be accepted, whose iterations along any "legitimately-defined" order should also be accepted provided that the proof of the well-orderedness have been accepted.
- (B) 2-fold autonomous progression of elementary comprehension: once (A) is accepted, the iteration (of the same type) of (A) itself should also be accepted.
- (C) general multi-fold autonomous progression.
- (A') - (C') the same but "elementary" replaced by "recognizably invariant" (under the generating process), namely, essentially Δ^1_1, denoted by Δ.

These investigations can be uniformed in the second order framework: elements of the given totality are of first order; subclasses (which are always being generated) of the given totality are of second order. What makes ω-predicativity exceptional is that well-foundedness is non-elementary in the context of ω whereas it is elementary in the others.

This motivates us to ask: among the results known in number theory, which survives when we replace ω by others, and which does not. The following is a partial list, where $T > S$ means $T \vdash S + \mathrm{Con}(S)$, where Γ-$\mathbf{TR}^k$ is the internalization of k-fold autonomous progression of Γ-comprehension (k omitted if $k = 1$) and where $k, j \geq 1$:

given ω	given $\mathcal{P}^n(\omega)$ $(n \geq 1)$ and $\mathcal{P}^n(V)$ $(n \geq 0)$
$\Delta\textbf{-TR}^{k+1} \leftrightarrow \Delta^1_0\textbf{-TR}^{j+2} \leftrightarrow \Delta^1_0\textbf{-TR}$	$\Delta\textbf{-TR}^{k+1}, \Delta^1_0\textbf{-TR}^{j+2} > \Delta^1_0\textbf{-TR}$
$\Delta^1_0\textbf{-LFP} > \Delta^1_0\textbf{-FP} \leftrightarrow \Pi^1_1\textbf{-Red} \leftrightarrow \Delta^1_0\textbf{-TR}$	$\Pi^1_1\textbf{-Red} > \Delta^1_0\textbf{-LFP} \leftrightarrow \Delta^1_0\textbf{-FP} > \Delta^1_0\textbf{-TR}$
$ID_1 \to \mathrm{Con}(\Delta^1_0\textbf{-TR})$; $\quad \Delta^1_0\textbf{-TR} \to \mathrm{Con}(\widehat{ID}_1)$	$ID_k \leftrightarrow \widehat{ID}_k \to \mathrm{Con}(\Delta^1_0\textbf{-TR})$
$\Delta^1_0\textbf{-TR} \to \Delta^1_1\textbf{-CA}, \Sigma^1_1\textbf{-Coll}$	$\Delta^1_0\textbf{-TR} \not\to \Delta^1_1\textbf{-CA}, \Sigma^1_1\textbf{-Coll}$
$\Pi^1_{k+1}\textbf{-CA} > \Pi^1_k\textbf{-CA}$; $\qquad \Delta\textbf{-TR}^{k+2} \leftrightarrow \Delta\textbf{-TR}$	
$\mathrm{Con}(\Delta^1_0\textbf{-TR}) \leftrightarrow \mathrm{Con}(\Delta^1_0\textbf{-TR} + \Delta\textbf{-CA}) \leftrightarrow \mathrm{Con}(\Delta^1_0\textbf{-TR} + \Sigma^1_1\textbf{-Coll})$	

Furthermore, the comparison with the case where totality is given only to finite entities will be provided. In this new case, the framework is two-sorted bounded arithmetic, where (A)-(C') characterize the polynomial-time computability, and the relations among the principles appearing in the table above differ from those in the previous cases.