*Representing an ordering when the population varies*

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*Abstract*. This note describes a domain of distributions of wellbeing, in which different distributions may have different populations. It proves a representation theorem for an ordering defined on this domain.

We are often interested in an ordering of distributions of wellbeing (or something else, such as income) across a population of people. Sometimes we want to allow different distributions to have different populations. But if they do, the standard representation theorems, which allow us to represent the ordering by a real-valued function, do not apply.

Charles Blackorby, Walter Bossert and David Donaldson [1] have recently published a proof of a new representation theorem for an ordering like this. That theorem assumed anonymity, but a subsequent paper by the same authors [2, theorem 4] proves a similar theorem without assuming anonymity. In this present note, I prove a theorem that is equivalent to their second, more general theorem. I use a framework that I think is neater than theirs, and my proof is shorter.

Let there be a number of possible people, each of whom may or may not exist. The number may be infinite. However,

**Assumption 1**: the number of possible people is countable.

Index the possible people by $p = 1, 2, \ldots$ If $p$ exists, let $g_p$ be her wellbeing, which I assume to be a real number. If $p$ does not exist, let $g_p$ be $\Omega$, which is some arbitrary non-numerical object. So $g_p \in R \cup \{\Omega\}$, where $R$ is the set of real numbers. In effect, '$\Omega$' is a symbol whose only role is to indicate that a particular possible person does not actually exist. The vector $g = (g_1, g_2, \ldots)$ specifies which people exist and the wellbeings of those who do. I call it a *distribution*. The *population* of a distribution is the set of people who exist in the distribution. It is the *set* of people, not the *number* of people in this set.

Let $\succeq$ be the relation 'at least as good as' defined on distributions. Let $F$ be the field of this relation (that is to say, the union of the domain and the range of the relation).

**Assumption 2**: $\succeq$ is transitive and complete on $F$.

**Assumption 3**: $F$ contains only distributions whose populations are finite.

Take any finite population $\pi$. Let $n$ be the number of its members. Consider the set of distributions in $F$ that have the population $\pi$; call it $F_\pi$. The members of $F_\pi$ are vectors that have a real number in each of the $n$ places that are assigned to a member of the population, and $\Omega$s elsewhere. Each member of $F_\pi$ has $\Omega$s in the same places.

Take any vector $g$ in $F_\pi$. Delete all the $\Omega$s from $g$. This will leave an $n$-dimensional vector; call it $s(g)$. $s(\,)$ is a one-one mapping, which maps $F_\pi$ on to a subspace of $n$-dimensional Euclidean space. Each open set in the topology of this subspace has an image in $F_\pi$ under the inverse mapping $s^{-1}(\,)$. The set of these images constitutes a topology for $F_\pi$. In effect, by ignoring the $\Omega$s, we give $F_\pi$ the topology of a subspace of $n$-dimensional Euclidean space.

Let $\Pi$ be the set of all finite populations. Each member of $\Pi$ is a finite subset of the set of possible people, which is countable. The number of finite subsets of a countable set is countable. Therefore $\Pi$ is countable.

$F$ is $\bigcup_{\pi\in\Pi} F_\pi$. It is the union of a countable number of spaces. I shall call each $F_\pi$ a 'part' of $F$. (Some of these parts may be empty.) Each part has the topology of a subspace of Euclidean space. For each part, take some basis for its topology. $F$ has a topology whose basis is the union of these bases.

A space with this sort of structure may be unfamiliar. It is a union of unconnected parts, each of which has the topology of a subspace of Euclidean space, but these parts are not embedded in a bigger Euclidean space. For an analogy, think of a pair of parallel lines. Each line has a two-dimensional Euclidean topology, a basis for which is the set of open intervals in the line. There is a space that consists of the union of these two lines, with a topology whose basis is the union of their two bases. This basis consists of all the open intervals in one line or in the other. The topology of $F$ is like that.

As it happens, this pair of parallel lines is embedded in a plane, which has a two-dimensional Euclidean topology. The union of the lines can be regarded as a subspace of the plane. Regarded that way, it would have the same topology as the one I defined. But the union does not have to be regarded that way. I defined its topology independently of the fact that it is embedded in a plane.

Unlike a pair of lines, the set $F$ cannot necessarily be embedded in a finite-dimension Euclidean space. Since the set of possible people may be infinite, there may be no upper bound on the size of the population. So there may be no upper bound on the dimensionality of the parts of $F$. Consequently, it may not be possible to fit all these parts into a finite-dimension space. Still, $F$ has a topology that is defined in the way I defined it.

> **Assumption 4 (continuity)**: for all $g \in F$ the sets $\{h \in F | \ h \succeq g\}$ and $\{h \in F | \ g \succeq h\}$ are closed.

'Closed', of course, means closed relative to the topology of $F$. An equivalent condition is this: for any $g$, whenever either of these sets $\{h \in F | \ h \succeq g\}$ or $\{h \in F | \ g \succeq h\}$ intersects with a particular part $F_\pi$ of $F$, the intersection is closed in $F_\pi$, relative to the topology of $F_\pi$. My assumption is equivalent to what Blackorby, Bossert and Donaldson [2] call 'unconditional continuity'.

Within each part of $F$, it is a familiar continuity assumption, ruling out such familiar discontinuities as lexical orderings. When we take more than one part of $F$ together, continuity has another effect. To see what it is, take a simple example. Suppose there are just two possible people, and think about an ordering that has these features:

If $g_1 > g_2$, then $(g_1, \Omega) \succ (\Omega, g_2)$.

If $g_1 < g_2$, then $(g_1, \Omega) \prec (\Omega, g_2)$.

If $g_1 = g_2$, then $(g_1, \Omega) \succ (\Omega, g_2)$.

This ordering is inconsistent with the assumption of continuity. The set $\{h \in F | \ h \succeq (g_1, \Omega)\}$ of distributions that are at least as good as $(g_1, \Omega)$ includes $(\Omega, g_2)$ for all $g_2$ greater than $g_1$, but it does not include $(\Omega, g_1)$. So this set is not closed.

The ordering I described gives value to people's wellbeing. It also gives some value to the matter of which person exists: when quantities of wellbeing are equal, it prefers the first person to exist rather than the second. But this second value is lexically dominated by the value of wellbeing. Continuity rules out this type of lexical ordering, as well as more familiar types.

**Theorem**. Given the four assumptions, $\succeq$ can be represented by a continuous, real-valued function on $F$.

**Proof**. A space is *perfectly separable* if its topology has a countable basis. Any subspace of Euclidean space is perfectly separable – a countable basis for its topology is the set of intersections of the subspace with open spheres that have rational centres and rational radii. Therefore, $F_\pi$ is perfectly separable, for each $\pi \in \Pi$. The union of the countable bases of all the $F_\pi$s is countable, since $\Pi$ is countable. This union forms a countable basis for the topology of $F$. So $F$ is perfectly separable.

The following theorem was proved by Debreu [3, theorem II]. Let $F$ be a perfectly separable space and $\succeq$ a transitive, complete and continuous relation on $F$. Then there exists on $F$ a continuous, real-valued function that represents $\succeq$. That establishes my theorem.

*References*

[1] Blackorby, Charles, Walter Bossert and David Donaldson,‘Population ethics and the existence of value functions’, *Journal of Public Economics*, 82 (2001), pp. 301–8.

[2] Blackorby, Charles, Walter Bossert and David Donaldson,‘A representation theorem for domains with discrete and continuous variables’, unpublished.

[3] Debreu, Gerard, ‘Representation of a preference ordering by a numerical function’, in *Decision Processes*, edited by R. M. Thrall, C. H. Coombs and R. L. Davis, Wiley. 1954, pp. 159–65.