If knowledge is defined as justified true belief, then the following equivalence will hold for all sentences $A$:

$$(JTB) \quad K[A] \leftrightarrow JB[A] \land A$$

In this equivalence $K$ and $JB$ are predicates. For sentence $A$, the expression $[A]$ may be understood as a designator for the proposition expressed by $A$ in our fixed language or for the sentence $A$, depending on what kind of objects are assumed to be knowable or justifiable.\(^1\)

The condition $JB[A]$ can be split up further into a belief and a justification condition or just understood as doxastic justification. It is crucial for what follows that $JB[A]$ is understood as true justified belief, not as an arbitrary other condition. Later we invoke a further assumption on $JB$ that we can defend only for justified true belief, but not for other conditions such as ‘warrant’ or some causal condition.

In (JTB) the truth condition is not expressed using a predicate, but only by adding the sentence $A$. Consequently (JTB) becomes a schema. If a predicate for truth were used, (JTB) could be stated as a single universally quantified principle using truth in the sense of truth-theoretic deflationism as generalising device. Here we dispense with the truth predicate in order to dispel any suspicions that

\(^1\)Halbach (2016) defends the use of predicates for knowledge, justification, and belief in the debate about the Gettier problem. We do not rehearse the arguments here.
our reasoning below involves truth-theoretic paradoxes. Our account can be recast with a truth predicate and T-sentences for all truth-free instances.\(^2\)

The left-to-right direction of (JTB)

\[(\text{Nec}) \quad K[A] \rightarrow JB[A] \land A\]

is usually seen as less problematic. In particular the factivity of knowledge (knowledge implies truth), which is entailed by it, is widely taken to be analytic of the concept of knowledge.

Gettier (1963) provides counterexamples against their sufficiency, that is against the right-to-left direction:

\[(\text{Suf}) \quad JB[A] \land A \rightarrow K[A]\]

We produce a new counterexample to (Suf) by reasoning that relies purely on syntactic assumptions. In particular, we assume that there is a closed term \(g\) such that \(g = [\neg Kg]\) is provable. It does not matter the term is obtained using Gödel's diagonal lemma or some other way. Abbreviating \(\neg Kg\) as \(G\), this identity implies the following equivalence:

\[(\text{Knower}) \quad G \leftrightarrow \neg K[G]\]

A sentence \(G\) can be derived in the usual settings using the Gödel diagonal lemma. The sentence is known to lead to contradiction with certain assumptions

\(^2\)There is a certain irony in the fact that we treat truth as a trivial sentential operator and justification and belief as predicates. This is exactly the opposite of Gettier's (1963, p. 121) original formulation:

\[S \text{ knows that } P \iff (i) \text{ } P \text{ is true} \]
\[\text{ (ii) } S \text{ believes that } P, \text{ and} \]
\[\text{ (iii) } S \text{ is justified in believing that } P.\]

The letter \(P\) in Gettier's formulation seems to be a propositional variable, because it is combined with 'that', except for the truth condition (i) where it function as a normal objectual variable. If a propositional variable is used, the truth predicate in (i) becomes superfluous: If \(P\) is a propositional variable, \(P\) by itself is perfectly sufficient as truth condition. We suspect that Gettier added the truth predicate, because it would be disappointing if a definition of knowledge of true justified belief did not contain the word 'true'. Strangely, most epistemologists, for instance Ichikawa and Steup (2014), have stuck to Gettier's unhappy mix of predicates and sentential operators. For a proper explicit definition all notions – truth, justification, belief, and knowledge – ought to be treated as predicates (Halbach 2016). Our version (JTB), in contrast, is to be understood as a schema.

For our arguments only \(K\) has to be a predicate, because Knower cannot be obtained with an operator. \(JB\) in contrast can be replaced with a corresponding sentential operator.
on the knowledge predicate. We do not rely on any specific assumption on the knowledge predicate implicit in (JTB) other than the Factivity of knowledge.

The unproblematic direction (Nec) of (JTB) implies the factivity of knowledge, that is, $K[A] \rightarrow A$.

$$
\begin{align*}
K[G] & \rightarrow G & \text{Factivity} \\
K[G] & \rightarrow \neg G & \text{Knower} \\
(1) & \neg K[G] & \text{two preceding lines} \\
(2) & G & \text{Knower} \\
(3) & JB[G] & \text{crucial assumption}
\end{align*}
$$

The last step labelled (3) is the crucial assumption in our argument. We have proved $G$ and we have come to believe $G$ on the basis of this proof. Moreover, all the premisses in the proof of $G$ – diagonalization and (JTB) – are justified. Therefore we have $JB[G]$. This is an empirical fact: If for whatever reason we failed to believe $G$, $JB[G]$ would not be true. We certainly do not assume a general rule that allows us to infer $JB[A]$ from the availability of a proof of $A$ from justified premisses. For the moment being we assume the step from (2) to (3) is sound. We come back to this point to later.

The sentence $G$ is a Gettier sentence: It is a belief that is true (2) and justified (3), but not known (1). From JTB we have so far used only factivity of knowledge, which follows from its left-to-right direction (Nec). It can be shown that (Nec), (3), and (Knower) are jointly consistent. Assuming also the sufficiency direction (Suf) of (JTB) leads to a contradiction, because as a Gettier sentence $G$ is a counterexample to the sufficiency of true justified belief for knowledge:

$$
\begin{align*}
(4) & \quad JB[G] \land G & \text{from (2) and (3)} \\
(5) & \quad K[G] & \text{(Suf)}
\end{align*}
$$

The last line is a contradiction with (1).

**A Paradox Resolved**

There is an obvious objection one might have against the step from (2) to (3): If the step is acceptable, then one might wonder whether we cannot directly infer $K[G]$ instead of $JB[G]$. Then the entire detour via JTB would not be needed to arrive at a contradiction, because $K[G]$ contradicts (1) and only the factivity of knowledge $K[A] \rightarrow A$ is required for the proof. The resulting paradox, the Montague–Tarski paradox for $K$, is a version of the liar paradox for knowledge: It is well known that the axiom schema $K[A] \rightarrow A$ and the
rule of necessitation (from $A$ conclude $K[A]$) are inconsistent in the presence of diagonalization.\footnote{This is Montague’s Montague (1963) paradox for $K$, which is not be confused with the Kaplan–Montague (1960) paradox. Obviously Montague’s paradox is a slight strengthening of the liar paradox. Since Tarski (1936) was the first to present the liar paradox in a setting as above, we call it the Montague–Tarski paradox. It can be applied to necessity, knowledge, apriority, and various other notions.} Thus, one might argue, our argument shows at best that knowledge is a paradoxical notion or that it is not definable, if one aims to solve the paradox in the same way Tarski (1936) ’solved’ the liar paradox; but it does not show that there is something wrong with (JTB).

We do not find this objection convincing. One can accept the step from (2) to $JB[G]$ without accepting also the step from (2) to $K[G]$. All premises in the proof are justified and the logical steps in the proof preserve justification under a suitable understanding. In general having a correct proof from justified premises does not mean that the conclusion is known. Thus we resolve the Montague–Tarski paradox for $K$ by rejecting the rule of necessitation for $K$: The inference from a proof of $G$ to $K[G]$ is not correct.

Inferring $K[G]$ from (2) only seems plausible because it follows that the belief in $G$ is justified and $G$ does not look like a Gettier sentence: There are no false premises, for instance. However, our argument shows that it is a Gettier example and we cannot conclude $K[G]$ from $JB[G]$ and $G$. We can have necessitation for $JB$. Rejecting (Suf) then blocks the step to $K[G]$. Therefore a rejection of the sufficiency direction (Suf) of (JTB) also solves the Montague–Tarski paradox for $K$.

**The Intensional Argument**

There is a further objection to our crucial assumption: It is empirical and *ad hominem*. Given the strange derivation, nobody may ever believe $G$. In response we weaken our assumption. From $G$ we merely conclude that it is possible to be justified in the belief $G$. This weakened version is sufficient for our argument.

This can be seen as follows. We assume that the metaphysical necessity operator $\Box$ is governed by the S5 rules of propositional modal logic and we formalise necessity as an *operator* in order to dispel the worry that our argument might turn on the well-known difficulties of treating necessity as a predicate. $\Box$ may be thought of as expressing conceptual or metaphysical necessity.

Under this reading the following modalized version of (JTB) ought to be correct, if (JTB) is an adequate definition of knowledge:

\[(JTB+) \quad \Box (K[A] \leftrightarrow JB[A] \land A)\]
This implies the modalized form $\Box(K[A] \rightarrow A)$ of factivity.

By diagonalisation there is a sentence \( G_+ \) such that

\begin{align*}
(\text{Knower}^+) & \quad G_+ \leftrightarrow \neg \Diamond K[G_+] \\
\end{align*}

Then we reason as follows:

\begin{align*}
\neg G_+ & \rightarrow \Diamond K[G_+] & \text{Knower}^+ \\
\neg G_+ & \rightarrow \Diamond G_+ & \text{modalized factivity} \\
\neg G_+ & \rightarrow G_+ & \text{S5} \\
G_+ & \\
\end{align*}

The penultimate line (6) follows because \( G_+ \) is of the form \( \neg \Diamond A \); and in S5 \( \Diamond \neg A \) is equivalent to \( \Diamond A \). In S5 \( \neg A \) is also equivalent to \( \Box \neg \Diamond A \), which yields the following:

\begin{align*}
(8) & \quad \Box G_+ \\
\end{align*}

We now use the modalized version of our crucial assumption: \( G_+ \) is demonstrated from justified premisses, so it can be justifiedly believed:

\begin{align*}
(9) & \quad \Diamond JB[G_+] \\
\end{align*}

Finally, since \( G_+ \leftrightarrow \Box \neg K[G_+] \) holds by (Knower+), line (7) implies the following:

\begin{align*}
(10) & \quad \Box \neg K[G_+] \\
\end{align*}

Together (8), (9), and (10) imply the following in S5:

\begin{align*}
(11) & \quad \Diamond (G_+ \land JB[G_+] \land \neg K[G_+]) \\
\end{align*}

This means there can be a counterexample to (JTB). This is inconsistent with the modalized version (JTB+) of (JTB).

**Conclusion**

Our Gettier sentences \( G \) and \( G_+ \) refute the adequacy of the traditional tripartite definition of knowledge as true justified belief. We do not suggest that they can be used to refute the adequacy of arbitrary other definitions of knowledge. In particular, if justification is replaced with another condition, our arguments does not necessarily go through: If JB is substituted with some other notion \( W \),
it is at least not clear why we should be able to conclude \( W[G] \) in what the called the crucial assumption; it is also not clear why we should be able to infer the modal variant \( W[G_+] \) from (7). In fact, if our arguments applied to arbitrary conditions, they would overgenerate: They would imply that we have counterexamples against (JT) and (JT+) with \( J \) replaced with \( K \) (see Huemer 2005 and Halbach 2016). But clearly a counterexample to \( K[A] \leftrightarrow K[A] \land A \) would be a counterexample to the factivity of knowledge.

Whether our arguments can be applied to a given definition of knowledge depends on whether the analogues of the transition from (2) to (3) can be vindicated, that is, from \( G \) to \( JB[G] \) (or their modal counterparts). For instance our reasoning does apply to no false lemmata solutions (Clark 1963), because the derivations of (3) and (9) do not contain false lemmata.

The general upshot of our paper is that any adequate solution of the Gettier problem that implies \( \Box (K[A] \leftrightarrow W[A] \land A) \) for some condition \( W \) will have to block the move analogous to the step from (7) to (9) on top of handling all the other known Gettier examples.

BIBLIOGRAPHY


