Market Selection with Multiple Equilibria

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September 14, 2018

Abstract

According to the “market selection hypothesis”, markets select for traders with more accurate beliefs, but this ignores the effect of beliefs on markets. I model market selection in general equilibrium when the economy has multiple equilibria. Given a particular equilibrium, I show conditions under which the evolution of agents’ consumption shares is described by the replicator dynamics. I then define a noisy process that moves between multiple equilibria, and that can be used to select between them as noise vanishes—an evolutionary-game-theoretic technique that I apply to select the equilibrium at target inflation in a standard Taylor-rule model. Journal of Economic Literature Classification: C72, D84, E31, E58.

Key Words: rational expectations; market selection; evolutionary game theory; replicator dynamics; equilibrium selection; Taylor rule.

*The paper benefited from discussions with Francesca Arduini, Camilla Egginton, Martin Ellison, Narayana Kocherlakota, Michael Mandler, Heinrich Nax, Jonathan Newton, Joe Perkins, Bary Pradelski, Ricardo Reis, Helene Rey, Larry Samuelson, Bill Sandholm, Harald Uhlig, Adrien Vigier, Michael Woodford and Peyton Young. Email thomas.norman@magd.ox.ac.uk.
1 Introduction

The questions of whether and which rational-expectations equilibria will occur are central to the macroeconomic application of general-equilibrium models, but they are questions that are difficult to address in the standard setup of homogeneous expectations. With heterogeneous agents, Sandroni (2000) and Blume and Easley (2006) offer a foundation for the “market selection hypothesis” (Alchian 1950, Friedman 1953, Cootner 1964, Fama 1965) that market forces will lead rational traders to flourish at the expense of irrational ones. They show that, if agents are equally patient and at least one has rational expectations, then a complete-markets economy satisfying certain conditions will eventually be dominated by correct beliefs.¹ ² The true path of the economy is exogenous in their models, but in practice may not be unique, as there may be multiple equilibria owing to heterogeneity (Shapley and Shubik 1977, Bergstrom, Shimomura, and Yamato 2009) or the imposition of side conditions such as a Taylor rule (Kehoe, Levine, and Romer 1992). Hence, whilst they address the question of whether rational-expectations equilibria will be played, they remain silent on which.

Here, I seek to select between multiple rational-expectations equilibria by extending the Blume–Easley model to allow it to move in and out of such equilibria, and measuring the likelihood of each. Given a particular equilibrium path, I establish in Section 2 a connection with evolutionary game theory by exhibiting a special case where the consumption shares of different beliefs evolve according to the “replicator dynamics” (Taylor and Jonker 1978). Under this selection process, beliefs flourish if and only if they outperform the economy’s average belief in their ability to predict the evolving state. I then introduce noise into the consumers’ endowments in Section 3, allowing the economy to move in and out of the various equilibria over time. Weak conditions on such noise yield an ergodic distribution on the economy’s Negishi weights, which can be used to select the most likely equilibrium in the long run as the noise vanishes—an evolutionary game-theoretic selection technique known as “stochastic stability” (Foster and Young 1990).

I then apply this model in Section 4 to a dynamic complete-markets exchange economy with multiple equilibria, and to the problem of determinacy under Taylor rules. For the latter, McCallum (1981) suggests that an interest-rate target that varies endogenously with economic conditions might overturn Sargent and Wallace’s (1975) clas-

¹There are important limits to this market selection; even in complete markets, when discount factors differ, patience can compensate for the effect of bad forecasts by inducing higher savings. More generally, Kogan, Ross, Wang, and Westerfield (2006, 2017) provide conditions under which irrational traders survive in complete markets, and show that even if they vanish, they can still have an impact on asset prices.

²As well as providing an important theoretical benchmark, the case of complete markets should not necessarily be dismissed empirically (Kreps 1982, Duffie and Huang 1985, Rios-Rull 1996, Krusell and Smith, Jr. 1998, Levine and Zame 2002, Badel and Huggett 2014).
sic price-level indeterminacy, and Clarida, Gali, and Gertler (2000) provide evidence supporting the view that an “active” Taylor rule that responds disproportionately to off-target inflation accomplishes this. However, Cochrane (2011) argues that the unique “locally bounded” equilibrium delivered by such a rule remains indeterminate in the face of a continuum of explosive equilibria—a most profound equilibrium-selection problem. Here, I find that the locally bounded solution favored by New Keynesians is selected over the explosive equilibria unless either beliefs or noise are asymmetric with respect to target inflation.

Multiple equilibria are endemic in the rational expectations literature, and there have been many attempts to select between them. In infinite-horizon models, a popular equilibrium-selection criterion is “stability” (Blanchard and Kahn 1980, Obstfeld and Rogoff 1986), in the sense that the model’s variables converge to a stationary state, often owing to transversality conditions. But some models have multiple stable equilibria (Calvo 1978), others have none that are stable (McCallum 1999, pp. 625–6), and others still have nominal explosions that transversality conditions cannot rule out (Cochrane 2011). A second criterion requires that a solution be derived under the assumption that agents’ expectations are a function of a minimal set of state variables (Wallace 1980, McCallum 1983), though the basis for this assumption and its resultant exclusion of “non-fundamental” solutions is open to question. More recently, the criterion of “expectational stability” or “learnability” has been proposed (Evans and Honkapohja 2001), and used by McCallum (2009) to select the level of inflation targeted by a Taylor rule, although Cochrane (2009) contests this result. As Sections 3 and 4 show, market selection offers another avenue for selection between multiple rational-expectations equilibria.

2 Market Selection

Blume and Easley (2006) analyze an infinite-horizon general-equilibrium model in which consumers allocate their wealth across states of the world each period. Whereas a standard rational-expectations model would equip a representative agent with correct beliefs about the evolving state of the world, they allow heterogeneity in beliefs, which then flourish or diminish according to the consumption success that they bring. These beliefs need not be consistent with the true process generating states of the world, nor with common knowledge of all consumers’ beliefs. In particular, price-taking

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3 For a review, see Driskill (2006).
4 Transversality conditions rule out explosions in real variables by requiring their present value to converge to zero as time goes to infinity.
5 Such common knowledge would imply well-known “no-trade” results (Milgrom and Stokey 1982), rendering the market selection of beliefs fundamentally problematic.
6 The economy will thus be in “temporary equilibrium” (Hicks 1939, Grandmont 1977), as in Garcia-
consumers need not acquire any information about each others’ beliefs from market prices; such inferences and their resulting information aggregation are instead captured in the notion of rational expectations (Lucas 1972). However, the path of the economy is assumed exogenous and unique by Blume and Easley; here, I explore their model when there may be multiple such equilibrium paths.

2.1 The model

There is a space $S$ of states of the world, with sequences of states in discrete time denoted $\sigma = (\sigma_0, \ldots) \in \Sigma$ and called paths of the economy. A state of the world here is in the Arrow–Debreu decision-theoretic sense of a resolution of ex ante uncertainty, rather than the macroeconomic (or evolutionary-game-theoretic) sense of “state” as a sufficient statistic for the evolution of the process. The “true” probability measure on the measurable space $\Sigma$ (together with its product sigma-field) is denoted $p$. Let $\sigma^t = (\sigma_0, \ldots, \sigma_t)$ denote the partial history through date $t$ of the path $\sigma$, $\Sigma^t$ the set of such $t$-length partial histories, and $\mathcal{F}_t$ the product sigma-field of events measurable at date $t$. For any probability measure $q$ on $\Sigma$, $q_t(\sigma) = q(\{\sigma_0 \times \cdots \times \sigma_t\} \times S \times S \times \cdots)$ is the (marginal) probability of the partial history $\sigma^t$, and $q_t(\sigma|\sigma^{t-1}) = q(\{\sigma_0 \times \cdots \times \sigma_t\} \times S \times S \times \cdots | \sigma^{t-1})$ is the (conditional) probability of the state $\sigma_t$ following $\sigma^{t-1}$.

An economy contains $I$ consumers, each with consumption set $\mathbb{R}_+$. A consumption plan $c^i : \Sigma \to \prod_{t=0}^\infty \mathbb{R}_+$ is a sequence of $\mathcal{F}_t$-measurable $\mathbb{R}_+$-valued functions $\{c^i_t(\sigma)\}_{t=0}^\infty$. Consumer $i$’s endowment stream is a particular known consumption plan denoted $\omega^i$, her beliefs the probability measure $p^i$ on $\Sigma$, and her utility function

$$U_i(c) = E_{p^i}\left\{\sum_{t=0}^\infty \beta^i_t u^i_\sigma(c^i_t(\sigma))\right\},$$

where $\beta^i_t \in (0, 1)$ is a discount factor and $u^i_\sigma : \mathbb{R}_+ \to [-\infty, \infty)$ is a payoff function on consumptions. The function $u^i_\sigma$ may be path- (and hence state-) dependent, but since this will not arise until Section 4, for now I adopt the simpler notation $u^i$. I will say that consumer $i$ has rational expectations if $p^i = p$.

Like Blume and Easley, I exploit the fact that, if $c^* = (c^{1^*}, \ldots, c^{J^*})$ is a Pareto-optimal allocation of resources, then there is a vector of welfare weights $(\lambda^1, \ldots, \lambda^I) \gg 0$ Schmidt and Woodford (2018) (but with market selection of beliefs).

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8This is a relaxation of Blume and Easley’s assumption of a finite state space.
9Note that this utility function relies implicitly on the consumer’s satisfaction of, e.g., the Savage (1954) axioms.
10This is stronger than the concept of rational expectations originally assumed in the macroeconomic literature (Muth 1961), which merely required the first moment of beliefs to be correct—see, e.g., Shiller (1978), p. 4.
such that \( c^* \) solves the problem

\[
\max_{(c^1, \ldots, c^I)} \sum_i \lambda_i U^i(c),
\]

s.t.

\[
\sum_i c^i - \omega \leq 0
\]

\[
\forall t, \sigma, i \quad c^i_t(\sigma) \geq 0,
\]

where \( \omega_t = \sum_i \omega_t^i \).\(^{11}\)

The following assumptions are basic to the model:

**Axiom 1** (i) The payoff functions \( u^i : \mathbb{R}_+ \rightarrow [-\infty, \infty) \) are \( C^1 \), strictly concave, and strictly monotonic. (ii) The payoff functions \( u^i \) satisfy an Inada condition at 0; that is, \( u^i(c) \rightarrow \infty \) as \( c \downarrow 0 \).

**Axiom 2** We have \( \infty > F = \sup_{t, \sigma} \sum_i \omega_t^i(\sigma) \geq \inf_{t, \sigma} \sum_i \omega_t^i(\sigma) = F > 0 \).

These are two of Blume and Easley’s first three axioms.\(^{12}\) The Inada condition rules out corner solutions, guaranteeing that Pareto-optimal consumption of each trader is either \( p^i \)-almost surely positive or almost surely zero, and is discussed in Blume and Easley’s Section 4. Note that I do not assume their Axiom 3—that if a partial history is possible, all consumers believe it to be possible—as I do not require market-clearing and do not wish to rule out the possibility of consumers vanishing in finite time; in this paper, vanished consumers will be able subsequently to return to positive consumption when I introduce noise to endowments in Section 3.

The first-order conditions for the solution to (2) are, for all \( \sigma \) and \( t \), as follows:

1. there is a number \( \theta_t(\sigma) > 0 \) such that if \( p_t^i(\sigma) > 0 \), then

\[
\lambda_i \beta_t^i u^i(c_t^i(\sigma)) p_t^i(\sigma) - \theta_t(\sigma) = 0;
\]

2. if \( p_t^i(\sigma) = 0 \), then \( c_t^i(\sigma) = 0 \).

I assume equal discount factors \( \beta_i = \beta_j = \beta \), under which these conditions imply that, if \( i \) and \( j \) have not vanished, then

\[
\frac{u^i(c_t^i(\sigma))}{u^j(c_t^j(\sigma))} = \frac{\lambda_j p_t^i(\sigma)}{\lambda_i p_t^i(\sigma)}.
\]

\(^{11}\)Note that the first constraint here implies free disposal and nonstorable endowments.

\(^{12}\)For failures of market selection absent Blume and Easley’s conditions, see Kogan, Ross, Wang, and Westerfield (2006). See also Massari (2013).
(2) is a standard Negishi (1960) problem, the solution to which characterizes the consumption dynamics of a complete-markets exchange economy in competitive equilibrium by the First Welfare Theorem. Thus, many standard models fit within this framework. For instance, in Sandroni’s (2000) analysis of the Lucas (1978) “tree” model, the state of the world determines the value of the dividend produced by the trees, whilst a consumer’s endowment stream is determined by her share of claims to those (non-storable) dividends. However, I single out the following example that will have an important role in the rest of the paper.

**Example** The model captures a heterogeneous version of the complete-markets cashless economy of Woodford (2003, §2.1)—perhaps the simplest infinite-horizon model with complete markets and money. In particular, let a *money plan* and an *asset plan* be sequences \( m^i_t : \Sigma \to \prod_{t=0}^{\infty} \mathbb{R}_+ \) and \( b^i_t : \Sigma \to \prod_{t=0}^{\infty} \mathbb{R}_+ \) of \( \mathcal{F}_t \)-measurable \( \mathbb{R}_+ \)-valued functions \( \{m^i_t(\sigma)\}_t \) and \( \{b^i_t(\sigma)\}_t \) capturing respectively consumer \( i \)'s nominal holdings of money (the economy’s unit of account) and of all other financial assets. If \( \gamma_t : S \to [0, \infty] \) is the relative state price of consumption in period \( t \), and \( \varsigma_s^t(\sigma) \) is the cylinder set \( \{\sigma^{t-1} \times s\} \times S \times S \times \cdots \), Arrow–Debreu equilibrium requires that each consumer \( i \) solve the problem

\[
\max_{c^i, m^i, b^i} \mathbb{E}_{p^i} \left\{ \sum_{t=0}^{\infty} \beta^i_t u^i(c^i_t(\sigma)) \right\}, \quad \text{s.t. } \forall t, \sigma, i:\n
\sum_{s \in S} \gamma_t(s)c^i_t(\varsigma^s_t(\sigma)) + m^i_t(\sigma) + b^i_t(\sigma) \leq \sum_{s \in S} \gamma_t(s)\omega^i_t(\varsigma^s_t(\sigma)) + a^i_t(\varsigma^s_t(\sigma));
\]

\[
\frac{\partial}{\partial m^i_{t-1}} u^i(c^i_t(\sigma)) p^i_t(\sigma) = 0,
\]

for given \( m^i_{t-1} \), and where \( a^i_t(\varsigma^s_t(\sigma)) \) is the total nominal value of all assets (including money) at the beginning of period \( t \) in state \( s \) (given \( \sigma^{t-1} \)). The first-order conditions for the optimal choice of consumption are, for all \( \sigma, t \) and \( i \):

1. there is a number \( \alpha_i > 0 \) such that if \( p^i_t(\sigma) > 0 \), then

\[
\beta^i_t u^i(c^i_t(\sigma)) p^i_t(\sigma) - \alpha_i p^i_t(\sigma) D_{0,t}(\sigma) \gamma_t(\sigma_i) = 0,
\]

where \( D_{0,t} \) is the nominal stochastic discount factor between periods 0 and \( t \);

2. if \( p^i_t(\sigma) = 0 \), then \( c^i_t(\sigma) = 0 \).

\[13\] See, for instance, Kehoe and Levine (1985, p. 437), Kehoe (1989, p. 368) or Kehoe, Levine, and Romer (1992, §2). Negishi aggregation of course only captures equilibrium behavior, and is likely inadequate for welfare analysis (see, e.g., Mas-Colell, Whinston, and Green 1995, §4.D), but it is quite sufficient for this paper’s equilibrium-characterization purposes.
Under equal discount factors, $\beta_i = \beta_j$, it follows that
\[
\frac{u^i(c^i_t(\sigma))}{u^j(c^j_t(\sigma))} = \frac{\alpha_i p^i_t(\sigma)}{\alpha_j p^j_t(\sigma)},
\]
coinciding with the solution to (2) with welfare weights $\lambda_i = 1/\alpha_i$. Negishi provided an algorithm for finding these welfare weights, using savings functions that measure the degree to which budget constraints do not bind, and iteratively searching for their zeros.

**Remark 1** For a given endowment stream, equilibrium in this framework exists (Bewley 1972). It may also appear to be unique: each $u^i$ is assumed strictly concave, a sum of strictly concave functions is strictly concave, and no single strictly concave maximization problem can have multiple solutions (by Berge’s maximum theorem). But even with strict concavity of $u^i$ there may be multiple equilibria, as in the (static-economy) heterogeneous quasilinear-utility examples of Shapley and Shubik (1977), Mas-Colell, Whinston, and Green (1995, p. 521) and Bergstrom, Shimomura, and Yamato (2009), and the heterogeneous CES-utility example of Kehoe (1991, pp. 2066-7). This is captured by the possibility of multiple equilibrium welfare weights in (2).

For the rest of this section, I assume logarithmic utility $u^i(c) = \ln c$, so that (3) becomes
\[
\frac{c^i_t(\sigma)}{c^j_t(\sigma)} = \frac{\lambda_j p^i_t(\sigma)}{\lambda_i p^j_t(\sigma)}.
\]
Such an equation holds for each unordered pair of consumers, giving $I!/2(I-2)!$ equations, of which $I - 1$ are independent, the solution to which has
\[
x^i_t(\sigma) = \lambda_i \left( \frac{p^i_t(\sigma)}{\overline{p}_t(\sigma)} \right),
\]
where $\overline{p}_t(\sigma) \equiv \sum_i \lambda_ip^i_t(\sigma)$ and $x^i_t(\sigma) \equiv c^i_t(\sigma)/\sum_{j=1}^I c^j_t(\sigma)$. The population profile $x_t(\sigma) \equiv (x^1_t(\sigma), \ldots, x^I_t(\sigma))$ of consumption shares of the consumers in period $t$ belongs to the $(I - 1)$-dimensional simplex. This population profile will correspond to the usual macroeconomic (or evolutionary-game-theoretic) notion of the “state” of the process qua a sufficient statistic for its evolution; in this paper, the “state” terminology is reserved for the realizations of uncertainty in the model.

\[14\] Dana (1993) gives sufficient conditions for uniqueness in complete markets (restricting coefficients of relative risk aversion to be less than 1).
Then
\[ x_i^t(\sigma) = \frac{\lambda_i p_i^t(\sigma|\sigma^{t-1}) p_{i-1}^t(\sigma)}{\bar{p}_i(\sigma|\sigma^{t-1}) \bar{p}_{t-1}(\sigma)} = \frac{p_i^t(\sigma|\sigma^{t-1})}{\bar{p}_i(\sigma|\sigma^{t-1})} x_{t-1}^i(\sigma). \] (6)

Intuitively, if a consumer’s conditional belief on next period’s state of the world is more accurate (in the sense of giving higher probability to the realized state) than the weighted-average belief \( \bar{p} \), then her consumption share grows. This sounds quite similar to the replicator dynamics (Taylor and Jonker 1978), but the weighted average here is timeless rather than evolving, and the beliefs condition on the entire partial history up to period \( t-1 \). However, the former issue is moot, as we can see by letting \( \hat{p}_t(\sigma|\sigma^{t-1}) \equiv \sum_i x_{t-1}^i(\sigma)p_i^t(\sigma|\sigma^{t-1}) \) be the evolving consumption-weighted-average conditional belief on the period-\( t \) state of the world:

**Lemma 1** For all \( t \) and \( \sigma \), \( \bar{p}_t(\sigma|\sigma^{t-1}) = \hat{p}_t(\sigma|\sigma^{t-1}) \).

The proof is immediate:
\[
\bar{p}_t(\sigma|\sigma^{t-1}) \equiv \frac{\bar{p}_t(\sigma)}{\bar{p}_{t-1}(\sigma)} = \frac{\sum_i \lambda_i p_i^t(\sigma|\sigma^{t-1}) p_{i-1}^t(\sigma)}{\bar{p}_{t-1}(\sigma)} = \sum_i x_{t-1}^i(\sigma)p_i^t(\sigma|\sigma^{t-1}) \equiv \hat{p}_t(\sigma|\sigma^{t-1}).
\]

Intuitively, if a consumer’s current consumption share is high, it is because she was more accurate than \( \bar{p} \) in predicting the current state. Note that \( \bar{p}_t(\sigma|\sigma^{t-1}) \) need not equal \( \sum_i \lambda_i p_i^t(\sigma|\sigma^{t-1}) \); instead, we can see from the second line above that it is a weighted average of the \( p_i^t(\sigma|\sigma^{t-1}) \)'s, but with weights of \( \lambda_i p_{i-1}^t(\sigma)/\bar{p}_{t-1}(\sigma) \). That these weights are equal to the consumption shares \( x_{t-1}^i(\sigma) \) is remarkable and not at all obvious a priori.

### 2.2 Evolutionary dynamics

To this point, I have only generalized the Blume and Easley framework, but in this subsection I impose two new axioms to give the process a Markov structure. I then establish a connection with evolutionary game theory by showing conditions under which the model’s evolving consumption shares are described by the replicator dynamics.

**Axiom 3** Suppose that there are (time-homogeneous) Markovian beliefs, in the sense that, for any \( p^i \in P \equiv \{p^1, \ldots, p^I\} \), there exists a conditional probability measure \( \rho^i \) such that, for all \( t \geq 1 \) and all \( \sigma \), \( p_i^t(\sigma|\sigma^{t-1}) = \rho^i(\sigma_t|x_{t-1}(\sigma), \sigma_{t-1}) \).
In words, consumer beliefs on next period’s state of the world must depend only on the current consumption shares and state of the world. This is a restriction on the set $P$ of possible beliefs, which is exogenously given as part of the description of the economy. It is, however, consistent with the following assumption.

**Axiom 4** For all $t$ and conditional on the set of beliefs $P$, suppose that the state of the world $\sigma_t \in S$ is determined by a (possibly unknown) time-homogeneous Markovian state $T : (x_{t-1}(\sigma), \sigma_{t-1}) \mapsto \Delta(S)$, where $\Delta(S)$ is the set of probability measures on $S$.

For instance, the state of the world $\sigma_t$ might be a function of $\hat{E}_t \sigma_t \equiv E_{\hat{p}}(\sigma_t | x_{t-1}(\sigma), \sigma_{t-1})$, the expected state in period $t$ under $\hat{p}_t(\sigma|\sigma^{t-1}) = \hat{\rho}(\sigma_t|x_{t-1}(\sigma), \sigma_{t-1}) = \sum_i x_{i-1}(\sigma) \rho^i(\sigma_t|x_{t-1}(\sigma), \sigma_{t-1})$, as is the case with inflation in the New Classical aggregate-supply relation (see, e.g., Woodford 2003, p. 159). Or $\sigma_t$ might be a function of $\hat{E}_t \sigma_{t+1}$, as in the New Keynesian Taylor-rule model explored in Section 4.15 Whilst thus appealing, the imposition of such a Markovian state raises issues of equilibrium existence and multiplicity, discussed further in Appendix A. Meanwhile, Appendix B discusses conditioning the state and beliefs on the current state alone—a technique that I will exploit in Section 3.

The function $T$ is said to be *deterministic* if, for all $t$ and all $\sigma$, it puts probability 1 on a particular state of the world (that may depend on $t$ and $\sigma$).

**Proposition 1** If $T$ is deterministic and the state $\sigma_t$ depends on the current state $\sigma_{t-1}$ only via the consumption shares $x_{t-1}(\sigma)$, then the population profile $x_t(\sigma)$ evolves according to the replicator dynamics.

The proof is again immediate; under the conditions of the proposition, some state $s(x_{t-1}(\sigma)) \in S$ occurs for sure in period $t$, so that (6) becomes

$$x^i_t(\sigma) = \frac{\rho^i(s(x_{t-1}(\sigma))|x_{t-1}(\sigma))}{\hat{\rho}(s(x_{t-1}(\sigma))|x_{t-1}(\sigma))} x^i_{t-1}(\sigma). \quad (7)$$

This equation defines the discrete-time replicator dynamics for a hypothetical *belief game* $G$, with:

- player set consisting of the economy’s $I$ consumers;

- common action set $P$, from which each player $i$ “chooses” $\rho^i$; and

- payoffs of $\rho^i(s(x)|x)$.

15 The mapping $T$ may or may not be founded in optimizing behaviour; for instance, government policy may affect the state of the world, and is often posited as an exogenous influence on macroeconomic models, as in Subsection 4.2 below.
To be clear, this game is not actually played, but is merely a useful device for characterizing the dynamics of (7).\footnote{The issue of equilibrium existence manifests itself here, as the belief game $G$ may have no Nash equilibrium. Since $S$ is finite, $s(\cdot)$ must be either constant or discontinuous, so that for nontrivial economies the payoff functions in the belief game $G$ will be discontinuous, and hence equilibrium existence may fail.} As in the standard replicator dynamics, player $i$’s problem is like facing a single opponent with mixed strategy $x$, but unlike the usual random-matching setting, player $i$’s expected payoffs need not be linear in $x$; we have a “playing the field” game (Maynard Smith 1982) where an action’s expected payoffs depend on some property of the whole population, here the (possibly nonlinear) state $s(\cdot)$.

**Remark 2** In the literature on “optimal beliefs” (Brunnermeier and Parker 2005, Brunnermeier, Gollier, and Parker 2007), agents choose their beliefs to their best advantage, but in the absence of effects arising from the interaction of beliefs, and where payoffs are exogenous utilities. Strategic interaction in belief choice is explored by Jouini, Napp, and Viossat (2013), who analyze the consequences of the replicator dynamics operating on beliefs in a general-equilibrium setting, again with exogenous utilities. They assume that beliefs flourish if they outperform average beliefs in the sense of giving higher expected utility in the resulting Walrasian equilibrium. This is reasonable if beliefs are determined by evolution and utilities coincide with biological fitnesses, or if agents imitate others’ beliefs (Björnerstedt and Weibull 1996, Schlag 1998) or choose myopically optimal beliefs (Hofbauer, Sorin, and Viossat 2009). Here, by contrast, I consider the case where beliefs are determined neither by evolution nor by choice, but instead are selected by the market. I have derived rather than assumed the replicator dynamics as a description of the market selection of beliefs, and the relevant game on which it acts is not one of utilities or biological fitnesses, but the probabilities that the beliefs assign to the realized path of the economy.

What is the importance of the assumptions here? The Markovian structure and deterministic $s$ give a deterministic Markovian evolutionary dynamic and time-homogeneous payoffs, whilst the assumed logarithmic utility and dependence of $s$ only on $x_{t-1}(\sigma)$ give the precise form of the replicator dynamics.\footnote{Whilst the “population” of beliefs is finite, the space of consumption shares is uncountable, as under the replicator dynamics.} Heterogeneous logarithmic utility functions would give a rescaled replicator dynamics, whereas other forms of utility would give different evolutionary dynamics. This provides a connection with evolutionary game theory on which I build in the next section.
3 Stochastic Stability

The model of the previous section describes the dynamics of consumption shares within a particular equilibrium, fixed by the welfare weights of the Negishi problem (2). In this section, I introduce noise in the consumers’ expected shares of the endowment stream, and thus in their Negishi weights. This yields an ergodic process that moves in and out of multiple equilibria over time, with a frequency captured by a unique invariant distribution. As the noise vanishes from the model, this distribution generically collapses on a single equilibrium, selecting it as infinitely more likely than any other. This evolutionary game-theoretic technique identifies the “stochastically stable” equilibrium (Foster and Young 1990).

To begin with, I construct a perturbed Markov chain \( \Xi_\varepsilon = \{\Xi^t_\varepsilon\}_{t=0,1,2,...} \) on the space of Negishi weights to describe dynamics in and out of equilibrium. Let \( \Lambda \equiv \{\kappa \in (0, \iota, 2\iota, \ldots, 1)^I : \sum_i \kappa_i = 1\} \), \( \iota > 0 \), be a discretization of the \((I-1)\)-dimensional simplex, to which the welfare weights \((\lambda_i)_{i=1,...,I}\) belong. The discretization is for analytical convenience, and there would be no substantive difference (aside from greater complexity) in the results to come if I used an infinite \( \Lambda \).\(^{18}\)

As we established in Subsection 2.1, the welfare weights are sufficient for characterizing the model’s equilibria. However, disequilibria cannot be captured by the solution to the Negishi problem (2), and so it is not immediately obvious that disequilibrium points of \( \Lambda \) have any meaning. Nevertheless, if we consider the Woodford cashless economy from Subsection 2.1, it is clear that the \( \alpha_i \)'s from the consumers’ utility maximization problems still exist in disequilibrium. I established there that \( \lambda_i = 1/\alpha_i \) in equilibrium, and hence I define \( \lambda_i \) to be \((1/\alpha_i)/\sum_{j=1}^I 1/\alpha_j\) in disequilibrium. Really then, this section models the evolution of the consumers’ respective Lagrange multipliers, in and out of equilibrium.

The unperturbed Markov chain \( \Xi = \{\Xi^t\}_{t=1,2,...} \) is defined on \( \Lambda \), and constructed from a set of transition probabilities \( \mathbb{P} = \{\mathbb{P}_{jk} : j, k \in \Lambda\} \), where \( \mathbb{P}_{jk} \) is a probability measure on \( \Lambda \).

**Axiom 5** For any \( \lambda \in \Lambda \), \( \mathbb{P}_{\lambda\lambda} = 1 \) if \( \lambda \) is an equilibrium set of welfare weights.

Thus, equilibrium welfare weights are absorbing under \( \Xi \). More specific processes can be adopted—as I do in Section 4 below—but this is all that I assume of \( \Xi \) in general.

Next I perturb \( \mathbb{P} \) to obtain a set \( \mathbb{P}^\varepsilon \) of perturbed transition probabilities. Let \( L \) be the cardinality of \( \Lambda \).

**Axiom 6** For all \( \varepsilon > 0 \) and all \( \lambda, \kappa \in \Lambda \), \( \mathbb{P}^\varepsilon_{\lambda\kappa} \geq L^{-1}\varepsilon^{||\kappa - \lambda||} \).

Thus, whatever the current welfare weights, there is some chance that each \( \kappa \in \Lambda \) will occur in the next period under the perturbed process. How are we to interpret this

\(^{18}\)See Newton (2015), for instance.
perturbed process? Looking at (15) in Appendix B, the welfare weight of consumer $i$ is determined by her expected share of the intertemporal endowment, to which the perturbation thus represents a random shock (or “mutation”), with larger shocks (as captured by the Euclidean distance between $\lambda$ and $\kappa$) less likely.

We are interested in the case in which

$$
\lim_{t \to \infty} \max_{\kappa \in \Lambda} |P^t_{\lambda \kappa} - \mu(\kappa)| = 0,
$$

where $\mu$ is an invariant measure of the process, i.e., a measure on $\Lambda$ with the property

$$
\mu(\kappa) = \sum_{\lambda \in \Lambda} \mu(\lambda)P_{\lambda \kappa}, \quad \kappa \in \Lambda.
$$

If this limit holds, then the long-run behaviour of $\Xi$ is described by the invariant measure $\mu$, independent of the initial measure from which the process starts: If, for any initial measure $\xi$,

$$
\left| \sum_{\lambda \in \Lambda} \xi(\lambda)P^t_{\lambda \kappa} - \mu(\kappa) \right| \to 0, \quad t \to \infty,
$$

then $\mu$ is said to be ergodic.

**Lemma 2** The perturbed process $\Xi_\varepsilon$ has a unique invariant measure $\mu_\varepsilon$ on $\Lambda$, which is ergodic and has a limiting measure $\mu$ as $\varepsilon \to 0$. $\mu$ is an invariant measure of the unperturbed process $\Xi$, and $\mu(\Lambda) = 1$.

These are standard results in the stochastic stability literature (see, e.g., Young 1998), following from $\Xi_\varepsilon$’s irreducibility and aperiodicity. The noise leads the process to move over the whole space—in and out of each fixed point $\phi \in \Phi$ of (9) in Appendix A—and the long-run frequency of each equilibrium is described by $\mu$.

**Remark 3** Of course, $\mu$ gives Blume and Easley’s “true” $p$, so that if we assumed their axioms, their results would hold here without having to specify the prevailing equilibrium. In particular, we would have their Corollaries 1 and 2: that a trader with rational expectations survives $p$-almost surely; and that a trader who survives almost surely in the presence of rational expectations must have a belief with respect to which $p$ is absolutely continuous. The latter means that, in the presence of rational expectations, surviving beliefs must be asymptotically almost surely correct.\(^{19}\)

But we can go further than this and use $\mu$ to select between multiple equilibria of $\Xi$: If $\lambda$ has positive weight in the limiting measure, $\mu(\lambda) > 0$, then it is called stochastically

\(^{19}\)See Kalai and Lehrer (1994b) for the equivalence of absolute continuity with this “merging” of probability measures, applied to a macroeconomic setting in Kalai and Lehrer (1994a), and discussed in Blume and Easley (2006, p. 942).
stable (Foster and Young 1990), and is our prediction for long-run equilibrium. Moreover, there exist techniques in the stochastic stability literature for easily identifying such equilibria. Specifically, let $\Omega$ be the recurrent classes of $\Xi$, and $L = \{1, \ldots, \bar{L}\}$ index the elements of $\Omega$. A graph on $L$ is an $l$-graph if each $k \neq l$ has a single exiting directed edge, and the graph has no cycles. Let $\mathcal{G}(l)$ denote the set of all $l$-graphs. Letting $j \rightarrow k$ denote a directed edge from $j$ to $k$, define:

$$V(l) = \min_{g \in \mathcal{G}(l)} \sum_{(j \rightarrow k) \in g} V(j,k),$$

$$L_{\min} = \{l \in L : V(l) = \min_{k \in L} V(k)\},$$

where the resistance $V(j,k)$ measures the difficulty of moving from $j$ to $k$, in the sense that its probability is at most $\varepsilon^{V(j,k)}$.

**Lemma 3** A recurrent class $j \in \Lambda$ is stochastically stable if and only if it belongs to $L_{\text{min}}$.

In words, an equilibrium is stochastically stable if and only if it has the minimum-resistance $l$-graph among $l \in L$. This is again a standard result in the stochastic stability literature.

### 4 Applications

In this section, I apply the model developed above to two economies with multiple equilibria; the first owing to heterogeneous utility functions amongst the consumers, and the second owing to the added side condition of a Taylor rule.

#### 4.1 Multiple equilibria under heterogeneous utility

I begin with an example adapted from Kubler and Schmedders (2002), of an infinite-horizon exchange economy with complete markets but multiple equilibria. There are two consumers with common discount factor $\beta_1 = \beta_2 = 0.75$. There are three states of the world $S = \{1, 2, 3\}$, which determine individual endowments $\omega_t^1 = (\omega^1_t(\varsigma_1^1(\sigma)), \omega^1_t(\varsigma_2^1(\sigma)), \omega^1_t(\varsigma_3^1(\sigma))) = (4, 12, 1)$ and $\omega_t^2 = (4, 1, 12)$ for all $t$ and $\sigma$. The consumers have state-dependent utility functions $u^i_t = \zeta^i e^{1-\eta}/1 - \eta$, with multipliers $(\zeta_1^1, \zeta_2^1, \zeta_3^1) = (1, 1024, 1)$ and $(\zeta_2^1, \zeta_2^2, \zeta_3^2) = (1, 1, 1024)$, and coefficient of relative risk aversion $\eta = 5$. The transition probabilities are iid, with each state equally likely, i.e. $T(x_{t-1}(\sigma), \sigma_{t-1}) = (1/3, 1/3, 1/3)$ for all $t$ and $\sigma$. There are three Arrow securities, with dividend vectors $d_1 = (d_1(1), d_1(2), d_1(3)) = (1, 0, 0)$, $d_2 = (0, 1, 0)$ and $d_3 = (0, 0, 1)$, of which the consumers possess no initial endowment.
Consumer $i$'s first-order conditions in this case imply that

$$c_i^t(\sigma) = \left( \frac{0.75^t c_i^t p_i^t(\sigma)}{\alpha_t p_t(\sigma) D_{0,t}(\sigma) \gamma_i(\sigma_t)} \right)^{0.2}.$$  

Substituting this into the market-clearing condition,

$$\gamma_i(\sigma_t) = \frac{0.75^t}{p_t(\sigma) D_{0,t}(\sigma)} \left( \frac{\sum_{j=1}^I (\lambda_j c_{i,j} p_{j,t}(\sigma))^0.2}{\omega_t(\sigma)} \right)^5,$$

and

$$c_i^t(\sigma) = \omega_t(\sigma) \frac{(\lambda_i c_{i,t} p_{i,t}(\sigma))^{0.2}}{\sum_{j=1}^I (\lambda_j c_{j,t} p_{j,t}(\sigma))^{0.2}}.$$  

Excess expenditure—the difference between $i$'s present value of consumption and her present value of endowment at time $t = 0$—is in turn

$$\sum_{t=0}^\infty \sum_{\sigma^{t-1} \in \Sigma^{t-1}} \sum_{s \in S} p_t(s^t(\sigma)) D_{0,t}(s^t(\sigma)) \gamma_t(s) \left[ c_i^t(s^t(\sigma)) - \omega_i^t(s^t(\sigma)) \right]$$

$$= \sum_{t=0}^\infty \sum_{\sigma^{t-1} \in \Sigma^{t-1}} \sum_{s \in S} 0.75^t \left( \frac{\sum_{j=1}^I (\lambda_j c_{i,j} p_{j,t}(s^t(\sigma)))^{0.2}}{\omega_t(s^t(\sigma))} \right)^5 \left[ c_i^t(s^t(\sigma)) - \omega_i^t(s^t(\sigma)) \right].$$

Under rational expectations, there are three equilibria of this economy, with respective consumption vectors

$$c^1_i = (c_i^1(s_i^1(\sigma)), c_i^1(s_i^2(\sigma)), c_i^1(s_i^3(\sigma))) = (4, 10.4, 2.6) \quad \text{and} \quad c^2_i = (4, 2.6, 10.4);$$

$$c^1_i = (5.083637, 11.369409, 3.945724) \quad \text{and} \quad c^2_i = (2.916363, 1.630591, 9.054276);$$

$$c^1_i = (2.916363, 9.054276, 1.630591) \quad \text{and} \quad c^2_i = (5.083637, 3.945724, 11.369409);$$

for all $t$ and $\sigma$. The first of these is symmetric, and derived using equal Negishi weights $\lambda_1 = \lambda_2 = 0.5$. The second is asymmetric in favor of consumer 1, and the third is essentially the same asymmetric equilibrium but in favor of consumer 2; these equilibria are derived using Negishi weights $(\lambda_1, \lambda_2) = (0.9415, 0.0585)$ and $(\lambda_1, \lambda_2) = (0.0585, 0.9415)$, respectively. At these weights, no consumers are in either excess demand or excess supply. By Walras’ Law, these equilibria are the zeroes of the graph in Figure 1, which depicts consumer 1’s excess expenditure as a function of her Negishi weight.

But which (if any) of these equilibria is stochastically stable? To answer this question, I will have to be more specific about the unperturbed dynamic $\Xi$. Outside of equilibrium, market clearing will of course not hold, and there will instead be excess demand (in the sense that the present value of consumption exceeds the present value of the endowment stream) or excess supply (vice versa). Looking at (13) and (14) in
Appendix B, we can see that an individual consumer is in excess demand (resp., supply) if her welfare weight $\lambda_i$ is “too high”. This intuition forms the basis for the Negishi (1960) algorithm for finding equilibrium welfare weights, whereby agents in excess demand have their welfare weights reduced and vice versa for agents in excess supply.

Thus, with disequilibrium welfare weights, application of the Negishi algorithm leads consumers to have their excess demand/supply reduced (in absolute value). For the purpose of this example, let us suppose that this property is satisfied by $\Xi$; in particular, assume that for all $t$ and $i$, if the present value of consumer $i$’s consumption in period $t$ exceeds (resp., falls short of) the present value of her endowment, then her welfare weight falls (resp., rises) in period $t + 1$, i.e. $\Xi_{t+1}^i < \Xi_t^i$ (resp., $\Xi_{t+1}^i > \Xi_t^i$).

Notice that the arrows in Figure 1 indicate the direction of the unperturbed dynamic $\Xi$; the weight of consumer 1 decreases from one period to the next if she has positive excess demand, and increases if she has positive excess supply. Thus, the symmetric equilibrium with $\lambda_1 = 0.5$ is very unstable in the long run, requiring only an arbitrarily small endowment shock to escape. The asymmetric equilibria with $\lambda_1 = 0.0585$ and $\lambda_1 = 0.9415$ look more stable, but their relative stability is determined by the comparison of $y$ and $z$ in Figure 1; for instance, the path of least resistance from $\lambda_1 = 0.0585$
to $\lambda_1 = 0.9415$ has resistance $y$, and that from $\lambda_1 = 0.9415$ to $\lambda_1 = 0.0585$ has resistance $z$. In this case, $y = z = 0.4415$ and hence both of the asymmetric equilibria are stochastically stable.

Suppose now though that the consumers have different beliefs on the likelihood of the three states, consumer 1 retaining rational expectations ($1/3, 1/3, 1/3$), but consumer 2 believing the iid transition probabilities to be ($0.4, 0.3, 0.3$). There are again three equilibria of this economy, with $\lambda_1 = 0.421, \lambda_1 = 0.935$ and $\lambda_1 = 0.07$, and respective period-0 consumption vectors

\[
\begin{align*}
    c^1_0 &= (3.799771, 10.310127, 2.512398); \quad \text{and} \quad c^2_0 = (4.200229, 2.689873, 10.487602); \\
    c^1_0 &= (5.080138, 11.367432, 3.941917); \quad \text{and} \quad c^2_0 = (2.919862, 1.632568, 9.058083); \\
    c^1_0 &= (2.919862, 9.215649, 1.717236); \quad \text{and} \quad c^2_0 = (5.080138, 3.784351, 11.282764).
\end{align*}
\]

However, without rational expectations for consumer 2, these consumption vectors are no longer constant through time. In the $\lambda_1 = 0.421$ equilibrium, for instance, consumer

Figure 2: Negishi map under heterogeneous expectations
1’s period-1 consumptions are

\[
\begin{align*}
    c_1^1(1,1,\ldots) &= 3.727100 & c_1^1(1,2,\ldots) &= 10.231506 & c_1^1(1,3,\ldots) &= 2.439316; \\
    c_1^1(2,1,\ldots) &= 3.841830 & c_1^1(2,2,\ldots) &= 10.354802 & c_1^1(2,3,\ldots) &= 2.555384; \\
    c_1^1(3,1,\ldots) &= 3.841830; & c_1^1(3,2,\ldots) &= 10.354802 & c_1^1(3,3,\ldots) &= 2.555384.
\end{align*}
\]

This delivers a rise in consumer 1’s \( p \)-expected period-1 consumption relative to period 0, consistent with the vanishing of the belief furthest from the truth (in relative entropy) in Blume and Easley’s (2006, §3.1) iid economy. The Negishi map for this case is altered as depicted in Figure 2, so that \( y = 0.351 < 0.514 = z \) and the high \( \lambda_1 \) equilibrium becomes uniquely stochastically stable (it has a larger “basin of attraction”). However, if consumer 2’s belief is instead \((0.33, 0.33, 0.34)\), so that it is biased optimistically towards her preferred state 3, then \( y > z \) and the low \( \lambda_1 \) equilibrium (now at \( \lambda_1 = 0.053 \)) becomes stochastically stable, as depicted in Figure 3.
4.2 Determinacy with Taylor rules

Smooth economies with a finite number of goods are of course generically determinate (Debreu 1970); they almost always have a finite set of equilibrium outcomes. However, infinite-horizon dynamic models fail this condition, leaving the troubling possibility of indeterminacy. As Kehoe, Levine, and Romer (1990, p. 2) observe, an indeterminate model “offers little guidance about what should be observed”, and renders it impossible “to condition on the equilibrium values that are observed and to perform comparative statics analysis because small changes in the underlying parameters can lead to large and discontinuous changes in the observed outcomes”.

In economies with a finite number of infinitely lived consumers, equilibria are generically determinate (Kehoe and Levine 1985, Muller and Woodford 1988, Kehoe, Levine, and Romer 1990, Shannon 1999, Shannon and Zame 2002). However, in the presence of side conditions capturing taxes, externalities or endogenous variables, determinacy may fail (Kehoe, Levine, and Romer 1992), as discussed in Appendix A. A prominent example of such a side condition leading to indeterminacy is provided by interest-rate-targeting regimes (Sargent and Wallace 1975). In such circumstances, a policy regime may be designed under which equilibrium is determinate, as is argued to be the case under an “active” Taylor rule satisfying the Taylor Principle (Clarida, Galí, and Gertler 2000, Woodford 2003).

However, strictly speaking, indeterminacy still prevails under the Taylor Principle (Cochrane 2011), leaving a profound problem of multiple equilibria. Here, I use market selection to choose between these equilibria. This situation can be modelled with Subsection 2.1’s example of the Woodford cashless economy; as Woodford puts it, “neither the usefulness nor the validity of the [cashless model] depends on a claim that monetary frictions do not exist in actual present-day economies” (Woodford 2003, p. 62). From (5) we know that, under equal discount factors, for all \( t, i \) and \( \sigma \),

\[
\sum_{s \in S} \frac{\beta u^i(c_{t+1}^i(\sigma))p_{t+1}^i(\sigma)}{u^i(c_{t}^i(\sigma))} = \sum_{s \in S} \frac{\beta u^i(c_{t+1}^s(\sigma))p_{t+1}^s(\sigma|\sigma')}{u^i(c_{t}^s(\sigma))} = \sum_{s \in S} \frac{\bar{p}_{t+1}(\sigma|\sigma')D_{t+1}(\sigma_{t+1})\gamma_{t+1}(\sigma_{t+1})}{\gamma_t(\sigma_t)}
\]

\[
\sum_{s \in S} \bar{p}_{t+1}(\sigma|\sigma')d_{t+1}(s_{t+1}^s(\sigma)) = \mathbb{E}_t D_{t+1} \frac{\gamma_{t+1}}{\gamma_t}
\]

\[
\mathbb{E}_t d_{t+1} = \mathbb{E}_t D_{t+1} \frac{\gamma_{t+1}}{\gamma_t},
\]

\[20\text{For a fiscal instance of such a policy regime, see Woodford (1986).} \]
where
\[ d_{t,t+1}(s_{t+1}^s(\sigma)) = \frac{\beta u''(c_{t+1}(\sigma)) p_{t+1}(\sigma|\sigma^t)}{u''(c_t^i(\sigma))} \frac{p_{t+1}(\sigma|\sigma^t)}{\bar{p}_{t+1}(\sigma|\sigma^t)} \]
is the (unique) real stochastic discount factor prevailing in the cylinder set \( s_{t+1}^s(\sigma) \) under complete markets (Harrison and Kreps 1979), \( D_{t,t+1}(s_{t+1}^s(\sigma)) \) is the corresponding nominal stochastic discount factor, \( E_t \) is expectation with respect to the weighted-average belief \( \bar{p}_t(\sigma|\sigma^{t-1}) \), \( r_t \) is the real rate of interest, and \( R_t \) is the nominal rate of interest. Note that \( E_t \) is the relevant expectational operator owing to the existence of a representative trader with beliefs \( \bar{p} \) under logarithmic utility (see Back 2017, §21.3, or Rubinstein 1974). If \( d_{t,t+1} \) is uncorrelated with the ratio \( \Pi_t \equiv \gamma_{t+1}/\gamma_t \) of the price of consumption in period \( t+1 \) to that in period \( t \), then it follows that
\[ \frac{1}{1 + r_t} \frac{1}{\Pi_t} = \frac{1}{1 + R_t}. \]
Linearization—and focusing for simplicity on the limit \( \beta \to 1 \), so that \( \forall t, r_t = 0 \)—then gives the “Fisher equation,”
\[ R_t = \frac{1}{E_t} \pi_t, \]
where \( \pi_t = \Pi_t - 1 \) is the period-(\( t + 1 \)) rate of inflation.
Clearly the economy’s path is undetermined as it stands, so I follow Cochrane (2011) in adding a Taylor rule with a target rate of inflation \( \pi^* \),
\[ R_t = \pi^* + \frac{1}{a}(\pi_t - \pi^*), \tag{8} \]
along with a “Ricardian” fiscal policy (that adjusts to satisfy the government budget constraint at any prices). Inflation in the model is then
\[ \pi_t = \pi^* + a(\bar{E}_t \pi_{t+1} - \pi^*). \]
Under the Taylor Principle \( 1/a > 1 \), there is a unique “locally bounded” equilibrium \( \pi_t = \pi^* \), illustrated in Figure 4. In this equilibrium, the economy follows a constant path on \( \pi^* \) for sure. However, there are many other equilibria satisfying \( \pi_{t+1} = \pi^* + (1/a)(\pi_t - \pi^*) \) (implying an explosive path for inflation), and hence we have the classic indeterminacy of inflation under interest-rate targeting (Sargent and Wallace 1975).  

---

21 Also, note by Lemma 1 above that the conditional belief \( \bar{p}_t(\sigma|\sigma^{t-1}) \) is the same as the consumption-weighted conditional belief \( \bar{p}_t(\sigma|\sigma^{t-1}) \), and hence that \( \bar{E}_t = \bar{E}_t \).
22 This will be true in the special case where there is no uncertainty about \( \Pi_t \) at time \( t \), and that where consumers are risk neutral (Sargent 1987, pp. 165-6).
23 Note that I omit the usual error terms from the model for simplicity, and thus deal with perfect-foresight equilibria. For an analysis of the noisy case, see Cochrane (2009).
24 The “locally bounded” solution is also the “minimum state variable” (MSV) solution (McCallum 1981, 2003).
We are led to wonder which equilibrium is stochastically stable. As Woodford (2003, p. 128) puts it, “it might seem that the existence of other equilibria with initial inflation rates arbitrarily close to the target rate should make it easy for the economy to ‘slip’ into one of those other equilibria”. But these equilibria “can only occur as a result of expectations of future inflation rates (at least in some states) that are even further from the target inflation rate”; thus, “the economy can only move to one of these alternative paths if expectations about the future change significantly, something that one may suppose would not easily occur”. Cochrane (2011, p. 582) takes issue with this view: “If you see a small change today in an unstable dynamic system, your expectations of the future may well change by a large amount. If you see the waiter trip, it is a good bet that the stack of plates he is carrying will crash.” My model provides a framework within which to evaluate these competing claims.

In particular, we can take the state of the world $\sigma_t$ in Section 2.1 to be the inflation rate $\pi_t$ here, and let the state be $T(x_{t-1}(\sigma), \sigma_{t-1}) = \pi^* + a(E_t\pi_{t+1} - \pi^*).$\footnote{Note that $x_{t-1}(\sigma)$ is sufficient to calculate $E_t$ owing to Lemma 1.} This side condition on the Negishii problem (2), and its associated indeterminacy, is a particular instance of the Kehoe, Levine, and Romer (1992) framework discussed in Appendix A.
Let $P$ consist of the Dirac measures \{\$\delta_*, \delta_-, \delta_+\$\} on the equilibrium paths

\[
\begin{align*}
&\left(\pi^*, \pi^*, \pi^*, \ldots\right) \\
&\left(\pi^* - 1, \pi^* - \frac{1}{a}, \pi^* - \frac{1}{a^2}, \ldots\right) \\
&\left(\pi^* + 1, \pi^* + \frac{1}{a}, \pi^* + \frac{1}{a^2}, \ldots\right).
\end{align*}
\]

Let $\lambda_*$, $\lambda_-$ and $\lambda_+$ be the unit vectors on the corresponding elements of $P$. Finally, suppose that $\Xi$ is such that: for all $t$, $i$ and $\sigma$, $\Xi_{t+1} = 0$ if $p_t^i(\sigma|\sigma^{t-1}) = 0$ and $\max_{j:\xi(\sigma)} p_t^j(\sigma|\sigma^{t-1}) > 0$; and if $\max_{j:\xi(\sigma)} p_t^j(\sigma|\sigma^{t-1}) = 0$, then $\Xi_{t+1} = \Xi_t$. In words, any consumer who put zero conditional probability on the realized state has her welfare weight fall to zero next period, unless all surviving consumers did so, in which case welfare weights remain unchanged next period.

**Proposition 2** If $P = \{\delta_*, \delta_-, \delta_+\}$, then $\lambda_*$ is stochastically stable.

**Proof.** I claim that $V(\lambda_-, \lambda_*) = V(\lambda_+, \lambda_*) = 1/2$ and all other resistances are 1. To see this, note that an equal-weight mixture of $\delta_-$ and $\delta_+$ gives $\mathbb{E}_t\pi_{t+1}$ (and hence a state) equal to that under $\lambda_*$ for all $t$; $\delta_*$ thus correctly predicts the path, and hence will dominate the population one period later as $\varepsilon \to 0$. Moreover, no other mixture of any two measures can generate $\mathbb{E}_t\pi_{t+1}$ under the remaining measure in this way. It follows that $\lambda_*$ has the minimum-resistance $l$-graph and hence is stochastically stable.

However, this finding might be sensitive to my choice of $P$. With this in mind, I now expand $P$ to include the Dirac measures \{\$\delta_-, \delta_+\$\} on the equilibrium paths

\[
\begin{align*}
&\left(\pi^* - 2, \pi^* - \frac{2}{a}, \pi^* - \frac{2}{a^2}, \ldots\right) \\
&\left(\pi^* + 2, \pi^* + \frac{2}{a}, \pi^* + \frac{2}{a^2}, \ldots\right).
\end{align*}
\]

**Proposition 3** If $P = \{\delta_*, \delta_-, \delta_+, \delta_-, \delta_+\}$, then $\lambda_*$ is stochastically stable.

**Proof.** I claim that $V(\lambda_*, \lambda_+) = V(\lambda_+, \lambda_-) = 1/2$; an equal-weight mixture of $\delta_*$ and $\delta_+$, for instance, gives $\mathbb{E}_t\pi_{t+1}$ equal to that under $\lambda_+$ for all $t$. But then a mutation away from $\lambda_-$ that puts weight $v$ on $\delta_+$ and weight $w$ on $\delta_*$ generates an expected period-$t$ state of

$$\pi^* + \frac{1}{a^{t-1}} (3v + w - 1).$$

If $v = (1 - w)/3$, then $\delta_*$ correctly predicts the path. Since $w$ can be arbitrarily small, it follows that $V(\lambda_-, \lambda_*) = 1/3$, and the same is true for $V(\lambda_+, \lambda_*)$. Since $V(\lambda_-, \lambda_-) = V(\lambda_+, \lambda_+) = 1$, $\lambda_*$ is again stochastically stable.

26Note, however, that the finding does not depend on the Taylor Principle.
To overturn the stochastic stability of $\lambda_*$ would require an asymmetry in $P$ around $\delta_*$, or an asymmetry of the mutation probabilities (in violation of Axiom 6) in the manner suggested by Bergin and Lipman (1996); they note that any equilibrium may be stochastically stable under an appropriately chosen mutation process, suggesting that it is important to think about the relative likelihood of mutations into different beliefs. If, for instance, consumers were sufficiently less likely to mutate into a deflationary belief than into a symmetric (around $\delta^*$) inflationary belief, then one of the explosive inflationary equilibria would be stochastically stable. However, without such an asymmetry, it is harder to go from target beliefs to extreme beliefs than vice versa, providing selection pressure in favour of the locally bounded solution.
Appendices

A Existence and Multiple Equilibria

Imposing a Markovian state of the world $T$ complicates the Blume and Easley model; the Negishī problem (2) is no longer a complete description of equilibrium. We can, however, employ the Kehoe, Levine, and Romer (1992) approach of characterizing equilibrium as the solution to a Negishī-style Pareto problem with side conditions.

Describing the standard Negishī problem (2) in the language of Kehoe, Levine, and Romer, the vector $c$ is chosen to maximize the distorted social welfare function $\sum_i \lambda_i U^i(c)$, given a vector of parameters $(\lambda_1, \ldots, \lambda_I)$, a constraint set $\left\{ c : \sum_i c^i - \omega \leq 0 \text{ and } \forall t, \sigma, c^i(\sigma) \geq 0 \right\}$, and the additional conditions that Negishī’s savings functions be zero (giving binding budget constraints). Note that the welfare weights in (15) below do not depend on the realized path of the economy. How can this be? Surely the realized path could determine whether consumer $i$ vanishes, and hence whether she receives positive weight in the Pareto problem? No, the fault in the logic here is that consumer $i$ could vanish and still receive positive expected utility under her (incorrect) belief $p^i$. Hence, when we place conditions on the evolution of the economy’s path, we have a particularly simple case of the Kehoe, Levine, and Romer framework where the parameters of the original Negishī problem are unaltered (by contrast with the Kehoe, Levine, and Romer examples with endogenous parameters).

But whilst the welfare weights of (2) are not affected by a Markovian state of the world, Blume and Easley’s “true” probability measure $p$ (necessary for the analysis of market selection of beliefs) is now endogenous. In particular, this true measure is given by the solution to an infinite-dimensional fixed-point problem,

$$ q_t(\sigma) = \int_{\Sigma} T(x_{t-1}(\sigma)) \, dq_{t-1}(\sigma), \quad t = 1, \ldots, \quad q_0(\sigma) = \phi_0, \quad (9) $$

where $\phi_0 \in \Delta(S)$. For any fixed point $\phi \in \Phi \subseteq \Delta(\Sigma)$ of (9), the first-order conditions of the Pareto problem (2) must hold for $\phi$ to be an equilibrium. Formally, I have added the side condition $p = \phi$, $\phi \in \Phi$, to (2), but since the Negishī problem makes no reference to $p$, its solution(s) remains unchanged and indeed Pareto optimal—unlike the examples of Kehoe, Levine, and Romer, where the welfare weights are endogenous and markets incomplete.

In short, Arrow–Debreu equilibrium is unaffected by endogenizing the economy’s
path, and Blume and Easley’s results continue to apply given a true path of play—but which true path? (9) may have no solutions, or it may have multiple solutions. Indeed, there can be a robust continuum of equilibria near a steady state (see Kehoe, Levine, and Romer 1992, §5), i.e. indeterminacy, as encountered in Subsection 4.2. There could be multiplicity arising from the various possible initial \( q_0(\sigma) \), or multiplicity for a given \( \phi_0 \). However, under each fixed point we have the same Pareto problem (2), whose Euler equations then interact with the fixed point to determine the dynamics of that equilibrium.

There is a substantial macroeconomic literature on the existence of Markov (or recursive) equilibrium in competitive-market economies. This is straightforward in the standard frictionless, representative-consumer case, and indeed under additional side conditions (capturing frictions or endogenous variables) over a finite horizon, but in infinite discrete time existence is an open question (Santos 2002, Kubler and Schmedders 2002).\(^{27}\) I could place conditions on \( T \) guaranteeing equilibrium existence (see, e.g., Stokey, Lucas, and Prescott 1989, §17.4), but since existence is not the focus of this paper I do not do so.

B Welfare Weights

Subsection 2.2’s dependence of the state on expectations is of course an important feature of modern macroeconomics. Could it be modelled by conditioning the state and beliefs on the current state alone—the defining characteristic of recursive competitive equilibrium (see, e.g. Ljungqvist and Sargent 2012, p. 277)? After all, under complete markets with rational expectations, the consumption allocation at time \( t-1 \) (and hence \( x_{t-1}(\sigma) \)) will depend only on the aggregate endowment realization in that period (and hence the state). To see this under logarithmic utility and equal discount factors in Subsection 2.1’s example of the Woodford cashless economy, note that (5) implies that, for all \( \sigma, t \) and \( i \),

\[
\beta^t \left( \frac{1}{c^i_t(\sigma)} \right) p^i_t(\sigma) = \alpha_i p_t(\sigma) D_{0,t}(\sigma) \gamma_t(\sigma)\]

\[
c^i_t(\sigma) = \frac{\beta^i p^i_t(\sigma)}{\alpha_i p_t(\sigma) D_{0,t}(\sigma) \gamma_t(\sigma)}.
\]

\(^{27}\)Hellwig (1982) provides an incomplete-markets example in which the unique rational-expectations equilibrium is not Markov.
Substituting this into the market-clearing condition,
\[
\frac{\beta^t}{p_t(\sigma)D_{0,t}(\sigma)\gamma_t(\sigma_t)} \sum_{j=1}^{I} \frac{p^t_j(\sigma)}{\alpha_i} = \omega_t(\sigma)
\]
\[
\gamma_t(\sigma_t) = \frac{\beta^t}{p_t(\sigma)D_{0,t}(\sigma)\omega_t(\sigma)} \frac{\beta^t}{\bar{p}_t(\sigma)}
\]
(11)

where the final equality follows from the existence of a representative trader with beliefs \( \bar{p} \) under logarithmic utility (Rubinstein 1974). Thus, given a partial history \( \sigma^{t-1} \), the equilibrium price \( \gamma_t(\sigma_t) \) (which is also the Lagrange multiplier \( \theta_t(\sigma) \) of the Negishi problem) is decreasing in the aggregate consumption endowment in state \( \sigma_t \). If all consumers have rational expectations, it follows from (10) and (11) that consumption,
\[
c_t^i(\sigma) = \lambda_i \omega_t(\sigma)
\]
depends only on the current state (via the aggregate endowment).

However, with heterogeneous beliefs,
\[
c_t^i(\sigma) = \lambda_i \omega_t(\sigma) \frac{p^t_i(\sigma)}{\bar{p}_t(\sigma)}
\]
(12)
depends also on both \( p^t_i(\sigma) \) and \( \bar{p}(\sigma) \), each of which depends on the entire partial history \( \sigma^t \). Nevertheless, if we know the welfare weights \( (\lambda_i)_{i \in I} \), we can still calculate the conditional expectation \( E_t \equiv E_{\bar{p}}(\cdot | \sigma^{t-1}) \). Letting \( n_{t+1}(\sigma) \) be the state-contingent value of nonmonetary assets in period \( t + 1 \), the absence of arbitrage opportunities implies the existence of a unique stochastic discount factor (or asset-pricing kernel) \( D_{t,t+1} \) with the property that \( b_t^i(\sigma) = E_t[D_{t,t+1}n_{t+1}(\sigma)] \) (Harrison and Kreps 1979). Letting \( R^m_t \) be the nominal interest rate paid on money balances held at the end of period \( t \), the beginning-of-period value of all assets is given by \( a_{t+1}(\sigma) = (1 + R^m_t)m_t(\sigma) + n_{t+1}(\sigma) \), and the budget constraint (4) becomes
\[
\sum_{s \in S} \gamma_t(s)c_t^i(\xi_t^s(\sigma)) + m_t^i(\xi_t^s(\sigma)) + E_t \left[ D_{t,t+1} \left( a_{t+1}^i(\xi_t^s(\sigma)) - (1 + R^m_t)m_t^i(\xi_t^s(\sigma)) \right) \right]
\]
\[
\leq \sum_{s \in S} \gamma_t(s)\omega_t^i(\xi_t^s(\sigma)) + a_t^i(\xi_t^s(\sigma))
\]
\[
\iff \sum_{s \in S} \gamma_t(s)c_t^i(\xi_t^s(\sigma)) + \nu_t m_t^i(\xi_t^s(\sigma)) + E_t \left[ D_{t,t+1}a_{t+1}^i(\xi_t^s(\sigma)) \right]
\]
\[
\leq \sum_{s \in S} \gamma_t(s)\omega_t^i(\xi_t^s(\sigma)) + a_t^i(\xi_t^s(\sigma)),
\]
where \( \nu_t \equiv (R_t - R^m_t)/(1 + R_t) \), and \( R_t \) is the riskless one-period nominal interest rate.
that solves $1/(1 + R_t) = E_t D_{t,t+1}$. This constraint (and a borrowing limit ruling out Ponzi schemes) is satisfied in each period if and only if

$$
\sum_{t=0}^{\infty} \sum_{\sigma^{-1} \in \Sigma^{t-1}} \sum_{s \in S} p_t(\zeta^s(\sigma)) D_{0,t}(\zeta^s(\sigma)) \left[ \gamma_t(s) c_t^i(\zeta^s(\sigma)) + \nu_t m_t^i(\zeta^s(\sigma)) \right]
\leq a_0 + \sum_{t=0}^{\infty} \sum_{\sigma^{-1} \in \Sigma^{t-1}} \sum_{s \in S} p_t(\zeta^s(\sigma)) D_{0,t}(\zeta^s(\sigma)) \gamma_t(s) \omega_t^i(\zeta^s(\sigma)),
$$

where $D_{t,T} = \prod_{\tau=t+1}^{T} D_{\tau-1,\tau}$ (see Woodford 2003, Proposition 2.1); this is the single budget constraint characteristic of complete-markets models. Substituting (10) into this constraint (which will bind at the optimum), and since $\nu_t$ must be 0 if money is to be held in positive quantities,

$$
\sum_{t=0}^{\infty} \sum_{\sigma^{-1} \in \Sigma^{t-1}} \sum_{s \in S} p_t(\zeta^s(\sigma)) D_{0,t}(\zeta^s(\sigma)) \left[ \frac{\beta^t p_t^i(\zeta^s(\sigma))}{\alpha_t} \right] = a_0 + \sum_{t=0}^{\infty} \sum_{\sigma^{-1} \in \Sigma^{t-1}} \sum_{s \in S} p_t(\zeta^s(\sigma)) D_{0,t}(\zeta^s(\sigma)) \gamma_t(s) \omega_t^i(\zeta^s(\sigma))
$$

$$
\Leftrightarrow \alpha_t = \frac{\sum_{t=0}^{\infty} \beta^t \sum_{\sigma^{-1} \in \Sigma^{t-1}} \sum_{s \in S} p_t(\zeta^s(\sigma))}{a_0 + \sum_{t=0}^{\infty} \sum_{\sigma^{-1} \in \Sigma^{t-1}} \sum_{s \in S} p_t(\zeta^s(\sigma)) D_{0,t}(\zeta^s(\sigma)) \gamma_t(s) \omega_t^i(\zeta^s(\sigma))}
$$

$$
\Rightarrow \lambda_t = \frac{a_0 + \sum_{t=0}^{\infty} \sum_{\sigma^{-1} \in \Sigma^{t-1}} \sum_{s \in S} p_t(\zeta^s(\sigma)) D_{0,t}(\zeta^s(\sigma)) \gamma_t(s) \omega_t^i(\zeta^s(\sigma))}{\sum_{t=0}^{\infty} \beta^t}
$$

(14)

Substituting (11) into (14) and setting $a_0 = 0$ gives

$$
\lambda_t = \sum_{t=0}^{\infty} \sum_{\sigma^{-1} \in \Sigma^{t-1}} \sum_{s \in S} p_t(\zeta^s(\sigma)) \left( \frac{\omega_t^i(\zeta^s(\sigma))}{\omega_t^s(\zeta^s(\sigma))} \right)
$$

(15)

so that consumer $i$'s welfare weight is determined by her expected share of the intertemporal endowment. We then have a system of $I$ equations in the $I$ unknown welfare weights; the classic Negishi reduction of an infinite-dimensional problem (with demands depending on the value of the infinite endowment stream) to a finite-dimensional one.

Thus, we are faced with the choice of calculating the welfare weights or keeping track of the consumption shares. In Section 3, I use the former.

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28The latter is because $E_t D_{t,t+1}$ prices the risk-free period-$(t + 1)$ asset; since the model has just one consumption good, this asset will simply pay 1 unit of consumption in each period-$(t + 1)$ state.
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